The Calculus of Computation

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Chapter 5: Program Correctness: Mechanics

Program A: LinearSearch with function specification

```
@pre 0 ≤ ℓ ∧ u < |a|
@post rv ↔ ∃i. ℓ ≤ i ≤ u ∧ a[i] = e
bool LinearSearch(int[] a, int ℓ, int u, int e) {
  for @ ⊤
    (int i := ℓ; i ≤ u; i := i + 1) {
    if (a[i] = e) return true;
  }
  return false;
}
```


Function LinearSearch searches subarray of array a of integers for specified value e.

Function specifications

- Function precondition (@*pre*)
  It behaves correctly only if $0 \leq \ell$ and $u < |a|$
- Function postcondition (@*post*)
  It returns true iff a contains the value e in the range $[\ell, u]$

for loop: initially set i to be ℓ,
execute the body and increment i by 1
as long as $i \leq u$

@ - program annotation

Program B: BinarySearch with function specification

```
@pre 0 ≤ ℓ ∧ u < |a| ∧ sorted(a, ℓ, u)
@post rv ↔ ∃i. ℓ ≤ i ≤ u ∧ a[i] = e
bool BinarySearch(int[] a, int ℓ, int u, int e) {
  if (ℓ > u) return false;
  else {
    int m := (ℓ + u) div 2;
    if (a[m] = e) return true;
    else if (a[m] < e) return BinarySearch(a, m + 1, u, e);
    else return BinarySearch(a, ℓ, m − 1, e);
  }
}
```

The recursive function BinarySearch searches sorted subarray a of integers for specified value e.

sorted: weakly increasing order, i.e.

$$\text{sorted}(a, \ell, u) \;\Leftrightarrow\; \forall i, j.\ \ell \le i \le j \le u \;\rightarrow\; a[i] \le a[j]$$

Defined in the combined theory of integers and arrays, $T_{\mathbb{Z} \cup A}$

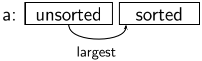
Function specifications

- Function precondition (@*pre*)
  It behaves correctly only if
  $0 \le \ell$ and $u < |a|$ and
  $\text{sorted}(a, \ell, u)$.
- Function postcondition (@*post*)
  It returns true iff a contains the value e in the range $[\ell, u]$

Program C: BubbleSort with function specification

```
@pre ⊤
@post sorted(rv, 0, |rv| − 1)
int[] BubbleSort(int[] a₀) {
  int[] a := a₀;
  for @ ⊤
    (int i := |a| − 1; i > 0; i := i − 1) {
    for @ ⊤
      (int j := 0; j < i; j := j + 1) {
      if (a[j] > a[j + 1]) {
        int t := a[j];
        a[j] := a[j + 1];
        a[j + 1] := t;
      }
    }
  }
  return a;
}
```


Function BubbleSort sorts integer array a



by "bubbling" the largest element of the left unsorted region of a toward the sorted region on the right.

Each iteration of the outer loop expands the sorted region by one cell.[1]

Function specification

- Function postcondition (@post):
  BubbleSort returns array rv sorted on the range $[0, |rv| - 1]$.

[1] Except the last iteration, which expands the sorted region by two cells, so that an entire array of length n is sorted in $n - 1$ iterations.

Sample execution of BubbleSort

2	3	4	1	2	5	6
j				i		
2	3	4	1	2	5	6
	j			i		
2	3	4	1	2	5	6
		j		i		
2	3	1	4	2	5	6
			j	i		
2	3	1	2	4	5	6
				j, i		
2	3	1	2	4	5	6
j			i			

Program Annotation

- Function Specifications

  function precondition (@pre)
  function postcondition (@post)

- Runtime Assertions

  e.g., @ $0 \le j < |a| \ \wedge\ 0 \le j+1 < |a|$
  $a[j] := a[j+1]$

- Loop Invariants

  e.g., @ $L : \ell \le i \ \wedge\ \forall j.\ \ell \le j < i \ \rightarrow\ a[j] \ne e$

  The L : gives a name to the formula, just like the F : we've used in other formulae.

Program A: LinearSearch with runtime assertions

```
@pre 0 ≤ ℓ ∧ u < |a|
@post rv ↔ ∃i. ℓ ≤ i ≤ u ∧ a[i] = e
bool LinearSearch(int[] a, int ℓ, int u, int e) {
  for
    @ L : ⊤
    (int i := ℓ; i ≤ u; i := i + 1) {
    @ 0 ≤ i < |a|;
    if (a[i] = e) return true;
  }
  return false;
}
```


Program B: BinarySearch with runtime assertions

```
@pre 0 ≤ ℓ ∧ u < |a| ∧ sorted(a, ℓ, u)
@post rv ↔ ∃i. ℓ ≤ i ≤ u ∧ a[i] = e
bool BinarySearch(int[] a, int ℓ, int u, int e) {
  if (ℓ > u) return false;
  else {
    @ 2 ≠ 0;
    int m := (ℓ + u) div 2;
    @ 0 ≤ m < |a|;
    if (a[m] = e) return true;
    else {
      @ 0 ≤ m < |a|;
      if (a[m] < e) return BinarySearch(a, m + 1, u, e);
      else return BinarySearch(a, ℓ, m − 1, e);
    }
  }
}
```


Program C: BubbleSort with runtime assertions

```
@pre ⊤
@post sorted(rv, 0, |rv| − 1)
int[] BubbleSort(int[] a0) {
  int[] a := a0;
  for
    @ L1 : ⊤
    (int i := |a| − 1; i > 0; i := i − 1) {
    for
      @ L2 : ⊤
      (int j := 0; j < i; j := j + 1) {
      @ 0 ≤ j < |a| ∧ 0 ≤ j + 1 < |a|;
      if (a[j] > a[j + 1]) {
        int t := a[j];
        a[j] := a[j + 1];
        a[j + 1] := t;
      }
    }
  }
  return a;
}
```

Loop Invariants

```
while
  @ F
  ⟨cond⟩ { ⟨body⟩ }
```

- apply ⟨*body*⟩ as long as ⟨*cond*⟩ holds
- assertion F holds at the beginning of every iteration evaluated before ⟨*cond*⟩ is checked

for
  @ F
  (⟨*init*⟩; ⟨*cond*⟩; ⟨*incr*⟩)
  {⟨*body*⟩}
	⟨*init*⟩;
	while
	  @ F
	  ⟨*cond*⟩ { ⟨*body*⟩; ⟨*incr*⟩ }

Program A: LinearSearch with loop invariants

```
@pre 0 ≤ ℓ ∧ u < |a|
@post rv ↔ ∃j. ℓ ≤ j ≤ u ∧ a[j] = e
bool LinearSearch(int[] a, int ℓ, int u, int e) {
  for
    @L : ℓ ≤ i ∧ (∀j. ℓ ≤ j < i → a[j] ≠ e)
    (int i := ℓ; i ≤ u; i := i + 1) {
    if (a[i] = e) return true;
  }
  return false;
}
```


Proving Partial Correctness

A function is partially correct if
when the program's precondition is satisfied on entry,
its postcondition is satisfied when the program halts/exits.

- A program + annotation is reduced to finite set of verification conditions (VCs), FOL formulae
- If all VCs are T-valid, then the program obeys its specification (partially correct)

Basic Paths: Loops

To handle loops, we break the program into basic paths

@ ← precondition or loop invariant

sequence of instructions
(with no loop invariants)

@ ← loop invariant, runtime assertion, or postcondition

Program A: LinearSearch I

Basic Paths of LinearSearch

(1)

@pre $0 \leq \ell \wedge u < |a|$

$i := \ell$;

@L : $\ell \leq i \wedge \forall j.\ \ell \leq j < i \ \rightarrow\ a[j] \neq e$

(2)

@L : $\ell \leq i \wedge \forall j.\ \ell \leq j < i \ \rightarrow\ a[j] \neq e$

assume $i \leq u$;

assume $a[i] = e$;

rv := true;

@post $rv \ \leftrightarrow\ \exists j.\ \ell \leq j \leq u \wedge a[j] = e$

Program A: LinearSearch II

(3)

@L : $\ell \leq i \wedge \forall j.\ \ell \leq j < i \ \rightarrow\ a[j] \neq e$

assume $i \leq u$;

assume $a[i] \neq e$;

$i := i + 1$;

@L : $\ell \leq i \wedge \forall j.\ \ell \leq j < i \ \rightarrow\ a[j] \neq e$

(4)

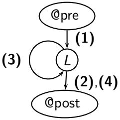
@L : $\ell \leq i \wedge \forall j.\ \ell \leq j < i \ \rightarrow\ a[j] \neq e$

assume $i > u$;

rv := false;

@post $rv \ \leftrightarrow\ \exists j.\ \ell \leq j \leq u \wedge a[j] = e$

Visualization of basic paths of LinearSearch



Program C: BubbleSort with loop invariants

```
@pre |a_0| > 0
@post sorted(rv, 0, |rv| - 1)
int[] BubbleSort(int[] a_0) {
  int[] a := a_0;
  for
```

$$@L_1 : \begin{bmatrix} 0 \leq i < |a| \\ \wedge \mathsf{partitioned}(a, 0, i, i+1, |a|-1) \\ \wedge \mathsf{sorted}(a, i, |a|-1) \end{bmatrix}$$

(int $i := |a| - 1$; $i > 0$; $i := i - 1$) {

```
      for
        @L_2 : [1 ≤ i < |a| ∧ 0 ≤ j ≤ i
                ∧partitioned(a, 0, i, i + 1, |a| − 1)
                ∧partitioned(a, 0, j − 1, j, j)
                ∧sorted(a, i, |a| − 1)]
        (int j := 0; j < i; j := j + 1) {
        if (a[j] > a[j + 1]) {
          int t := a[j];
          a[j] := a[j + 1];
          a[j + 1] := t;
        }
      }
    }
    return a;
  }
```


Partition

$$\begin{aligned}&\text{partitioned}(a, \ell_1, u_1, \ell_2, u_2)\\ &\quad\Leftrightarrow\ \forall i, j.\ \ell_1 \le i \le u_1 < \ell_2 \le j \le u_2\ \rightarrow\ a[i] \le a[j]\end{aligned}$$

in $T_{\mathbb{Z}} \cup T_{\mathsf{A}}$.

That is, each element of a in the range $[\ell_1, u_1]$ is $\le$ each element in the range $[\ell_2, u_2]$.

Basic Paths of BubbleSort

(1)

$@\text{pre } |a_0| > 0$

$a := a_0;$

$i := |a| - 1;$

$$@L_1 : \begin{bmatrix} 0 \le i < |a| \wedge \text{partitioned}(a, 0, i, i+1, |a|-1) \\ \wedge\ \text{sorted}(a, i, |a|-1) \end{bmatrix}$$

(2)

$$@L_1 : \begin{bmatrix} 0 \le i < |a| \wedge \text{partitioned}(a, 0, i, i+1, |a|-1) \\ \wedge\ \text{sorted}(a, i, |a|-1) \end{bmatrix}$$

`assume` $i > 0;$

$j := 0;$

$$@L_2 : \begin{bmatrix} 1 \le i < |a| \wedge 0 \le j \le i \wedge \text{partitioned}(a, 0, i, i+1, |a|-1) \\ \wedge\ \text{partitioned}(a, 0, j-1, j, j) \wedge \text{sorted}(a, i, |a|-1) \end{bmatrix}$$

(3)

$$@L_2 : \begin{bmatrix} 1 \le i < |a| \wedge 0 \le j \le i \wedge \text{partitioned}(a, 0, i, i+1, |a|-1) \\ \wedge\ \text{partitioned}(a, 0, j-1, j, j) \wedge \text{sorted}(a, i, |a|-1) \end{bmatrix}$$

`assume` $j < i;$

`assume` $a[j] > a[j+1];$

$t := a[j];$

$a[j] := a[j+1];$

$a[j+1] := t;$

$j := j + 1;$

$$@L_2 : \begin{bmatrix} 1 \le i < |a| \wedge 0 \le j \le i \wedge \text{partitioned}(a, 0, i, i+1, |a|-1) \\ \wedge\ \text{partitioned}(a, 0, j-1, j, j) \wedge \text{sorted}(a, i, |a|-1) \end{bmatrix}$$

(4)

$$@L_2 : \begin{bmatrix} 1 \leq i < |a| \land 0 \leq j \leq i \land \text{partitioned}(a, 0, i, i+1, |a|-1) \\ \land\ \text{partitioned}(a, 0, j-1, j, j) \land \text{sorted}(a, i, |a|-1) \end{bmatrix}$$

`assume` $j < i$;

`assume` $a[j] \leq a[j+1]$;

$j := j + 1$;

$$@L_2 : \begin{bmatrix} 1 \leq i < |a| \land 0 \leq j \leq i \land \text{partitioned}(a, 0, i, i+1, |a|-1) \\ \land\ \text{partitioned}(a, 0, j-1, j, j) \land \text{sorted}(a, i, |a|-1) \end{bmatrix}$$

(5)

$$@L_2 : \begin{bmatrix} 1 \leq i < |a| \land 0 \leq j \leq i \land \text{partitioned}(a, 0, i, i+1, |a|-1) \\ \land\ \text{partitioned}(a, 0, j-1, j, j) \land \text{sorted}(a, i, |a|-1) \end{bmatrix}$$

`assume` $j \geq i$;

$i := i - 1$;

$$@L_1 : \begin{bmatrix} 0 \leq i < |a| \land \text{partitioned}(a, 0, i, i+1, |a|-1) \\ \land\ \text{sorted}(a, i, |a|-1) \end{bmatrix}$$

(6)

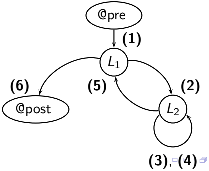
$$@L_1 : \begin{bmatrix} 0 \leq i < |a| \land \text{partitioned}(a, 0, i, i+1, |a|-1) \\ \land\ \text{sorted}(a, i, |a|-1) \end{bmatrix}$$

`assume` $i \leq 0$;

$rv := a$;

$@\text{post sorted}(rv, 0, |rv|-1)$

Visualization of basic paths of BubbleSort



Basic Paths: Function Calls

- Loops produce unbounded number of paths
  loop invariants cut loops to produce
  finite number of basic paths
- Reursive calls produce unbounded number of paths
  function specifications cut function calls

In BinarySearch

$@\text{pre } 0 \leq \ell \land u < |a| \land \text{sorted}(a, \ell, u)$ $\quad \ldots F[a, \ell, u, e]$

$\vdots$

$@R_1 : \ 0 \leq m+1 \land u < |a| \land \text{sorted}(a, m+1, u)$ $\quad \ldots F[a, m+1, u, e]$

`return` BinarySearch$(a, m+1, u, e)$

$\vdots$

$@R_2 : \ 0 \leq \ell \land m-1 < |a| \land \text{sorted}(a, \ell, m-1)$ $\quad \ldots F[a, \ell, m-1, e]$

`return` BinarySearch$(a, \ell, m-1, e)$

Program B: BinarySearch with function call assertions

```
@pre 0 ≤ ℓ ∧ u < |a| ∧ sorted(a, ℓ, u)
@post rv ↔ ∃i. ℓ ≤ i ≤ u ∧ a[i] = e
bool BinarySearch(int[] a, int ℓ, int u, int e) {
  if (ℓ > u) return false;
  else {
    int m := (ℓ + u) div 2;
    if (a[m] = e) return true;
    else if (a[m] < e) {
      @R₁ : 0 ≤ m + 1 ∧ u < |a| ∧ sorted(a, m + 1, u);
      return BinarySearch(a, m + 1, u, e);
    } else {
      @R₂ : 0 ≤ ℓ ∧ m − 1 < |a| ∧ sorted(a, ℓ, m − 1);
      return BinarySearch(a, ℓ, m − 1, e);
    }
  }
}
```


(1)

$@\mathsf{pre}\ 0 \le \ell \wedge u < |a| \wedge \mathsf{sorted}(a, \ell, u)$

$\mathtt{assume}\ \ell > u;$

$rv := \mathtt{false};$

$@\mathsf{post}\ rv \leftrightarrow \exists i.\ \ell \le i \le u \wedge a[i] = e$

(2)

$@\mathsf{pre}\ 0 \le \ell \wedge u < |a| \wedge \mathsf{sorted}(a, \ell, u)$

$\mathtt{assume}\ \ell \le u;$

$m := (\ell + u)\ \mathtt{div}\ 2;$

$\mathtt{assume}\ a[m] = e;$

$rv := \mathtt{true};$

$@\mathsf{post}\ rv \leftrightarrow \exists i.\ \ell \le i \le u \wedge a[i] = e$

(3)

$@\mathsf{pre}\ 0 \le \ell \wedge u < |a| \wedge \mathsf{sorted}(a, \ell, u)$

$\mathtt{assume}\ \ell \le u;$

$m := (\ell + u)\ \mathtt{div}\ 2;$

$\mathtt{assume}\ a[m] \ne e;$

$\mathtt{assume}\ a[m] < e;$

$@R_1 : 0 \le m + 1 \wedge u < |a| \wedge \mathsf{sorted}(a, m + 1, u)$

(5)

$@\mathsf{pre}\ 0 \le \ell \wedge u < |a| \wedge \mathsf{sorted}(a, \ell, u)$

$\mathtt{assume}\ \ell \le u;$

$m := (\ell + u)\ \mathtt{div}\ 2;$

$\mathtt{assume}\ a[m] \ne e;$

$\mathtt{assume}\ a[m] \ge e;$

$@R_2 : 0 \le \ell \wedge m - 1 < |a| \wedge \mathsf{sorted}(a, \ell, m - 1)$

(4)

$@\mathrm{pre}\ 0 \le \ell \wedge u < |a| \wedge \mathrm{sorted}(a, \ell, u)$

$\texttt{assume}\ \ell \le u;$

$m := (\ell + u)\ \texttt{div}\ 2;$

$\texttt{assume}\ a[m] \ne e;$

$\texttt{assume}\ a[m] < e;$

$\texttt{assume}\ v_1 \ \leftrightarrow\ \exists i.\ m + 1 \le i \le u \wedge a[i] = e;$

$rv := v_1;$

$@\mathrm{post}\ rv \ \leftrightarrow\ \exists i.\ \ell \le i \le u \wedge a[i] = e$

(6)

$@\mathrm{pre}\ 0 \le \ell \wedge u < |a| \wedge \mathrm{sorted}(a, \ell, u)$

$\texttt{assume}\ \ell \le u;$

$m := (\ell + u)\ \texttt{div}\ 2;$

$\texttt{assume}\ a[m] \ne e;$

$\texttt{assume}\ a[m] \ge e;$

$\texttt{assume}\ v_2 \ \leftrightarrow\ \exists i.\ \ell \le i \le m - 1 \wedge a[i] = e;$

$rv := v_2;$

$@\mathrm{post}\ rv \ \leftrightarrow\ \exists i.\ \ell \le i \le u \wedge a[i] = e$

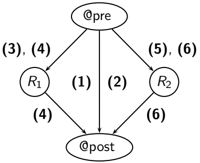


Figure: Visualization of basic paths of BinarySearch

Program States

Program counter pc holds current location of control

State s of P assignment of values to all variables (proper types) of P

Example:

$$s : \left\{ \begin{array}{l} pc \ \mapsto\ L_2,\ a \ \mapsto\ [0;1;2], \\ i \ \mapsto\ 3,\ j \mapsto 0 \end{array} \right\}$$

is a state of BubbleSort.

Reachable state s of P a state that can be reached during some computation of P

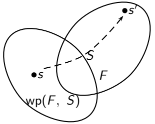
Example:

$$s : \left\{ \begin{array}{l} pc \ \mapsto\ L_2,\ a \ \mapsto\ [0;1;2], \\ i \ \mapsto\ 2,\ j \mapsto 0 \end{array} \right\}$$

is a reachable state of BubbleSort.

Weakest Precondition wp(F, S)

For FOL formula F, program statement S,
$s \models \text{wp}(F,\ S)$ iff
statement S is executed on state s to produce state s',
and $s' \models F$:



- $\text{wp}(F,\ \texttt{assume}\ c) \Leftrightarrow c \rightarrow F$
- $\text{wp}(F[v],\ v := e) \Leftrightarrow F[e]$
- For $S_1; \ldots; S_n$,
  $\text{wp}(F,\ S_1; \ldots; S_n) \Leftrightarrow \text{wp}(\text{wp}(F,\ S_n),\ S_1; \ldots; S_{n-1})$

Verification Conditions

Verification Condition of basic path

$@\ F$
$S_1;$
$\ldots$
$S_n;$
$@\ G$

is

$F \rightarrow \text{wp}(G,\ S_1; \ldots; S_n)$

Also denoted by

$\{F\}S_1; \ldots; S_n\{G\}$

That is, for every state s,

if $s \models F$
then $s' \models G$ (after the path $S_1; S_2; \ldots; S_n$ is executed)

Example: Basic path

(1)

$@\ F:\ x \geq 0$
$S_1:\ x := x + 1;$
$@\ G:\ x \geq 1$

The VC is $F \rightarrow \text{wp}(G,\ S_1)$. That is,

$$\begin{aligned} & \text{wp}(G,\ S_1) \\ \Leftrightarrow\ & \text{wp}(x \geq 1,\ x := x + 1) \\ \Leftrightarrow\ & (x \geq 1)\{x \mapsto x + 1\} \\ \Leftrightarrow\ & x + 1 \geq 1 \\ \Leftrightarrow\ & x \geq 0 \end{aligned}$$

Therefore the VC of path (1) is

$$x \geq 0 \rightarrow x \geq 0\,,$$

which is $T_{\mathbb{Z}}$-valid.

Example 1: Shortcut (backward substitution)

VC: $\boxed{\underbrace{x \geq 0}_{F} \rightarrow \underbrace{x \geq 0}_{\text{wp}(G, S_1)}}$

$@F:\ x \geq 0$

$x + 1 \geq 1$ $\quad$ *i.e.* $\quad x \geq 0$

$S_1:\ x := x + 1;$

$x \geq 1$

$@G:\ x \geq 1$

$\Uparrow$

Example: Basic path (2) of LinearSearch

(2)

$@L: \ F: \ \ell \leq i \wedge \forall j.\ \ell \leq j < i \ \rightarrow \ a[j] \neq e$
$S_1: \ \texttt{assume}\ i \leq u;$
$S_2: \ \texttt{assume}\ a[i] = e;$
$S_3: \ rv := \texttt{true};$
$@\mathrm{post}\ G: \ rv \ \leftrightarrow \ \exists j.\ \ell \leq j \leq u \wedge a[j] = e$

The VC is $\quad F \ \rightarrow \ \mathrm{wp}(G,\ S_1; S_2; S_3)$. That is,

$$
\begin{array}{ll}
 & \mathrm{wp}(G,\ S_1; S_2; S_3) \\
\Leftrightarrow & \mathrm{wp}(\mathrm{wp}(rv \ \leftrightarrow \ \exists j.\ \ell \leq j \leq u \wedge a[j] = e,\ rv := \texttt{true}),\ S_1; S_2) \\
\Leftrightarrow & \mathrm{wp}(\texttt{true} \ \leftrightarrow \ \exists j.\ \ell \leq j \leq u \wedge a[j] = e,\ S_1; S_2) \\
\Leftrightarrow & \mathrm{wp}(\exists j.\ \ell \leq j \leq u \wedge a[j] = e,\ S_1; S_2) \\
\Leftrightarrow & \mathrm{wp}(\mathrm{wp}(\exists j.\ \ell \leq j \leq u \wedge a[j] = e,\ \texttt{assume}\ a[i] = e),\ S_1) \\
\Leftrightarrow & \mathrm{wp}(a[i] = e \ \rightarrow \ \exists j.\ \ell \leq j \leq u \wedge a[j] = e,\ S_1) \\
\Leftrightarrow & \mathrm{wp}(a[i] = e \ \rightarrow \ \exists j.\ \ell \leq j \leq u \wedge a[j] = e,\ \texttt{assume}\ i \leq u) \\
\Leftrightarrow & i \leq u \ \rightarrow \ (a[i] = e \ \rightarrow \ \exists j.\ \ell \leq j \leq u \wedge a[j] = e)
\end{array}
$$

Therefore the VC of path (2) is

$$
\begin{array}{l}
\ell \leq i \ \wedge \ (\forall j.\ \ell \leq j < i \ \rightarrow \ a[j] \neq e) \\
\quad \rightarrow \ (i \leq u \ \rightarrow \ (a[i] = e \ \rightarrow \ \exists j.\ \ell \leq j \leq u \wedge a[j] = e))
\end{array}
\tag{1}
$$

or, equivalently,

$$
\begin{array}{l}
\ell \leq i \ \wedge \ (\forall j.\ \ell \leq j < i \ \rightarrow \ a[j] \neq e) \ \wedge \ i \leq u \ \wedge \ a[i] = e \\
\quad \rightarrow \ \exists j.\ \ell \leq j \leq u \ \wedge \ a[j] = e
\end{array}
\tag{2}
$$

according to the equivalence

$$
\begin{array}{ll}
 & F_1 \wedge F_2 \ \rightarrow \ (F_3 \ \rightarrow \ (F_4 \ \rightarrow \ F_5)) \\
\Leftrightarrow & (F_1 \wedge F_2 \wedge F_3 \wedge F_4) \ \rightarrow \ F_5\ .
\end{array}
$$

This formula (2) is $(T_{\mathbb{Z}} \cup T_{\mathsf{A}})$-valid.

Example 2: Shortcut (backward substitution)

VC: $\boxed{\underbrace{1 \leq i \wedge (\forall j. A[j])}_{F} \ \wedge i \leq u \ \wedge \ a[i] = e \rightarrow (\exists j. B[j])}$

$@L: \ F: \ 1 \leq i \ \wedge \ \forall j. \underbrace{1 \leq j < i \rightarrow a[j] \neq e}_{A[j]}$

$i \leq u \ \wedge \ a[i] = e \rightarrow (\exists j. B[j])$

$S_1: \ \text{assume}\ i \leq u;$

$a[i] = e \rightarrow (\exists j. B[j])$

⇑

Example 2: Shortcut (backward substitution), cont.

$S_1: \ \text{assume}\ i \leq u;$

$a[i] = e \rightarrow (\exists j. B[j])$

$S_2: \ \text{assume}\ a[i] = e;$

$\text{true} \leftrightarrow (\exists j. B[j]) \quad i.e. \quad (\exists j. B[j]))$

$S_3: \ rv := \text{true};$

$rv \leftrightarrow (\exists j. B[j])$

$@\mathrm{post}\ G: \ rv \leftrightarrow \exists j. \underbrace{1 \leq j \leq u \ \wedge \ a[j] = e}_{B[j]}$

⇑

P-invariant and P-inductive I

Consider program P with function f s.t.

function precondition F_{pre} and

initial location L_0.

A P-computation is a sequence of states

$s_0, s_1, s_2, \ldots$

such that

- $s_0[pc] = L_0$ and $s_0 \models F_{\text{pre}}$, and
- for each i, s_{i+1} is the result of executing the instruction at $s_i[pc]$ on state s_i.

where $s_i[pc]$ = value of pc given by state s_i.

P-invariant and P-inductive II

A formula F annotating location L of program P is P-invariant if for all P-computations $s_0, s_1, s_2, \ldots$ and for each index i,

$$s_i[pc] = L \quad \Rightarrow \quad s_i \models F$$

Annotations of P are P-invariant iff each annotation of P is P-invariant at its location.

Not Implementable: checking if F is P-invariant requires an infinite number of P-computations in general.

Annotations of P are P-inductive iff all VCs generated from the basic paths of program P are T-valid

$$P\text{-inductive} \Rightarrow P\text{-invariant}$$

In Practice: we check if the annotations are P-inductive.

Theorem (Verification Conditions)

If for every basic path

$@L_1 :\ F$
$S_1;$
$\vdots$
$S_n;$
$@L_j :\ G$

of program P, the verification condition

$$\{F\}S_1; \ldots ; S_n\{G\}$$

is T-valid, then the annotations are P-inductive, and therefore P-invariant.

Partial Correctness: For program P, if there is a P-invariant annotation, then P is partially correct.

Total Correctness

Total Correctness = Partial Correctness + Termination

For every input that satisfies F_{pre}, the program eventually halts and produces output that satisfies F_{post}.

Proving function termination:

- Choose set W with well-founded relation $\prec$
  Usually set of n-tuples of natural numbers with the lexicographic relation $<_n$
- Find function δ (ranking function) mapping
  program states $\rightarrow$ W
  such that δ decreases according to $\prec$ along every basic path.

Since $\prec$ is well-founded, there cannot exist an infinite sequence of program states. The program must terminate.

Showing decrease of ranking function

For basic path with ranking function

$$
\begin{array}{ll}
@\ F & \\
\downarrow\ \delta[\overline{x}] & \text{... ranking function} \\
S_1; & \\
\vdots & \\
S_k; & \\
\downarrow\ \kappa[\overline{x}] & \text{... ranking function}
\end{array}
$$

We must prove that
  the value of $\kappa \in W$ after executing $S_1; \cdots ; S_n$
  is less than
  the value of $\delta \in W$ before executing the statements
Thus, we show the verification condition

$$F \ \rightarrow\ \mathrm{wp}(\kappa \prec \delta[\overline{x}_0],\ S_1; \cdots ; S_k)\{\overline{x}_0\ \mapsto\ \overline{x}\}\ .$$

Example: BubbleSort — loops

Choose $(\mathbb{N}^2, <_2)$ as well-founded set

```
@pre ⊤
@post ⊤
int[] BubbleSort(int[] a0) {
  int[] a := a0;
  for
    @L1 : i + 1 ≥ 0
    ↓ (i + 1, i + 1)            ...ranking function δ1
    (int i := |a| − 1; i > 0; i := i − 1) {
```



```
    for
      @L2 : i + 1 ≥ 0 ∧ i − j ≥ 0
      ↓ (i + 1, i − j)            ...ranking function δ2
      (int j := 0; j < i; j := j + 1) {
      if (a[j] > a[j + 1]) {
        int t := a[j];
        a[j] := a[j + 1];
        a[j + 1] := t;
      }
    }
  }
  return a;
}
```


We have to prove

- loop invariants are inductive (we don't show here)
- function decreases along each basic path.

The relevant basic paths:

(1)

```
@L1 : i + 1 ≥ 0
↓L1 : (i + 1, i + 1)
assume i > 0;
j := 0;
↓L2 : (i + 1, i − j)
```

Path **(1)**:

$$i + 1 \geq 0 \wedge i > 0 \rightarrow (i + 1, i - 0) <_2 (i + 1, i + 1)$$

(2, 3)

$@L_2 : \ i+1 \geq 0 \wedge i-j \geq 0$
$\downarrow L_2 : \ (i+1, i-j)$
`assume` $j < i$;
$\ldots$
$j := j+1$;
$\downarrow L_2 : \ (i+1, i-j)$

Paths **(2)** and **(3)**:

$$i+1 \geq 0 \wedge i-j \geq 0 \wedge j < i \rightarrow (i+1, i-(j+1)) <_2 (i+1, i-j)$$

(4)

$@L_2 : \ i+1 \geq 0 \wedge i-j \geq 0$
$\downarrow L_2 : \ (i+1, i-j)$
`assume` $j \geq i$;
$i := i-1$;
$\downarrow L_1 : \ (i+1, i+1)$

Path **(4)**:

$$i+1 \geq 0 \wedge i-j \geq 0 \wedge j \geq i \rightarrow ((i-1)+1, (i-1)+1) <_2 (i+1, i-j)$$

All VCs are valid. Hence, BubbleSort always halts.

Construction of last VC

The verification condition for Path (4) is generated as follows:

$$\begin{aligned} & \mathrm{wp}((i+1, i+1) <_2 (i_0+1, i_0-j_0), \ \texttt{assume } j \geq i; i := i-1) \\ \Leftrightarrow \quad & \mathrm{wp}(((i-1)+1, (i-1)+1) <_2 (i_0+1, i_0-j_0), \ \texttt{assume } j \geq i) \\ \Leftrightarrow \quad & j \geq i \ \rightarrow \ (i, i) <_2 (i_0+1, i_0-j_0) \end{aligned}$$

Replace back $(i_0, j_0) \ \rightarrow \ (i, j)$:

$$j \geq i \ \rightarrow \ (i, i) <_2 (i+1, i-j),$$

producing the VC

$$i+1 \geq 0 \ \wedge \ i-j \geq 0 \ \wedge \ j \geq i \ \rightarrow \ (i, i) <_2 (i+1, i-j).$$

Example 3: Shortcut (backward substitution)

VC: $\boxed{i+1 \geq 0 \wedge i-j \geq 0 \wedge j \geq i \rightarrow (i, i) <_2 (i+1, i-j)}$

$$i+1 \geq 0 \wedge i-j \geq 0 \wedge j \geq i \rightarrow (i, i) <_2 (i_0+1, i_0-j_0)$$

$@L_2 : \ i+1 \geq 0 \wedge i-j \geq 0$

$$j \geq i \rightarrow (i, i) <_2 (i_0+1, i_0-j_0)$$

$\downarrow L_2 : \ (i+1, i-j)$

$$j \geq i \rightarrow (i, i) <_2 (i_0+1, i_0-j_0)$$

assume $j \geq i$;

$$(i, i) <_2 (i_0+1, i_0-j_0)$$

$i := i-1$;

$$(i+1, i+1) <_2 (i_0+1, i_0-j_0)$$

$\downarrow L_1 : \ (i+1, i+1)$ $\Uparrow$

Example 3: Shortcut (backward substitution)

VC: $\boxed{i+1 \geq 0 \wedge i-j \geq 0 \wedge j \geq i \rightarrow (i,i) <_2 (i+1, i-j)}$

$@L_2: \ i+1 \geq 0 \wedge i-j \geq 0$

$j \geq i \rightarrow (i,i) <_2 (i+1, i-j)$

$\downarrow L_2: \ (i+1, i-j)$

$j \geq i \rightarrow (i,i) <_2 \ ?$

assume $j \geq i$;

$(i,i) <_2 \ ?$

$i := i-1$;

$(i+1, i+1) <_2 \ ?$

$\downarrow L_1: \ (i+1, i+1)$ $\Uparrow$

Example: Binary Search — recursive calls

Choose $(\mathbb{N}, <)$ as well-founded set and ranking function $\delta : u - \ell + 1$

```
@pre u − ℓ + 1 ≥ 0
@post ⊤
↓ u − ℓ + 1      ... ranking function δ
bool BinarySearch(int[] a, int ℓ, int u, int e) {
  if (ℓ > u) return false;
  else {
    int m := (ℓ + u) div 2;
    if (a[m] = e) return true;
    else if (a[m] < e) return
      @R1 : u − (m + 1) + 1 ≥ 0
      BinarySearch(a, m + 1, u, e);
    else return
      @R2 : (m − 1) − ℓ + 1 ≥ 0
      BinarySearch(a, ℓ, m − 1, e);
  }
}
```


Show $@R_1$ and $@R_2$ are P-invariant

Show decrease in $u - \ell + 1$:

(1)

@pre $u - \ell + 1 \geq 0$
$\downarrow u - \ell + 1$
assume $\ell \leq u$;
$m := (\ell + u)$ div 2;
assume $a[m] \neq e$;
assume $a[m] < e$;
$\downarrow u - (m+1) + 1$

Verification condition:

$$u - \ell + 1 \geq 0 \ \wedge \ \ell \leq u \ \wedge \ \cdots$$
$$\rightarrow \quad u - (((\ell + u) \ \mathtt{div}\ 2) + 1) + 1 < u - \ell + 1$$

Show decrease in $u - \ell + 1$:

(2)

@pre $u - \ell + 1 \geq 0$
$\downarrow u - \ell + 1$
assume $\ell \leq u$;
$m := (\ell + u)$ div 2;
assume $a[m] \neq e$;
assume $a[m] \geq e$;
$\downarrow (m-1) - \ell + 1$

Verification condition:

$$u - \ell + 1 \geq 0 \ \wedge \ \ell \leq u \ \wedge \ \cdots$$
$$\rightarrow \quad (((\ell + u) \ \mathtt{div}\ 2) - 1) - \ell + 1 < u - \ell + 1$$

Note: two other basic paths (... `return false` and ... `return true`) are irrelevant to the termination argument (recursion ends at each).

Both VCs are $T_{\mathbb{Z}}$-valid. Thus BinarySearch halts on all input in which ℓ is initially at most $u + 1$.