CS256/Winter 2009 Lecture #3

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TEMPORAL LOGIC(S)

Languages that can specify the behavior of a reactive program.

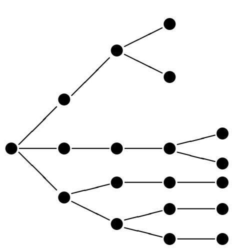
Two views:

(1) the program generates a set of sequences of states

- the models of temporal logic are infinite sequences of states
- LTL (linear time temporal logic) [Manna, Pnueli] approach

(2) the program generates a tree, where the branching points represent nondeterminism in the program

- the models of temporal logic are infinite trees
- CTL (computation tree logic) [Clarke, Emerson] at CMU Also CTL*.



Temporal logic: underlying assertion language

Assertion language $\mathcal{L}$:

first-order language over interpreted typed symbols

(functions and relations over concrete domains)

`Example:` $x > 0 \;\rightarrow\; x + 1 > y$

$x, y \in \mathbf{Z}^+$

formulas in $\mathcal{L}$ called:

state formulas or assertions

Temporal logic: underlying assertion language (Con't)

A state formula is evaluated over a single state to yield a truth value.

For state s and state formula p

$$s \Vdash p \quad \text{if} \quad s[p] = \mathrm{T}$$

We say:

p holds at s
s satisfies p
s is a p-state

> `Example:`
>
> For state $s : \{x : 4,\ y : 1\}$
>
> $s \Vdash x = 0 \lor y = 1$
> $s \nVdash x = 0 \land y = 1$
> $s \Vdash \exists z.\ x = z^2$

Temporal logic: underlying assertion language (Con't)

p is state-satisfiable if

$s \Vdash p$ for some state s

p is state-valid if

$s \Vdash p$ for all states s

p and q are state-equivalent if

$s \Vdash p$ iff $s \Vdash q$ for all states s

> `Example:` $(x, y :$ integer$)$
>
> state-valid: $x \geq y \ \leftrightarrow \ x+1 > y$
>
> state-equivalent: $x = 0 \ \rightarrow \ y = 1$
> and
> $x \neq 0 \ \lor \ y = 1$

TEMPORAL LOGIC (TL)

A formalism for specifying sequences of states

TL = assertions + temporal operators

- assertions (state formulas):
  First-order formulas
  describing the properties of a single state

- temporal operators
  Fig 0.15

Future Temporal Operators

$\square\, p$ – Henceforth p
$\diamond\, p$ – Eventually p
$p\,\mathcal{U}\,q$ – p Until q
$p\,\mathcal{W}\,q$ – p Waiting-for (Unless) q
$\bigcirc\, p$ – Next p

Past Temporal Operators

$\boxminus\, p$ – So-far p
$\diamondminus\, p$ – Once p
$p\,\mathcal{S}\,q$ – p Since q
$p\,\mathcal{B}\,q$ – p Back-to q
$\ominus\, p$ – Previously p
$\circledtilde\, p$ – Before p

Fig. 0.15. The temporal operators

future temporal operators

$$\longleftarrow \text{ past } \longrightarrow | \longleftarrow \text{ future } \longrightarrow\longrightarrow\longrightarrow$$

0 $\qquad\qquad\uparrow$ present

$\Diamond q$	—	Eventually q	q 0 ↑
$\Box p$	—	Henceforth p	$p\ p\ p\ p$ ...... 0 ↑
$p\,\mathcal{U}\,q$	—	p Until q	$p\ p\ p\ p\ p\ q$ 0 ↑
$p\,\mathcal{W}\,q$	—	p Wait-for (Unless) q	$\Box p \ \vee\ p\,\mathcal{U}\,q$
$\bigcirc p$	—	Next p	p 0 ↑

past temporal operators

$\diamondminus q$	—	Once q	q 0 ↑
$\boxminus p$	—	So-far p	$p\ p\ p\ p\ p\ p$ 0 ↑
$p\,\mathcal{S}\,q$	—	p Since q	$q\ p\ p\ p\ p\ p$ 0 ↑
$p\,\mathcal{B}\,q$	—	p Back-to q	$\boxminus p \ \vee\ p\,\mathcal{S}\,q$
$\ominus p$	—	Previously p (false at position 0)	p 0 ↑
$\widetilde{\ominus}\, p$	—	Before p (true at position 0)	

Temporal Logic: Syntax

- Every assertion is a temporal formula
- If p and q are temporal formulas (and u is a variable), so are:

$\neg p \qquad p \vee q \qquad p \wedge q \qquad p \rightarrow q \qquad p \leftrightarrow q$

$\exists u.p \qquad \forall u.p$

$\Box p \qquad \Diamond p \qquad p\,\mathcal{U}\,q \qquad p\,\mathcal{W}\,q \qquad \bigcirc p$

$\boxminus p \qquad \diamondminus p \qquad p\,\mathcal{S}\,q \qquad p\,\mathcal{B}\,q \qquad \ominus p \qquad \widetilde{\ominus}\, p$

`Example:`

$$\Box(x > 0 \rightarrow \diamondminus\, y = x)$$

$$p\,\mathcal{U}\,q \rightarrow \Diamond q$$

Temporal Logic: Semantics

Temporal formulas are evaluated over <u>a model</u> (an infinite sequence of states)

$$\sigma :\ s_0,\ s_1,\ s_2,\ \ldots$$

- The semantics of temporal logic formula p at a position $j \geq 0$ in a model σ,

$$(\sigma, j) \vDash p$$

"formula p holds at position j of model σ", is defined by induction on p:

$$\sigma :\quad s_0,\ \ s_1,\ \ \ldots,\quad \underset{\substack{\uparrow \\ (\sigma, j)}}{s_j,}\quad \ldots$$

Temporal Logic: Semantics (Con't)

For state formula (assertion) p
(i.e., no temporal operators)

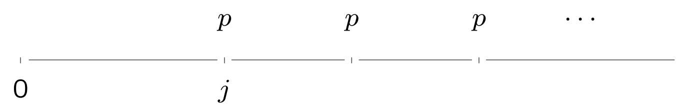
- $(\sigma, j) \vDash p \quad \Longleftrightarrow \quad s_j \Vdash p$

For a temporal formula p:

- $(\sigma, j) \vDash \neg p \quad \Longleftrightarrow \quad (\sigma, j) \nvDash p$
- $(\sigma, j) \vDash p \vee q \quad \Longleftrightarrow \quad (\sigma, j) \vDash p$ or $(\sigma, j) \vDash q$

Temporal Logic: Semantics (Con't)

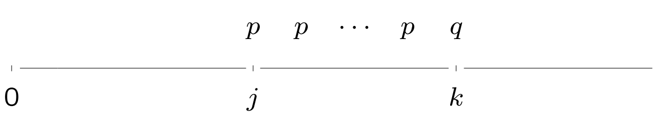
- $(\sigma, j) \vDash \square\, p \quad \Longleftrightarrow$
  for all $k \geq j,\ (\sigma, k) \vDash p$



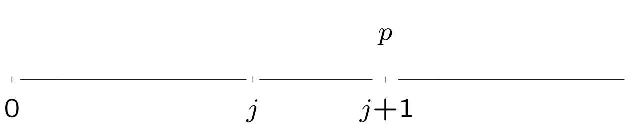
- $(\sigma, j) \vDash \Diamond\, p \quad \Longleftrightarrow$
  for some $k \geq j,\ (\sigma, k) \vDash p$

Temporal Logic: Semantics (Con't)

- $(\sigma, j) \vDash p\, \mathcal{U}\, q \quad \Longleftrightarrow$
  for some $k \geq j,\ (\sigma, k) \vDash q$,
  and for all i, $j \leq i < k,\ (\sigma, i) \vDash p$

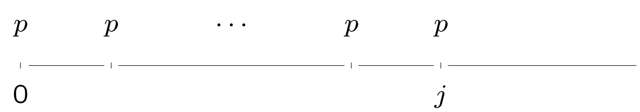


- $(\sigma, j) \vDash p\, \mathcal{W}\, q \quad \Longleftrightarrow$
  $(\sigma, j) \vDash p\, \mathcal{U}\, q$ or $(\sigma, j) \vDash \square\, p$
- $(\sigma, j) \vDash \bigcirc\, p \quad \Longleftrightarrow$
  $(\sigma, j+1) \vDash p$



Temporal Logic: Semantics (Con't)

- $(\sigma, j) \vDash \boxminus\, p \quad \Longleftrightarrow$
  for all k, $0 \leq k \leq j,\ (\sigma, k) \vDash p$



- $(\sigma, j) \vDash \diamondminus\, p \quad \Longleftrightarrow$
  for some k, $0 \leq k \leq j,\ (\sigma, k) \vDash p$

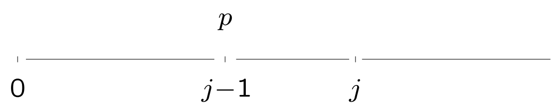
Temporal Logic: Semantics (Con't)

- $(\sigma, j) \models p\,\mathcal{S}\,q \iff$
  for some k, $0 \leq k \leq j$, $(\sigma, k) \models q$
  and for all i, $k < i \leq j$, $(\sigma, i) \models p$

- $(\sigma, j) \models p\,\mathcal{B}\,q \iff$
  $(\sigma, j) \models p\,\mathcal{S}\,q$ or $(\sigma, j) \models \boxminus\, p$

Temporal Logic: Semantics (Con't)

- $(\sigma, j) \models \circleddash\, p \iff$
  $j \geq 1$ and $(\sigma, j-1) \models p$



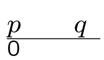
- $(\sigma, j) \models \widetilde{\circleddash}\, p \iff$
  either $j = 0$ or else $(\sigma, j-1) \models p$

Simple Examples

Given temporal formula φ, describe model σ, such that

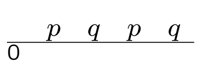
$$(\sigma, 0) \models \varphi$$

$p \rightarrow \Diamond\, q$



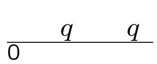
if initially p then eventually q

$\Box(p \rightarrow \Diamond\, q)$



every p is eventually followed by a q

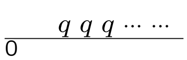
$\Box\,\Diamond\, q$



every position is eventually followed by a q,
i.e.,
infinitely many q's

Simple Examples (Con't)

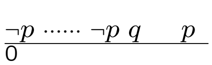
$\Diamond\,\Box\, q$



eventually permanently q,
i.e.,
finitely many $\neg q$'s

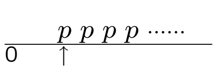
$\Box\,\Diamond\, p \rightarrow \Box\,\Diamond\, q$
if there are infinitely many p's
then there are infinitely many q's

$(\neg p)\,\mathcal{W}\,q$



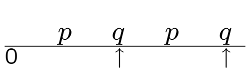
q precedes p (if p occurs)

$\Box(p \rightarrow \bigcirc\, p)$



once p, always p

$\Box(q \rightarrow \diamondminus\, p)$



every q is preceded by a p

Nested Waiting-for Formulas

$$\boxed{q_1\ \mathcal{W}\ q_2\ \mathcal{W}\ q_3\ \mathcal{W} q_4}$$

stands for

$$q_1\ \mathcal{W}\ (q_2\ \mathcal{W}\ (q_3\ \mathcal{W}\ q_4))$$

intervals of continuous q_i

$$\overbrace{q_1\cdots q_1}\ \ \overbrace{q_2\cdots q_2}\ \ \overbrace{q_3\cdots q_3}\ \ q_4$$
$$0\quad \uparrow$$

- possibly empty interval

$$\overbrace{q_1\cdots q_1}\ \ \overbrace{q_3\cdots q_3}\ \ q_4$$
$$0\qquad \uparrow$$

- possibly infinite interval

$$\overbrace{q_1\cdots q_1}\ \ \overbrace{q_2\cdots q_2}\ \ q_3q_3q_3\cdots\cdots\cdots q_3\cdots\cdots$$
$$0\quad \uparrow \qquad\qquad\qquad\qquad \rightarrow$$

Abbreviation:

$p \Rightarrow q$ for $\square(p \rightarrow q)$ "p entails q"

> `Example:`
> $p \Rightarrow \Diamond\, q$
> stands for
> $\square(p \rightarrow \Diamond\, q)$

Past/Future Formulas

Past Formula –
formula with no future operators

Future Formula –
formula with no past operators

A state formula is both a past and a future formula.

Definitions

- For temporal formula p, sequence σ and position $j \geq 0$:

  $(\sigma, j) \vDash p$: p holds at position j of σ
  σ satisfies p at j
  j is a p-position in σ.

- For temporal formula p and sequence σ,

$$\sigma \vDash p \quad \text{iff} \quad (\sigma, 0) \vDash p$$

$\sigma \vDash p$: p holds on σ
σ satisfies p

Satisfiable/Valid

For temporal formula p,

- p is satisfiable if $\sigma \vDash p$ for some sequence (model) σ
- p is valid if $\sigma \vDash p$ for all sequences (models) σ

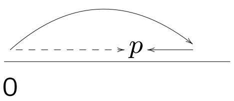
p is valid iff $\neg p$ is unsatisfiable

> `Example:` (x : integer)
> $\Diamond(x = 0)$ is satisfiable
> $\Diamond(x = 0) \vee \square(x \neq 0)$ is valid
> $\Diamond(x = 0) \wedge \square(x \neq 0)$ is unsatisfiable

Equivalence

For temporal formulas p and q:

p is equivalent to q, written $p \sim q$
if $p \leftrightarrow q$ is valid
(i.e., p and q have the same truth-value at the first position of every model)

> `Example:`
>
> $\Diamond p \sim \Diamond \diamondminus p$
>
> 
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$\varphi \sim \psi$: for any σ,
$(\sigma, 0) \vDash \varphi$ iff $(\sigma, 0) \vDash \psi$.

φ valid: for any σ, $(\sigma, 0) \vDash \varphi$.

Therefore,
φ, ψ valid $\Rightarrow \varphi \sim \psi$.

φ unsatisfiable: for any σ, $(\sigma, 0) \nvDash \varphi$.

For the same reason,
$\varphi,\ \psi$ unsatisfiable $\Rightarrow \varphi \sim \psi$.

first

Characterizes the first position.

first: $\neg \ominus \mathrm{T}$

$(\sigma, j) \vDash$ *first*: true for $j = 0$
false for $j > 0$

Then

- $\mathrm{T} \sim \Box \mathrm{T} \sim first$
- T, $\Box \mathrm{T}$, $first$ are valid

Assume V={integer x}
$first : \neg \ominus (x = 0 \vee x \neq 0)$
$\mathrm{T} : (x = 0 \vee x \neq 0)$
$\Box \mathrm{T} : \Box (x = 0 \vee x \neq 0)$

For arbitrary σ:
$(\sigma, 0) \vDash first \quad (\sigma, 0) \vDash \mathrm{T} \quad (\sigma, 0) \vDash \Box \mathrm{T}$
$(\sigma, j) \nvDash first \quad (\sigma, j) \vDash \mathrm{T} \quad (\sigma, j) \vDash \Box \mathrm{T}$ for $j > 0$

Congruence

For temporal formulas p and q:
p is congruent to q, written $p \approx q$
if $\Box (p \leftrightarrow q)$ is valid
$\varphi \approx \psi$: for any σ, j, $(\sigma, j) \vDash \varphi$ iff $(\sigma, j) \vDash \psi$

> `Example:`
>
> $\mathrm{T} \approx \Box \mathrm{T}$
> $\mathrm{T} \not\approx first$ — T may be true in the second state, but $first$ is not
>
> $\Diamond p \not\approx \Diamond \diamondminus p$ — because $\Rightarrow$, but $\not\Leftarrow$
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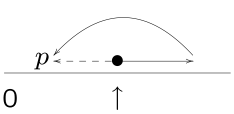
> 
>
>
> $\Box p \approx \neg \Diamond \neg p$
> $\neg \bigcirc p \approx \bigcirc \neg p$

Note
$A \approx B$ *iff* $A \Rightarrow B$ and $B \Rightarrow A$ are valid
$A \sim B$ *iff* $A \rightarrow B$ and $B \rightarrow A$ are valid

Congruences

"conjunction character" — match well with $\wedge$
"disjunction character" — match well with $\vee$

$\Box$ and $\boxminus$ have conjunction character

$\Diamond$ and $\diamondminus$ have disjunction character

$\mathcal{U}$, $\mathcal{W}$, $\mathcal{S}$, $\mathcal{B}$ first argument has conjunction character
second argument has disjunction character

$$\Box (p \wedge q) \approx \Box p \wedge \Box q$$

$$\Diamond (p \vee q) \approx \Diamond p \vee \Diamond q$$

$$p \,\mathcal{U}\, (q \vee r) \approx (p \,\mathcal{U}\, q) \vee (p \,\mathcal{U}\, r)$$

$$(p \wedge q) \,\mathcal{U}\, r \approx (p \,\mathcal{U}\, r) \wedge (q \,\mathcal{U}\, r)$$

$$p \,\mathcal{W}\, (q \vee r) \approx (p \,\mathcal{W}\, q) \vee (p \,\mathcal{W}\, r)$$

$$(p \wedge q) \,\mathcal{W}\, r \approx (p \,\mathcal{W}\, r) \wedge (q \,\mathcal{W}\, r)$$

Expansions

$$\square\, p \;\approx\; (p \,\wedge\, \bigcirc \square\, p)$$

$$\diamond\, p \;\approx\; (p \,\vee\, \bigcirc \diamond\, p)$$

$$p\,\mathcal{U}\,q \;\approx\; \Big[q \,\vee\, (p \,\wedge\, \bigcirc(p\,\mathcal{U}\,q))\Big]$$

$$\boxminus\, p \;\approx\; (p \,\wedge\, \widetilde{\ominus}\, \boxminus\, p)$$

$$\diamondminus\, p \;\approx\; (p \,\vee\, \ominus\, \diamondminus\, p)$$

$$p\,\mathcal{S}\,q \;\approx\; \Big[q \,\vee\, (p \,\wedge\, \ominus(p\,\mathcal{S}\,q))\Big]$$

Strict Operators

(present not included)

$$\begin{array}{ccccc} [\longleftrightarrow] & & \bullet & & [\longrightarrow \\ s_0 \quad s_{j-1} & & \uparrow & & s_{j+1} \\ & & s_j & & \end{array}$$

$\widehat{\square} p$	$\approx$	$\bigcirc \square\, p$	$\widehat{\boxminus} p$	$\approx$	$\widetilde{\ominus}\, \boxminus\, p$
$\widehat{\diamond} p$	$\approx$	$\bigcirc \diamond\, p$	$\widehat{\diamondminus} p$	$\approx$	$\ominus \diamondminus\, p$
$p\,\widehat{\mathcal{U}}\,q$	$\approx$	$\bigcirc(p\,\mathcal{U}\,q)$	$p\,\widehat{\mathcal{S}}\,q$	$\approx$	$\ominus(p\,\mathcal{S}\,q)$
$p\,\widehat{\mathcal{W}}\,q$	$\approx$	$\bigcirc(p\,\mathcal{W}\,q)$	$p\,\widehat{\mathcal{B}}\,q$	$\approx$	$\widetilde{\ominus}(p\,\mathcal{B}\,q)$

Next and Previous Values of Exps

When evaluating x at position $j \geq 0$

x refers to $s_j[x]$

x^+ refers to $s_{j+1}[x]$

x^- refers to $\begin{cases} s_{j-1}[x] & \text{if } j > 0 \\ s_0[x] & \text{if } j = 0 \end{cases}$

Example:

$$\sigma\colon\ \langle x\colon 0\rangle,\ \langle x\colon 1\rangle,\ \langle x\colon 2\rangle,\ \ldots$$

satisfies

$$x = 0 \,\wedge\, \square(x^+ = x+1) \,\wedge\, \bigcirc \square(x = x^- + 1)$$

Temporal Logic: Substitutivity

The ability to substitute equals for equals in a formula and obtain a formula with identical meaning.

- For state formula $\phi(u)$

  if $p \sim q$ then $\phi(p) \sim \phi(q)$

Example:

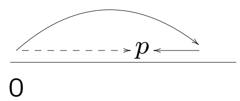
Consider state formula $\phi(u)\colon\ r \wedge u$

Since $\quad \diamond\, p \;\sim\; \diamond \diamondminus\, p$

then $\quad r \wedge \diamond\, p \;\sim\; r \wedge \diamond \diamondminus\, p.$

Temporal Logic: Substitutivity (Con't)

This does not hold if $\phi(u)$ is a temporal formula.

> `Example:`
>
> Consider temporal formula $\phi(u)$: $\square\, u$
>
> $$\begin{array}{ll} & \Diamond\, p \;\sim\; \Diamond \diamondminus p \\ \text{but} & \square \Diamond\, p \not\sim \square \Diamond \diamondminus p \end{array}$$
>
> 
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- For temporal formula $\phi(u)$

  if $p \approx q$ then $\phi(p) \approx \phi(q)$

> `Example:`
>
> Consider the temporal formula $\phi(u)$: $q\,\mathcal{U} u$
>
> | | |
> |---|---|
> | Since | $\square\, p \;\approx\; \neg \Diamond \neg p$ |
> | therefore | $q\,\mathcal{U}(\square\, p) \;\approx\; q\,\mathcal{U}(\neg \Diamond \neg p)$ |