CS156: The Calculus of Computation

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Autumn 2008

Chapter 5: Program Correctness: Mechanics

Program A: LinearSearch with function specification

```
@pre $0 \le \ell \wedge u < |a|$
@post $rv \leftrightarrow \exists i.\ \ell \le i \le u \wedge a[i] = e$
bool LinearSearch(int[] $a$, int $\ell$, int $u$, int $e$) {
  for @ $\top$
    (int $i := \ell$; $i \le u$; $i := i + 1$) {
    if ($a[i] = e$) return true;
  }
  return false;
}
```

Function LinearSearch searches subarray of array a of integers for specified value e.

Function specifications

- Function precondition (@*pre*)
  It behaves correctly only if $0 \le \ell$ and $u < |a|$
- Function postcondition (@*post*)
  It returns true iff a contains the value e in the range $[\ell, u]$

for loop: initially set i to be ℓ,
execute the body and increment i by 1
as long as $i \le u$

@ - program annotation

Program B: BinarySearch with function specification

```
@pre $0 \le \ell \wedge u < |a| \wedge \mathrm{sorted}(a, \ell, u)$
@post $rv \leftrightarrow \exists i.\ \ell \le i \le u \wedge a[i] = e$
bool BinarySearch(int[] $a$, int $\ell$, int $u$, int $e$) {
  if ($\ell > u$) return false;
  else {
    int $m := (\ell + u)$ div 2;
    if ($a[m] = e$) return true;
    else if ($a[m] < e$) return BinarySearch($a, m + 1, u, e$);
    else return BinarySearch($a, \ell, m - 1, e$);
  }
}
```

The recursive function BinarySearch searches sorted subarray a of integers for specified value e.

sorted: weakly increasing order, i.e.

$$\text{sorted}(a, \ell, u) \;\Leftrightarrow\; \forall i, j.\ \ell \leq i \leq j \leq u \;\rightarrow\; a[i] \leq a[j]$$

Defined in the combined theory of integers and arrays, $T_{\mathbb{Z} \cup A}$

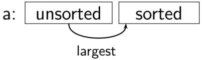
Function specifications

- Function precondition (@*pre*)
  It behaves correctly only if
  $0 \leq \ell$ and $u < |a|$ and
  $\text{sorted}(a, \ell, u)$.
- Function postcondition (@*post*)
  It returns true iff a contains the value e in the range $[\ell, u]$

Program C: BubbleSort with function specification

```
@pre $\top$
@post sorted($rv, 0, |rv| - 1$)
int[] BubbleSort(int[] $a_0$) {
  int[] $a := a_0$;
  for @ $\top$
    (int $i := |a| - 1$; $i > 0$; $i := i - 1$) {
    for @ $\top$
      (int $j := 0$; $j < i$; $j := j + 1$) {
      if ($a[j] > a[j + 1]$) {
        int $t := a[j]$;
        $a[j] := a[j + 1]$;
        $a[j + 1] := t$;
      }
    }
  }
  return $a$;
}
```

Function BubbleSort sorts integer array a



by “bubbling” the largest element of the left unsorted region of a toward the sorted region on the right.

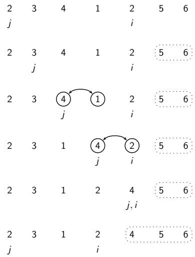
Each iteration of the outer loop expands the sorted region by one cell.[1]

Function specification

- Function postcondition (@post):
  BubbleSort returns array rv sorted on the range $[0, |rv| - 1]$.

[1] Except the last iteration, which expands the sorted region by two cells, so that an entire array of length n is sorted in $n - 1$ iterations.

Sample execution of BubbleSort



2 3 4 1 2 5 6
j i

2 3 4 1 2 5 6
j i

2 3 4 1 2 5 6
j i

2 3 1 4 2 5 6
j i

2 3 1 2 4 5 6
j, i

2 3 1 2 4 5 6
j i

Program Annotation

- Function Specifications

  function precondition (@pre)
  function postcondition (@post)

- Runtime Assertions

  e.g., @ $0 \leq j < |a| \;\wedge\; 0 \leq j+1 < |a|$
  $a[j] := a[j+1]$

- Loop Invariants

  e.g., @ $L : \ell \leq i \;\wedge\; \forall j.\ \ell \leq j < i \;\rightarrow\; a[j] \neq e$

  The L : gives a name to the formula, just like the F : we've used in other formulae.

Program A: LinearSearch with runtime assertions

```
@pre $0 \leq \ell \;\wedge\; u < |a|$
@post $rv \;\leftrightarrow\; \exists i.\ \ell \leq i \leq u \wedge a[i] = e$
bool LinearSearch(int[] $a$, int $\ell$, int $u$, int $e$) {
  for
    @ $L : \top$
    (int $i := \ell$; $i \leq u$; $i := i+1$) {
    @ $0 \leq i < |a|$;
    if ($a[i] = e$) return true;
  }
  return false;
}
```

Program B: BinarySearch with runtime assertions

```
@pre $0 \leq \ell \;\wedge\; u < |a| \;\wedge\; \mathrm{sorted}(a, \ell, u)$
@post $rv \;\leftrightarrow\; \exists i.\ \ell \leq i \leq u \wedge a[i] = e$
bool BinarySearch(int[] $a$, int $\ell$, int $u$, int $e$) {
  if ($\ell > u$) return false;
  else {
    @ $2 \neq 0$;
    int $m := (\ell + u)$ div 2;
    @ $0 \leq m < |a|$;
    if ($a[m] = e$) return true;
    else {
      @ $0 \leq m < |a|$;
      if ($a[m] < e$) return BinarySearch($a, m+1, u, e$);
      else return BinarySearch($a, \ell, m-1, e$);
    }
  }
}
```

Program C: BubbleSort with runtime assertions

```
@pre $\top$
@post sorted($rv, 0, |rv| - 1$)
int[] BubbleSort(int[] $a_0$) {
  int[] $a := a_0$;
  for
    @ $L_1 : \top$
    (int $i := |a| - 1$; $i > 0$; $i := i - 1$) {
    for
      @ $L_2 : \top$
      (int $j := 0$; $j < i$; $j := j + 1$) {
      @ $0 \leq j < |a| \wedge 0 \leq j+1 < |a|$;
      if ($a[j] > a[j+1]$) {
        int $t := a[j]$;
        $a[j] := a[j+1]$;
        $a[j+1] := t$;
      }
    }
  }
  return $a$;
}
```

Loop Invariants

```
while
  @ $F$
  $\langle cond \rangle$ { $\langle body \rangle$ }
```

- apply $\langle body \rangle$ as long as $\langle cond \rangle$ holds
- assertion F holds at the beginning of every iteration evaluated before $\langle cond \rangle$ is checked

```
for
  @ $F$
  ($\langle init \rangle$; $\langle cond \rangle$; $\langle incr \rangle$)
  {$\langle body \rangle$}
```

```
$\langle init \rangle$;
while
  @ $F$
  $\langle cond \rangle$ { $\langle body \rangle$; $\langle incr \rangle$ }
```

Program A: LinearSearch with loop invariants

```
@pre $0 \leq \ell \wedge u < |a|$
@post $rv \leftrightarrow \exists j.\ \ell \leq j \leq u \wedge a[j] = e$
bool LinearSearch(int[] $a$, int $\ell$, int $u$, int $e$) {
  for
    @$L$ : $\ell \leq i \wedge (\forall j.\ \ell \leq j < i \rightarrow a[j] \neq e)$
    (int $i := \ell$; $i \leq u$; $i := i + 1$) {
    if ($a[i] = e$) return true;
  }
  return false;
}
```

Proving Partial Correctness

A function is partially correct if
when the function's precondition is satisfied on entry,
its postcondition is satisfied when the function halts/exits.

- A function + annotation is reduced to finite set of verification conditions (VCs), FOL formulae
- If all VCs are T-valid, then the function obeys its specification (partially correct)

Basic Paths: Loops

To handle loops, we break the function into basic paths

@ ← precondition or loop invariant

sequence of instructions
(with no loop invariants)

@ ← loop invariant, runtime assertion, or postcondition

Program A: LinearSearch I

Basic Paths of LinearSearch

(1)

```
@pre $0 \le \ell \wedge u < |a|$
$i := \ell$;
@$L$ : $\ell \le i \wedge \forall j.\ \ell \le j < i \ \rightarrow\ a[j] \ne e$
```

(2)

```
@$L$ : $\ell \le i \wedge \forall j.\ \ell \le j < i \ \rightarrow\ a[j] \ne e$
assume $i \le u$;
assume $a[i] = e$;
$rv$ := true;
@post $rv \ \leftrightarrow\ \exists j.\ \ell \le j \le u \wedge a[j] = e$
```

Program A: LinearSearch II

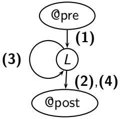
(3)

```
@$L$ : $\ell \le i \wedge \forall j.\ \ell \le j < i \ \rightarrow\ a[j] \ne e$
assume $i \le u$;
assume $a[i] \ne e$;
$i := i + 1$;
@$L$ : $\ell \le i \wedge \forall j.\ \ell \le j < i \ \rightarrow\ a[j] \ne e$
```

(4)

```
@$L$ : $\ell \le i \wedge \forall j.\ \ell \le j < i \ \rightarrow\ a[j] \ne e$
assume $i > u$;
$rv$ := false;
@post $rv \ \leftrightarrow\ \exists j.\ \ell \le j \le u \wedge a[j] = e$
```

Visualization of basic paths of LinearSearch



Program C: BubbleSort with loop invariants

```
@pre $\top$
@post sorted$(rv, 0, |rv| - 1)$
int[] BubbleSort(int[] $a_0$) {
  int[] $a := a_0$;
  for
      @$L_1$ : $\begin{bmatrix} -1 \le i < |a| \\ \wedge \text{partitioned}(a, 0, i, i+1, |a|-1) \\ \wedge \text{sorted}(a, i, |a|-1) \end{bmatrix}$
      (int $i := |a| - 1$; $i > 0$; $i := i - 1$) {
```

```
    for
      $@L_2 : \begin{bmatrix} 1 \le i < |a| \wedge 0 \le j \le i \\ \wedge \text{partitioned}(a, 0, i, i+1, |a|-1) \\ \wedge \text{partitioned}(a, 0, j-1, j, j) \\ \wedge \text{sorted}(a, i, |a|-1) \end{bmatrix}$
      (int $j := 0$; $j < i$; $j := j + 1$) {
      if ($a[j] > a[j+1]$) {
        int $t := a[j]$;
        $a[j] := a[j+1]$;
        $a[j+1] := t$;
      }
    }
  }
  return $a$;
}
```

Partition

$$\text{partitioned}(a, \ell_1, u_1, \ell_2, u_2) \\ \Leftrightarrow\ \forall i, j.\ \ell_1 \le i \le u_1 < \ell_2 \le j \le u_2\ \rightarrow\ a[i] \le a[j]$$

in $T_{\mathbb{Z}} \cup T_{\mathsf{A}}$.

That is, each element of a in the range $[\ell_1, u_1]$ is $\le$ each element in the range $[\ell_2, u_2]$.

Basic Paths of BubbleSort

(1)

```
@pre $\top$
$a := a_0$;
$i := |a| - 1$;
$@L_1 : \begin{bmatrix} -1 \le i < |a| \wedge \text{partitioned}(a, 0, i, i+1, |a|-1) \\ \wedge\ \text{sorted}(a, i, |a|-1) \end{bmatrix}$
```

(2)

```
$@L_1 : \begin{bmatrix} -1 \le i < |a| \wedge \text{partitioned}(a, 0, i, i+1, |a|-1) \\ \wedge\ \text{sorted}(a, i, |a|-1) \end{bmatrix}$
assume $i > 0$;
$j := 0$;
$@L_2 : \begin{bmatrix} 1 \le i < |a| \wedge 0 \le j \le i \wedge \text{partitioned}(a, 0, i, i+1, |a|-1) \\ \wedge\ \text{partitioned}(a, 0, j-1, j, j) \wedge \text{sorted}(a, i, |a|-1) \end{bmatrix}$
```

(3)

```
$@L_2 : \begin{bmatrix} 1 \le i < |a| \wedge 0 \le j \le i \wedge \text{partitioned}(a, 0, i, i+1, |a|-1) \\ \wedge\ \text{partitioned}(a, 0, j-1, j, j) \wedge \text{sorted}(a, i, |a|-1) \end{bmatrix}$
assume $j < i$;
assume $a[j] > a[j+1]$;
$t := a[j]$;
$a[j] := a[j+1]$;
$a[j+1] := t$;
$j := j + 1$;
$@L_2 : \begin{bmatrix} 1 \le i < |a| \wedge 0 \le j \le i \wedge \text{partitioned}(a, 0, i, i+1, |a|-1) \\ \wedge\ \text{partitioned}(a, 0, j-1, j, j) \wedge \text{sorted}(a, i, |a|-1) \end{bmatrix}$
```

(4)

$$@L_2 : \left[\begin{array}{l} 1 \le i < |a| \land 0 \le j \le i \land \mathsf{partitioned}(a, 0, i, i+1, |a|-1) \\ \land\ \mathsf{partitioned}(a, 0, j-1, j, j) \land \mathsf{sorted}(a, i, |a|-1) \end{array}\right]$$

assume $j < i$;

assume $a[j] \le a[j+1]$;

$j := j + 1$;

$$@L_2 : \left[\begin{array}{l} 1 \le i < |a| \land 0 \le j \le i \land \mathsf{partitioned}(a, 0, i, i+1, |a|-1) \\ \land\ \mathsf{partitioned}(a, 0, j-1, j, j) \land \mathsf{sorted}(a, i, |a|-1) \end{array}\right]$$

(5)

$$@L_2 : \left[\begin{array}{l} 1 \le i < |a| \land 0 \le j \le i \land \mathsf{partitioned}(a, 0, i, i+1, |a|-1) \\ \land\ \mathsf{partitioned}(a, 0, j-1, j, j) \land \mathsf{sorted}(a, i, |a|-1) \end{array}\right]$$

assume $j \ge i$;

$i := i - 1$;

$$@L_1 : \left[\begin{array}{l} -1 \le i < |a| \land \mathsf{partitioned}(a, 0, i, i+1, |a|-1) \\ \land\ \mathsf{sorted}(a, i, |a|-1) \end{array}\right]$$

(6)

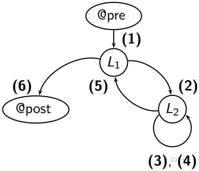
$$@L_1 : \left[\begin{array}{l} -1 \le i < |a| \land \mathsf{partitioned}(a, 0, i, i+1, |a|-1) \\ \land\ \mathsf{sorted}(a, i, |a|-1) \end{array}\right]$$

assume $i \le 0$;

$rv := a$;

@post $\mathsf{sorted}(rv, 0, |rv| - 1)$

Visualization of basic paths of BubbleSort



Basic Paths: Function Calls

- Loops produce unbounded number of paths
  loop invariants cut loops to produce
  finite number of basic paths
- Reursive calls produce unbounded number of paths
  function specifications cut function calls

In BinarySearch

@pre $0 \le \ell \land u < |a| \land \mathsf{sorted}(a, \ell, u)$ $\ldots F[a, \ell, u, e]$

$\vdots$

$@R_1 :\ 0 \le m + 1 \land u < |a| \land \mathsf{sorted}(a, m+1, u)$ $\ldots F[a, m+1, u, e]$

return BinarySearch$(a, m+1, u, e)$

$\vdots$

$@R_2 :\ 0 \le \ell \land m - 1 < |a| \land \mathsf{sorted}(a, \ell, m-1)$ $\ldots F[a, \ell, m-1, e]$

return BinarySearch$(a, \ell, m-1, e)$

Program B: BinarySearch with function call assertions

```
@pre $0 \le \ell \land u < |a| \land \mathrm{sorted}(a, \ell, u)$
@post $rv \leftrightarrow \exists i.\ \ell \le i \le u \land a[i] = e$
bool BinarySearch(int[] $a$, int $\ell$, int $u$, int $e$) {
  if ($\ell > u$) return false;
  else {
    int $m := (\ell + u)$ div 2;
    if ($a[m] = e$) return true;
    else if ($a[m] < e$) {
      @$R_1$ : $0 \le m + 1 \land u < |a| \land \mathrm{sorted}(a, m + 1, u)$;
      return BinarySearch($a, m + 1, u, e$);
    } else {
      @$R_2$ : $0 \le \ell \land m - 1 < |a| \land \mathrm{sorted}(a, \ell, m - 1)$;
      return BinarySearch($a, \ell, m - 1, e$);
    }
  }
}
```

(1)

```
@pre $0 \le \ell \land u < |a| \land \mathrm{sorted}(a, \ell, u)$
assume $\ell > u$;
$rv :=$ false;
@post $rv \leftrightarrow \exists i.\ \ell \le i \le u \land a[i] = e$
```

(2)

```
@pre $0 \le \ell \land u < |a| \land \mathrm{sorted}(a, \ell, u)$
assume $\ell \le u$;
$m := (\ell + u)$ div 2;
assume $a[m] = e$;
$rv :=$ true;
@post $rv \leftrightarrow \exists i.\ \ell \le i \le u \land a[i] = e$
```

(3)

```
@pre $0 \le \ell \land u < |a| \land \mathrm{sorted}(a, \ell, u)$
assume $\ell \le u$;
$m := (\ell + u)$ div 2;
assume $a[m] \ne e$;
assume $a[m] < e$;
@$R_1$ : $0 \le m + 1 \land u < |a| \land \mathrm{sorted}(a, m + 1, u)$
```

(5)

```
@pre $0 \le \ell \land u < |a| \land \mathrm{sorted}(a, \ell, u)$
assume $\ell \le u$;
$m := (\ell + u)$ div 2;
assume $a[m] \ne e$;
assume $a[m] \ge e$;
@$R_2$ : $0 \le \ell \land m - 1 < |a| \land \mathrm{sorted}(a, \ell, m - 1)$
```

(4)

```
@pre $0 \leq \ell \wedge u < |a| \wedge \text{sorted}(a, \ell, u)$
assume $\ell \leq u$;
$m := (\ell + u)$ div 2;
assume $a[m] \neq e$;
assume $a[m] < e$;
assume $v_1 \;\leftrightarrow\; \exists i.\ m + 1 \leq i \leq u \wedge a[i] = e$;
$rv := v_1$;
@post $rv \;\leftrightarrow\; \exists i.\ \ell \leq i \leq u \wedge a[i] = e$
```

(6)

```
@pre $0 \leq \ell \wedge u < |a| \wedge \text{sorted}(a, \ell, u)$
assume $\ell \leq u$;
$m := (\ell + u)$ div 2;
assume $a[m] \neq e$;
assume $a[m] \geq e$;
assume $v_2 \;\leftrightarrow\; \exists i.\ \ell \leq i \leq m - 1 \wedge a[i] = e$;
$rv := v_2$;
@post $rv \;\leftrightarrow\; \exists i.\ \ell \leq i \leq u \wedge a[i] = e$
```

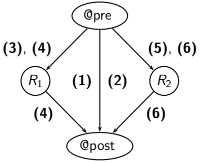


Figure: Visualization of basic paths of BinarySearch

Program States

Program counter pc holds current location of control

State s of P assignment of values to all variables (proper types) of P

Example:

$$s : \left\{ \begin{array}{l} pc \;\mapsto\; L_1,\; a \;\mapsto\; [2;0;1], \\ i \;\mapsto\; 2,\; j \mapsto 0,\; t \mapsto 2,\; rv \mapsto [\,] \end{array} \right\}$$

is a state of BubbleSort.

Reachable state s of P a state that can be reached during some computation of P

Example:

$$s : \left\{ \begin{array}{l} pc \;\mapsto\; L_1,\; a \;\mapsto\; [0;1;2], \\ i \;\mapsto\; 1,\; j \mapsto 1,\; t \mapsto 2 \end{array} \right\}$$

is a reachable state of BubbleSort.

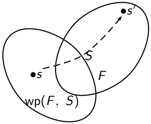
Weakest Precondition $\text{wp}(F, S)$

For FOL formula F, program statement S,

$s \models \text{wp}(F,\ S)$ iff

statement S is executed on state s to produce state s',

and $s' \models F$:



- $\text{wp}(F,\ \texttt{assume}\ c) \Leftrightarrow c \rightarrow F$
- $\text{wp}(F[v],\ v := e) \Leftrightarrow F[e]$
- For $S_1; \ldots; S_n$,
  $\text{wp}(F,\ S_1; \ldots; S_n) \Leftrightarrow \text{wp}(\text{wp}(F,\ S_n),\ S_1; \ldots; S_{n-1})$

Verification Conditions

Verification Condition of basic path

$@\ F$

$S_1;$

$\ldots$

$S_n;$

$@\ G$

is

$F \rightarrow \text{wp}(G,\ S_1; \ldots; S_n)$

Also denoted by

$\{F\}S_1; \ldots; S_n\{G\}$

That is, for every state s,

if $s \models F$

then $s' \models G$ (after the path $S_1; S_2; \ldots; S_n$ is executed)

Example: Basic path

(1)

$@\ F:\ x \geq 0$

$S_1:\ x := x + 1;$

$@\ G:\ x \geq 1$

The VC is $F \rightarrow \text{wp}(G,\ S_1)$. That is,

$$\begin{aligned} & \text{wp}(G,\ S_1) \\ \Leftrightarrow\ & \text{wp}(x \geq 1,\ x := x + 1) \\ \Leftrightarrow\ & (x \geq 1)\{x \mapsto x + 1\} \\ \Leftrightarrow\ & x + 1 \geq 1 \\ \Leftrightarrow\ & x \geq 0 \end{aligned}$$

Therefore the VC of path (1) is

$$x \geq 0 \rightarrow x \geq 0\ ,$$

which is $T_{\mathbb{Z}}$-valid.

Example: Basic path (2) of LinearSearch

(2)

$@L:\ F:\ \ell \leq i \wedge \forall j.\ \ell \leq j < i \rightarrow a[j] \neq e$

$S_1:\ \texttt{assume}\ i \leq u;$

$S_2:\ \texttt{assume}\ a[i] = e;$

$S_3:\ rv := \texttt{true};$

$@\text{post}\ G:\ rv \leftrightarrow \exists j.\ \ell \leq j \leq u \wedge a[j] = e$

The VC is $F \rightarrow \text{wp}(G,\ S_1; S_2; S_3)$. That is,

$$\begin{aligned} & \text{wp}(G,\ S_1; S_2; S_3) \\ \Leftrightarrow\ & \text{wp}(\text{wp}(rv \leftrightarrow \exists j.\ \ell \leq j \leq u \wedge a[j] = e,\ rv := \texttt{true}),\ S_1; S_2) \\ \Leftrightarrow\ & \text{wp}(\texttt{true} \leftrightarrow \exists j.\ \ell \leq j \leq u \wedge a[j] = e,\ S_1; S_2) \\ \Leftrightarrow\ & \text{wp}(\exists j.\ \ell \leq j \leq u \wedge a[j] = e,\ S_1; S_2) \\ \Leftrightarrow\ & \text{wp}(\text{wp}(\exists j.\ \ell \leq j \leq u \wedge a[j] = e,\ \texttt{assume}\ a[i] = e),\ S_1) \\ \Leftrightarrow\ & \text{wp}(a[i] = e \rightarrow \exists j.\ \ell \leq j \leq u \wedge a[j] = e,\ S_1) \\ \Leftrightarrow\ & \text{wp}(a[i] = e \rightarrow \exists j.\ \ell \leq j \leq u \wedge a[j] = e,\ \texttt{assume}\ i \leq u) \\ \Leftrightarrow\ & i \leq u \rightarrow (a[i] = e \rightarrow \exists j.\ \ell \leq j \leq u \wedge a[j] = e) \end{aligned}$$

Therefore the VC of path (2) is

$$\begin{aligned}&\ell \le i \;\wedge\; (\forall j.\ \ell \le j < i \;\rightarrow\; a[j] \neq e) \qquad (1)\\ &\rightarrow\; (i \le u \;\rightarrow\; (a[i] = e \;\rightarrow\; \exists j.\ \ell \le j \le u \wedge a[j] = e))\end{aligned}$$

or, equivalently,

$$\begin{aligned}&\ell \le i \;\wedge\; (\forall j.\ \ell \le j < i \;\rightarrow\; a[j] \neq e) \;\wedge\; i \le u \;\wedge\; a[i] = e \quad (2)\\ &\rightarrow\; \exists j.\ \ell \le j \le u \;\wedge\; a[j] = e\end{aligned}$$

according to the equivalence

$$\begin{aligned}&F_1 \wedge F_2 \;\rightarrow\; (F_3 \;\rightarrow\; (F_4 \;\rightarrow\; F_5))\\ \Leftrightarrow\quad &(F_1 \wedge F_2 \wedge F_3 \wedge F_4) \;\rightarrow\; F_5 \;.\end{aligned}$$

This formula (2) is $(T_{\mathbb{Z}} \cup T_{\mathsf{A}})$-valid.

P-invariant and P-inductive I

Consider program P with function f s.t.

function precondition F_{pre} and

initial location L_0.

A P-computation is a sequence of states

$s_0, s_1, s_2, \ldots$

such that

- $s_0[pc] = L_0$ and $s_0 \models F_{\text{pre}}$, and
- for each i, s_{i+1} is the result of executing the instruction at $s_i[pc]$ on state s_i.

where $s_i[pc]$ = value of pc given by state s_i.

P-invariant and P-inductive II

A formula F annotating location L of program P is P-invariant if for all P-computations $s_0, s_1, s_2, \ldots$ and for each index i,

$$s_i[pc] = L \quad\Rightarrow\quad s_i \models F$$

Annotations of P are P-invariant iff each annotation of P is P-invariant at its location.

Not Implementable: checking if F is P-invariant requires an infinite number of P-computations in general.

Annotations of P are P-inductive iff all VCs generated from the basic paths of program P are T-valid

$$P\text{-inductive} \;\Rightarrow\; P\text{-invariant}$$

In Practice: we check if the annotations are P-inductive.

Theorem (Verification Conditions)

If for every basic path

```
@L₁ : F
S₁;
⋮
Sₙ;
@Lⱼ : G
```

of program P, the verification condition

$$\{F\}S_1; \ldots; S_n\{G\}$$

is T-valid, then the annotations are P-inductive, and therefore P-invariant.

Partial Correctness: For program P, if there is a P-invariant annotation, then P is partially correct.

Total Correctness

Total Correctness = Partial Correctness + Termination

For every input that satisfies F_{pre}, the program eventually halts and produces output that satisfies F_{post}.

Proving function termination:

- Choose set W with well-founded relation $\prec$
  Usually set of n-tuples of natural numbers with the lexicographic extension $<_n$
- Find function δ (ranking function)
  mapping
  program states → W
  such that δ decreases according to $\prec$ along every basic path.

Since $\prec$ is well-founded, there cannot exist an infinite sequence of program states. The program must terminate.

Showing decrease of ranking function

For basic path with ranking function

```
@ $F$
$\downarrow \delta[\overline{x}]$        ... ranking function
$S_1$;
  ⋮
$S_k$;
$\downarrow \kappa[\overline{x}]$        ... ranking function
```

We must prove that
the value of $\kappa \in W$ after executing $S_1; \cdots ; S_n$
is less than
the value of $\delta \in W$ before executing the statements
Thus, we show the verification condition

$$F \;\rightarrow\; \text{wp}(\kappa \prec \delta[\overline{x}_0],\; S_1; \cdots ; S_k)\{\overline{x}_0 \;\mapsto\; \overline{x}\}\ .$$

Example: BubbleSort — loops

Choose $(\mathbb{N}^2, <_2)$ as well-founded set

```
@pre $\top$
@post $\top$
int[] BubbleSort(int[] $a_0$) {
  int[] $a := a_0$;
  for
    @$L_1$ : $i + 1 \geq 0$
    $\downarrow (i + 1,\; i + 1)$            ...ranking function $\delta_1$
    (int $i := |a| - 1$; $i > 0$; $i := i - 1$) {
```

```
      for
        @$L_2$ : $i + 1 \geq 0 \wedge i - j \geq 0$
        $\downarrow (i + 1,\; i - j)$            ...ranking function $\delta_2$
        (int $j := 0$; $j < i$; $j := j + 1$) {
        if ($a[j] > a[j + 1]$) {
          int $t := a[j]$;
          $a[j] := a[j + 1]$;
          $a[j + 1] := t$;
        }
      }
    }
    return $a$;
  }
```

We have to prove

- loop invariants are inductive (we don't show here)
- function decreases along each basic path.

The relevant basic paths:

(1)

```
@$L_1$ : $i+1 \geq 0$
$\downarrow L_1$ : $(i+1, i+1)$
assume $i > 0$;
$j := 0$;
$\downarrow L_2$ : $(i+1, i-j)$
```

Path **(1)**:

$$i+1 \geq 0 \land i > 0 \rightarrow (i+1, i-0) <_2 (i+1, i+1)$$

(2, 3)

```
@$L_2$ : $i+1 \geq 0 \land i-j \geq 0$
$\downarrow L_2$ : $(i+1, i-j)$
assume $j < i$;
...
$j := j+1$;
$\downarrow L_2$ : $(i+1, i-j)$
```

Paths **(2)** and **(3)**:

$$i+1 \geq 0 \land i-j \geq 0 \land j < i \rightarrow (i+1, i-(j+1)) <_2 (i+1, i-j)$$

(4)

```
@$L_2$ : $i+1 \geq 0 \land i-j \geq 0$
$\downarrow L_2$ : $(i+1, i-j)$
assume $j \geq i$;
$i := i-1$;
$\downarrow L_1$ : $(i+1, i+1)$
```

Path **(4)**:

$$i+1 \geq 0 \land i-j \geq 0 \land j \geq i \rightarrow ((i-1)+1, (i-1)+1) <_2 (i+1, i-j)$$

All VCs are valid. Hence, BubbleSort always halts.
TODO arrows

Construction of last VC

The verification condition for Path (4) is generated as follows:

$$\begin{aligned} & \mathrm{w}p((i+1, i+1) <_2 (i_0+1, i_0-j_0),\ \texttt{assume}\ j \geq i; i := i-1) \\ \Leftrightarrow\ & \mathrm{w}p(((i-1)+1, (i-1)+1) <_2 (i_0+1, i_0-j_0),\ \texttt{assume}\ j \geq i) \\ \Leftrightarrow\ & j \geq i\ \rightarrow\ (i, i) <_2 (i_0+1, i_0-j_0) \end{aligned}$$

Replace back $(i_0, j_0)\ \rightarrow\ (i, j)$:

$$j \geq i\ \rightarrow\ (i, i) <_2 (i+1, i-j),$$

producing the VC

$$i+1 \geq 0\ \land\ i-j \geq 0\ \land\ j \geq i\ \rightarrow\ (i, i) <_2 (i+1, i-j).$$

Example: Binary Search — recursive calls

Choose $(\mathbb{N}, <)$ as well-founded set and ranking function $\delta : u - \ell + 1$

```
@pre $u - \ell + 1 \geq 0$
@post $\top$
$\downarrow u - \ell + 1$      ... ranking function $\delta$
bool BinarySearch(int[] $a$, int $\ell$, int $u$, int $e$) {
  if ($\ell > u$) return false;
  else {
    int $m := (\ell + u)$ div 2;
    if ($a[m] = e$) return true;
    else if ($a[m] < e$) return
      @$R_1 : u - (m + 1) + 1 \geq 0$
      BinarySearch($a, m + 1, u, e$);
    else return
      @$R_2 : (m - 1) - \ell + 1 \geq 0$
      BinarySearch($a, \ell, m - 1, e$);
  }
}
```

Show @R_1 and @R_2 are P-invariant

Show decrease in $u - \ell + 1$:

(1)

```
@pre $u - \ell + 1 \geq 0$
$\downarrow u - \ell + 1$
assume $\ell \leq u$;
$m := (\ell + u)$ div 2;
assume $a[m] \neq e$;
assume $a[m] < e$;
$\downarrow u - (m + 1) + 1$
```

Verification condition:

$$\begin{array}{ll} & u - \ell + 1 \geq 0 \ \wedge \ \ell \leq u \ \wedge \ \cdots \\ \rightarrow & u - (((\ell + u) \texttt{ div } 2) + 1) + 1 < u - \ell + 1 \end{array}$$

Show decrease in $u - \ell + 1$:

(2)

```
@pre $u - \ell + 1 \geq 0$
$\downarrow u - \ell + 1$
assume $\ell \leq u$;
$m := (\ell + u)$ div 2;
assume $a[m] \neq e$;
assume $a[m] \geq e$;
$\downarrow (m - 1) - \ell + 1$
```

Verification condition:

$$\begin{array}{ll} & u - \ell + 1 \geq 0 \ \wedge \ \ell \leq u \ \wedge \ \cdots \\ \rightarrow & (((\ell + u) \texttt{ div } 2) - 1) - \ell + 1 < u - \ell + 1 \end{array}$$

Note: two other basic paths (... `return false` and ... `return true`) are irrelevant to the termination argument (recursion ends at each).

Both VCs are $T_{\mathbb{Z}}$-valid. Thus BinarySearch halts on all input in which ℓ is initially at most $u + 1$.