The Calculus of Computation

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Calculus of Computation?

> *It is reasonable to hope that the relationship between* **computation** *and* **mathematical logic** *will be as fruitful in the next century as that between* **analysis** *and* **physics** *in the last. The development of this relationship demands a concern for both applications and mathematical elegance.*

John McCarthy
A Basis for a Mathematical Theory of Computation, 1963

Assignment #1

- 1.1 e, f [10 points each]
- 1.2 s, x [10 points each]
- 1.3 (note typo: the last $\vee$ should be a $\wedge$) [30 points]
- 1.5 d [30 points]

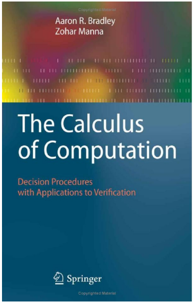
Textbook

THE CALCULUS OF COMPUTATION:
Decision Procedures with
Applications to Verification

by
Aaron Bradley
Zohar Manna

Springer 2007

There are two copies in CS-Math Library and you could also use `socrates.stanford.edu` to read the book according to its policy.



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The Calculus of Computation

Decision Procedures with Applications to Verification

Springer

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Topics: Overview

1. First-Order logic
2. Specification and verification
3. Satisfiability decision procedures

Part I: Foundations

1. Propositional Logic
2. First-Order Logic
3. First-Order Theories
4. Induction
5. Program Correctness: Mechanics
   Inductive assertion method, Ranking function method

Part II: Decision Procedures

7. Quantified Linear Arithmetic
   Quantifier elimination for integers and rationals
8. Quantifier-Free Linear Arithmetic
   Linear programming for rationals
9. Quantifier-Free Equality and Data Structures
10. Combining Decision Procedures
    Nelson-Oppen combination method
11. Arrays
    More than quantifier-free fragment

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Motivation

Motivation I

Decision Procedures are algorithms to decide formulae.
These formulae can arise

- in software verification.
- in hardware verification

Consider the following program:

```
for
  @ $\ell \leq i \leq u \wedge (rv \leftrightarrow \exists j.\ \ell \leq j < i \wedge a[j] = e)$
  (int $i := \ell$; $i \leq u$; $i := i + 1$) {
  if ($a[i] = e$) $rv$ := true;
}
```

How can we decide whether the formula is a loop invariant?

Motivation II

Prove:

```
assume $\ell \leq i \leq u \wedge (rv \leftrightarrow \exists j.\ \ell \leq j < i \wedge a[j] = e)$
assume $i \leq u$
assume $a[i] = e$
$rv$ := true;
$i := i + 1$
@ $\ell \leq i \leq u \wedge (rv \leftrightarrow \exists j.\ \ell \leq j < i \wedge a[j] = e)$
```

Motivation III

```
assume $\ell \le i \le u \land (rv \leftrightarrow \exists j.\ \ell \le j < i \land a[j] = e)$
assume $i \le u$
assume $a[i] \ne e$
$i := i + 1$
@ $\ell \le i \le u \land (rv \leftrightarrow \exists j.\ \ell \le j < i \land a[j] = e)$
```

A Hoare triple {P} S {Q} holds, iff

$$P \rightarrow wp(S, Q)$$

(wp denotes "weakest precondition")

Motivation IV

For assignments wp is computed by substitution:

```
assume $\ell \le i \le u \land (rv \leftrightarrow \exists j.\ \ell \le j < i \land a[j] = e)$
assume $i \le u$
assume $a[i] = e$
$rv$ := true;
$i := i + 1$
@ $\ell \le i \le u \land (rv \leftrightarrow \exists j.\ \ell \le j < i \land a[j] = e)$
```

Substituting $\top$ for rv and $i+1$ for i, the postcondition (denoted by the @ symbol) holds if and only if:

$$\begin{aligned} & \ell \le i \le u \land (rv \leftrightarrow \exists j.\ \ell \le j < i \land a[j] = e) \land i \le u \land a[i] = e \\ \rightarrow\ & \ell \le i+1 \le u \land (\top \leftrightarrow \exists j.\ \ell \le j < i+1 \land a[j] = e) \end{aligned}$$

Motivation V

We need an algorithm that decides whether this formula holds. If the formula does not hold, the algorithm should give a counterexample; e.g.,

$$\ell = 0, i = 1, u = 1, rv = \mathit{false}, a[0] = 0, a[1] = 1, e = 1.$$

We will discuss such algorithms in later lectures.

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Chapter 1: Propositional Logic (PL)

Propositional Logic (PL)

PL Syntax

Atom truth symbols $\top$ ("true") and $\bot$ ("false")
propositional variables $P, Q, R, P_1, Q_1, R_1, \ldots$

Literal atom α or its negation $\neg\alpha$

Formula literal or application of a
logical connective to formulae F, F_1, F_2

$\neg F$	"not"	(negation)
$F_1 \wedge F_2$	"and"	(conjunction)
$F_1 \vee F_2$	"or"	(disjunction)
$F_1 \rightarrow F_2$	"implies"	(implication)
$F_1 \leftrightarrow F_2$	"if and only if"	(iff)

Example:

formula $F : (P \wedge Q) \rightarrow (\top \vee \neg Q)$
atoms: $P, Q, \top$
literals: $P, Q, \top, \neg Q$
subformulae: $P, Q, \top, \neg Q, P \wedge Q, \top \vee \neg Q, F$
abbreviation
$F : P \wedge Q \rightarrow \top \vee \neg Q$

PL Semantics (meaning of PL)

Formula F + Interpretation I = Truth value (true, false)

Interpretation

$$I : \{P \mapsto \text{true}, Q \mapsto \text{false}, \cdots\}$$

Evaluation of F under I:

F	$\neg F$
0	1
1	0

where 0 corresponds to value false
1 true

F_1	F_2	$F_1 \wedge F_2$	$F_1 \vee F_2$	$F_1 \rightarrow F_2$	$F_1 \leftrightarrow F_2$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

Example:

$F : P \wedge Q \rightarrow P \vee \neg Q$
$I : \{P \mapsto \text{true}, Q \mapsto \text{false}\}$ i.e., $I[P] = \text{true}, I[Q] = \text{false}$

P	Q	$\neg Q$	$P \wedge Q$	$P \vee \neg Q$	F
1	0	1	0	1	1

1 = true 0 = false

F evaluates to true under I; i.e., $I[F] = \text{true}$.

Inductive Definition of PL's Semantics

$I \models F$ if F evaluates to true under I
$I \not\models F$ false

Base Case:

$I \models \top$ $I \not\models \bot$

$I \models P$ iff $I[P] = \text{true}$; i.e., P is true under I

$I \not\models P$ iff $I[P] = \text{false}$

Inductive Case:

$I \models \neg F$ iff $I \not\models F$

$I \models F_1 \wedge F_2$ iff $I \models F_1$ and $I \models F_2$

$I \models F_1 \vee F_2$ iff $I \models F_1$ or $I \models F_2$ (or both)

$I \models F_1 \rightarrow F_2$ iff $I \models F_1$ implies $I \models F_2$

$I \models F_1 \leftrightarrow F_2$ iff, $I \models F_1$ and $I \models F_2$,
or $I \not\models F_1$ and $I \not\models F_2$

Note:
$I \models F_1 \rightarrow F_2$ iff $I \not\models F_1$ or $I \models F_2$.
$I \not\models F_1 \rightarrow F_2$ iff $I \models F_1$ and $I \not\models F_2$.
$I \not\models F_1 \vee F_2$ iff $I \not\models F_1$ and $I \not\models F_2$.

Example of Inductive Reasoning:

$$F : P \wedge Q \rightarrow P \vee \neg Q$$

$$I : \{P \mapsto \text{true},\ Q \mapsto \text{false}\}$$

1. $I \models P$ since $I[P] = \text{true}$
2. $I \not\models Q$ since $I[Q] = \text{false}$
3. $I \models \neg Q$ by 2 and $\neg$
4. $I \not\models P \wedge Q$ by 2 and $\wedge$
5. $I \models P \vee \neg Q$ by 1 and $\vee$
6. $I \models F$ by 4 and $\rightarrow$ Why?

Thus, F is true under I.

Note: steps 1, 3, and 5 are nonessential.

Satisfiability and Validity

F satisfiable iff there exists an interpretation I such that $I \models F$.
F valid iff for all interpretations I, $I \models F$.

$$\boxed{F \text{ is valid iff } \neg F \text{ is unsatisfiable}}$$

Goal: devise an algorithm to decide validity or unsatisfiability of formula F.

Method 1: Truth Tables

Example $F : P \wedge Q \rightarrow P \vee \neg Q$

P	Q	$P \wedge Q$	$\neg Q$	$P \vee \neg Q$	F
0	0	0	1	1	1
0	1	0	0	0	1
1	0	0	1	1	1
1	1	1	0	1	1

Thus F is valid.

Example $F : P \vee Q \rightarrow P \wedge Q$

P	Q	$P \vee Q$	$P \wedge Q$	F	
0	0	0	0	1	← satisfying I
0	1	1	0	0	← falsifying I
1	0	1	0	0	
1	1	1	1	1	

Thus F is satisfiable, but invalid.

Method 2: Semantic Argument

- Assume F is not valid and I a falsifying interpretation: $I \not\models F$
- Apply proof rules.
- If no contradiction reached and no more rules applicable, F is invalid.
- If in every branch of proof a contradiction reached, F is valid.

Proof Rules for Semantic Arguments I

$$\frac{I \models \neg F}{I \not\models F} \qquad \frac{I \not\models \neg F}{I \models F}$$

$$\frac{I \models F \land G}{\begin{array}{l} I \models F \\ I \models G \end{array}} \leftarrow\text{and} \qquad \frac{I \not\models F \land G}{I \not\models F \;\mid\; I \not\models G}\ \underset{\nwarrow\text{or}}{}$$

$$\frac{I \models F \lor G}{I \models F \;\mid\; I \models G} \qquad \frac{I \not\models F \lor G}{\begin{array}{l} I \not\models F \\ I \not\models G \end{array}}$$

Proof Rules for Semantic Arguments II

$$\frac{I \models F \rightarrow G}{I \not\models F \;\mid\; I \models G} \qquad \frac{I \not\models F \rightarrow G}{\begin{array}{l} I \models F \\ I \not\models G \end{array}}$$

$$\frac{I \models F \leftrightarrow G}{I \models F \land G \;\mid\; I \not\models F \lor G} \qquad \frac{I \not\models F \leftrightarrow G}{I \models F \land \neg G \;\mid\; I \models \neg F \land G}$$

$$\frac{\begin{array}{l} I \models F \\ I \not\models F \end{array}}{I \models \bot}$$

Example: Prove

$$F :\ P \land Q \ \rightarrow\ P \lor \neg Q \quad \text{is valid.}$$

Let's assume that F is not valid and that I is a falsifying interpretation.

$$\begin{array}{llll}
1. & I \not\models & P \land Q \ \rightarrow\ P \lor \neg Q & \text{assumption} \\
2. & I \models & P \land Q & \text{1 and } \rightarrow \\
3. & I \not\models & P \lor \neg Q & \text{1 and } \rightarrow \\
4. & I \models & P & \text{2 and } \land \\
5. & I \not\models & P & \text{3 and } \lor \\
6. & I \models & \bot & \text{4 and 5 are contradictory}
\end{array}$$

Thus F is valid.

Example: Prove

$$F:\ (P \to Q) \land (Q \to R) \to (P \to R) \quad \text{is valid.}$$

Let's assume that F is not valid.

1.	$I \not\models F$	assumption
2.	$I \models (P \to Q) \land (Q \to R)$	1 and $\to$
3.	$I \not\models P \to R$	1 and $\to$
4.	$I \models P$	3 and $\to$
5.	$I \not\models R$	3 and $\to$
6.	$I \models P \to Q$	2 and $\land$
7.	$I \models Q \to R$	2 and $\land$

6.	$I \models P \to Q$	2 and $\land$
7.	$I \models Q \to R$	2 and $\land$
8a.	$I \not\models P$	6 and $\to$ (case a)
9a.	$I \models \perp$	4 and 8
8b.	$I \models Q$	6 and $\to$ (case b)
9ba.	$I \not\models Q$	7 and $\to$ (subcase ba)
10ba.	$I \models \perp$	8b and 9ba
9bb.	$I \models R$	7 and $\to$ (subcase bb)
10bb.	$I \models \perp$	5 and 9bb
9b.	$I \models \perp$	10ba and 10bb
8.	$I \models \perp$	9a and 9b

Our assumption is contradictory in all cases, so F is valid.

Example 3: Is

$$F:\ P \lor Q \to P \land Q$$

valid? Assume F is not valid:

1.	$I \not\models P \lor Q \to P \land Q$	assumption
2.	$I \models P \lor Q$	1 and $\to$
3.	$I \not\models P \land Q$	1 and $\to$
4a.	$I \models P$	2, $\lor$ (case a)
5aa.	$I \not\models P$	3, $\lor$ (subcase aa)
6aa.	$I \models \perp$	4a, 5aa
5ab.	$I \not\models Q$	3, $\lor$ (subcase ab)
6ab.	?	
5a.	?	

4b.	$I \models Q$	2, $\lor$ (case b)
5ba.	$I \not\models P$	3, $\lor$ (subcase ba)
6ba.	?	
5bb.	$I \not\models Q$	3, $\lor$ (subcase bb)
6bb.	$I \models \perp$	4b, 5bb
5b.	?	
5.	?	

We cannot derive a contradiction in both cases (4a and 4b), so we cannot prove that F is valid. To demonstrate that F is not valid, however, we must find a falsifying interpretation (here are two):

$$I_1:\ \{P \mapsto \text{true},\ Q \mapsto \text{false}\} \qquad I_2:\ \{Q \mapsto \text{true},\ P \mapsto \text{false}\}$$

Note: we have to derive a contradiction in all cases for F to be valid!

Equivalence

F_1 and F_2 are equivalent ($F_1 \Leftrightarrow F_2$)
iff for all interpretations I, $I \models F_1 \leftrightarrow F_2$

To prove $F_1 \Leftrightarrow F_2$, show $F_1 \leftrightarrow F_2$ is valid, that is, both $F_1 \rightarrow F_2$ and $F_2 \rightarrow F_1$ are valid.

F_1 entails F_2 ($F_1 \Rightarrow F_2$)
iff for all interpretations I, $I \models F_1 \rightarrow F_2$

Note: $F_1 \Leftrightarrow F_2$ and $F_1 \Rightarrow F_2$ are not formulae!!

Example: Show

$$P \rightarrow Q \Leftrightarrow \neg P \vee Q$$

i.e.

$$F : (P \rightarrow Q) \leftrightarrow (\neg P \vee Q) \textit{ is valid}.$$

Assume F is not valid, then we have two cases:

Case a: $I \not\models \neg P \vee Q$,
$I \models P \rightarrow Q$

Case b: $I \models \neg P \vee Q$,
$I \not\models P \rightarrow Q$

Derive contradictions in both cases.

Normal Forms

1. Negation Normal Form (NNF)

   $\neg, \wedge, \vee$ are the only boolean connectives allowed.

   Negations may occur only in literals of the form $\neg P$.

   To transform F into equivalent F' in NNF, apply the following template equivalences recursively (and left-to-right):

$$\neg\neg F_1 \Leftrightarrow F_1 \qquad \neg\top \Leftrightarrow \bot \qquad \neg\bot \Leftrightarrow \top$$

$$\left.\begin{array}{l}\neg(F_1 \wedge F_2) \Leftrightarrow \neg F_1 \vee \neg F_2 \\ \neg(F_1 \vee F_2) \Leftrightarrow \neg F_1 \wedge \neg F_2\end{array}\right\}\text{De Morgan's Law}$$

$$F_1 \rightarrow F_2 \Leftrightarrow \neg F_1 \vee F_2$$

$$F_1 \leftrightarrow F_2 \Leftrightarrow (F_1 \rightarrow F_2) \wedge (F_2 \rightarrow F_1)$$

   "Complete" syntactic restriction: every F has an equivalent F' in NNF.

Example: Convert

$$F : \neg(P \rightarrow \neg(P \wedge Q))$$

to NNF.

$$\begin{array}{lll} F' & : \neg(\neg P \vee \neg(P \wedge Q)) & \rightarrow \\ F'' & : \neg\neg P \wedge \neg\neg(P \wedge Q) & \text{De Morgan's Law} \\ F''' & : P \wedge P \wedge Q & \neg\neg \end{array}$$

F''' is equivalent to F ($F''' \Leftrightarrow F$) and is in NNF.

2. Disjunctive Normal Form (DNF)

Disjunction of conjunctions of literals

$$\bigvee_i \bigwedge_j \ell_{i,j} \quad \text{for literals } \ell_{i,j}$$

To convert F into equivalent F' in DNF,
transform F into NNF and then
use the following template equivalences (left-to-right):

$$\left.\begin{array}{lcl} (F_1 \vee F_2) \wedge F_3 & \Leftrightarrow & (F_1 \wedge F_3) \vee (F_2 \wedge F_3) \\ F_1 \wedge (F_2 \vee F_3) & \Leftrightarrow & (F_1 \wedge F_2) \vee (F_1 \wedge F_3) \end{array}\right\} \mathit{dist}$$

Note: formulae can grow exponentially as the distributivity laws are applied.

Example: Convert

$$F : \ (Q_1 \vee \neg\neg Q_2) \wedge (\neg R_1 \ \rightarrow \ R_2)$$

into equivalent DNF

$$\begin{array}{rll} F' & : (Q_1 \vee Q_2) \wedge (R_1 \vee R_2) & \text{in NNF} \\ F'' & : (Q_1 \wedge (R_1 \vee R_2)) \vee (Q_2 \wedge (R_1 \vee R_2)) & \text{dist} \\ F''' & : (Q_1 \wedge R_1) \vee (Q_1 \wedge R_2) \vee (Q_2 \wedge R_1) \vee (Q_2 \wedge R_2) & \text{dist} \end{array}$$

F''' is equivalent to F ($F''' \Leftrightarrow F$) and is in DNF.

3. Conjunctive Normal Form (CNF)

Conjunction of disjunctions of literals

$$\bigwedge_i \bigvee_j \ell_{i,j} \quad \text{for literals } \ell_{i,j}$$

To convert F into equivalent F' in CNF,
transform F into NNF and then
use the following template equivalences (left-to-right):

$$\begin{array}{lcl} (F_1 \wedge F_2) \vee F_3 & \Leftrightarrow & (F_1 \vee F_3) \wedge (F_2 \vee F_3) \\ F_1 \vee (F_2 \wedge F_3) & \Leftrightarrow & (F_1 \vee F_2) \wedge (F_1 \vee F_3) \end{array}$$

A disjunction of literals is called a clause.

Example: Convert

$$F : P \ \leftrightarrow \ (Q \ \rightarrow \ R)$$

to an equivalent formula F' in CNF.

First get rid of $\leftrightarrow$:

$$F_1 : (P \ \rightarrow \ (Q \ \rightarrow \ R)) \wedge ((Q \ \rightarrow \ R) \ \rightarrow \ P)$$

Now replace $\rightarrow$ with $\vee$:

$$F_2 : (\neg P \vee (\neg Q \vee R)) \wedge (\neg(\neg Q \vee R) \vee P)$$

Drop unnecessary parentheses and apply De Morgan's Law:

$$F_3 : (\neg P \vee \neg Q \vee R) \wedge ((\neg\neg Q \wedge \neg R) \vee P)$$

Simplify double negation (now in NNF):

$$F_4 : (\neg P \vee \neg Q \vee R) \wedge ((Q \wedge \neg R) \vee P)$$

Distribute disjunction over conjunction (now in CNF):

$$F' : (\neg P \vee \neg Q \vee R) \wedge (Q \vee P) \wedge (\neg R \vee P)$$

Equisatisfiability

Definition

F and F' are *equisatisfiable*, iff

$$F \text{ is satisfiable if and only if } F' \text{ is satisfiable}$$

Every formula is equisatifiable to either $\top$ or $\bot$.

Goal: Decide satisfiability of PL formula F

Step 1: Convert F to equisatisfiable formula F' in CNF

Step 2: Decide satisfiability of formula F' in CNF

Step 1: Convert F to equisatisfiable formula F' in CNF I

There is an *efficient conversion* of F to F' where

- F' is in CNF and
- F and F' are equisatisfiable

Note: efficient means polynomial in the size of F.

Basic Idea:

- Introduce a new variable P_G for every subformula G of F, unless G is already an atom.

Step 1: Convert F to equisatisfiable formula F' in CNF II

- For each subformula

$$G : \; G_1 \circ G_2,$$

produce a small formula

$$P_G \; \leftrightarrow \; P_{G_1} \circ P_{G_2}.$$

Here $\circ$ denotes an arbitrary connective ($\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$); if the connective is $\neg$, G_1 should be ignored.

Step 1: Convert F to equisatisfiable formula F' in CNF III

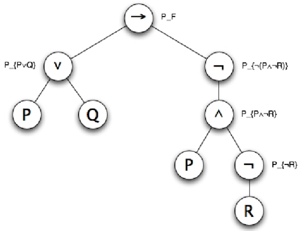


Figure: Parse tree for $F : P \vee Q \;\rightarrow\; \neg(P \wedge \neg R)$

Step 1: Convert F to equisatisfiable formula F' in CNF IV

- Convert each of these (small) formulae separately to an equivalent CNF formula

$$\mathrm{CNF}(P_G \;\leftrightarrow\; P_{G_1} \circ P_{G_2}) \;.$$

Let S_F be the set of all non-atom subformulae G of F (including F itself). The formula

$$P_F \wedge \bigwedge_{G \in S_F} CNF(P_G \;\leftrightarrow\; P_{G_1} \circ P_{G_2})$$

is equisatisfiable to F. (Why?)

The number of subformulae is linear in the size of F.
The time to convert one small formula is constant!

Example: CNF I

Convert

$$F:\; P \vee Q \;\rightarrow\; P \wedge \neg R$$

to an equisatisfiable formula in CNF.

Introduce new variables: P_F, $P_{P\vee Q}$, $P_{P\wedge\neg R}$, $P_{\neg R}$.

Create new formulae and convert them to equivalent formulae in CNF separately:

- $F_1 = \mathrm{CNF}(P_F \;\leftrightarrow\; (P_{P\vee Q} \;\rightarrow\; P_{P\wedge\neg R}))$:

$$(\neg P_F \vee \neg P_{P\vee Q} \vee P_{P\wedge\neg R}) \wedge (P_F \vee P_{P\vee Q}) \wedge (P_F \vee \neg P_{P\wedge\neg R})$$

- $F_2 = \mathrm{CNF}(P_{P\vee Q} \;\leftrightarrow\; P \vee Q)$:

$$(\neg P_{P\vee Q} \vee P \vee Q) \wedge (P_{P\vee Q} \vee \neg P) \wedge (P_{P\vee Q} \vee \neg Q)$$

Example: CNF II

- $F_3 = \mathrm{CNF}(P_{P\wedge\neg R} \;\leftrightarrow\; P \wedge P_{\neg R})$:

$$(\neg P_{P\wedge\neg R} \vee P) \wedge (\neg P_{P\wedge\neg R} \vee P_{\neg R}) \wedge (P_{P\wedge\neg R} \vee \neg P \vee \neg P_{\neg R})$$

- $F_4 = \mathrm{CNF}(P_{\neg R} \;\leftrightarrow\; \neg R)$:

$$(\neg P_{\neg R} \vee \neg R) \wedge (P_{\neg R} \vee R)$$

$P_F \wedge F_1 \wedge F_2 \wedge F_3 \wedge F_4$ is in CNF and equisatisfiable to F.

Step 2: Decide the satisfiability of PL formula F' in CNF

Boolean Constraint Propagation (BCP)

If a clause contains one literal ℓ,

Set ℓ to $\top$: $\cdots \wedge \overset{\top}{\cancel{\ell}} \wedge \cdots$

Remove all clauses containing ℓ: $\cdots \wedge \cancel{(\cdots \vee \ell \vee \cdots)} \wedge \cdots$

Remove $\neg\ell$ in all clauses: $\cdots \wedge (\cdots \vee \cancel{\neg\ell} \vee \cdots) \wedge \cdots$

based on the unit resolution

$$\frac{\ell \qquad \neg\ell \vee C}{C} \;\leftarrow \text{clause}$$

Pure Literal Propagation (PLP)

If P occurs only positive (without negation), set it to $\top$.
If P occurs only negative set it to $\bot$.
Then do the simplifications as in Boolean Constraint Propagation

Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Decides the satisfiability of PL formulae in CNF

Decision Procedure DPLL: Given F in CNF

```
let rec DPLL $F$ =
  let $F'$ = BCP $F$ in
  let $F''$ = PLP $F'$ in
  if $F'' = \top$ then true
  else if $F'' = \bot$ then false
  else
     let $P$ = CHOOSE vars($F''$) in
     (DPLL $F''\{P \mapsto \top\}$) $\vee$ (DPLL $F''\{P \mapsto \bot\}$)
```

Simplification

Simplify according to the template equivalences (left-to-right) [exercise 1.2]

$$\begin{array}{ccc} \neg\bot \Leftrightarrow \top & \neg\top \Leftrightarrow \bot & \neg\neg F \Leftrightarrow F \\ F \wedge \top \Leftrightarrow F & F \wedge \bot \Leftrightarrow \bot & \cdots \\ F \vee \top \Leftrightarrow \top & F \vee \bot \Leftrightarrow F & \cdots \end{array}$$

Example I

Consider

$$F: \ (\neg P \vee Q \vee R) \wedge (\neg Q \vee R) \wedge (\neg Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R).$$

Branching on Q

On the first branch, we have

$$F\{Q \ \mapsto \ \top\}: \ (R) \wedge (\neg R) \wedge (P \vee \neg R).$$

By unit resolution,

$$\frac{R \qquad (\neg R)}{\bot},$$

so $F\{Q \ \mapsto \ \top\} \ = \ \bot \ \Rightarrow$ false.

Example II

Recall

$$F: \ (\neg P \vee Q \vee R) \wedge (\neg Q \vee R) \wedge (\neg Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R).$$

On the other branch, we have

$$F\{Q \ \mapsto \ \bot\}: \ (\neg P \vee R).$$

Furthermore, by PLP,

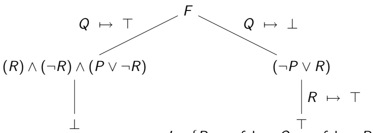
$$F\{Q \ \mapsto \ \bot, \ R \ \mapsto \ \top, \ P \ \mapsto \ \bot\} \ = \ \top \ \Rightarrow \ \text{true}$$

Thus F is satisfiable with satisfying interpretation

$$I: \ \{P \ \mapsto \ \text{false}, \ Q \ \mapsto \ \text{false}, \ R \ \mapsto \ \text{true}\}.$$

Example

$F: \; (\neg P \vee Q \vee R) \wedge (\neg Q \vee R) \wedge (\neg Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R)$



F

$Q \mapsto \top$

$Q \mapsto \bot$

$(R) \wedge (\neg R) \wedge (P \vee \neg R)$

$(\neg P \vee R)$

$\bot$

$R \mapsto \top$

$\top$

$I: \; \{P \mapsto \text{false}, \; Q \mapsto \text{false}, \; R \mapsto \text{true}\}$

(No matter what P is)