

CS156: The Calculus of Computation

Zohar Manna
Autumn 2008

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Grading

- ▶ Homeworks (60%)
 - ▶ weekly (totally 8)
 - ▶ no late assignments
 - ▶ no collaboration
- ▶ Final Exam (40%)
 - ▶ open book and notes
 - ▶ date: Monday, December 8th, 8:30-11:30 a.m.

Coverage

- ▶ Skip * sections
- ▶ Skip Chapter 6 and 12 of the book
- ▶ Skip complexity remarks

Website

- ▶ cs156.stanford.edu

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*It is reasonable to hope that the relationship between **computation** and **mathematical logic** will be as fruitful in the next century as that between **analysis** and **physics** in the last. The development of this relationship demands a concern for both applications and mathematical elegance.*

John McCarthy

A Basis for a Mathematical Theory of Computation, 1963

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Assignment #1 (due Mon, Sept 29)

- ▶ 1.1 e, f [10 points each]
- ▶ 1.2 s, x [10 points each]
- ▶ 1.3 (note typo: the last $\vee$ should be a $\wedge$) [30 points]
- ▶ 1.5 d [30 points]

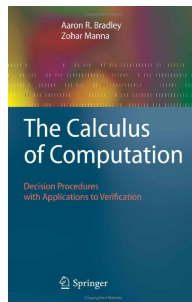
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THE CALCULUS OF COMPUTATION: Decision Procedures with Applications to Verification

by
Aaron Bradley
Zohar Manna

Springer 2007

There are two copies in CS-Math Library and you could also use
`socrates.stanford.edu` to read the book according to its policy.



Topics: Overview

1. First-Order logic
2. Specification and verification
3. Satisfiability decision procedures

Part I: Foundations

1. Propositional Logic
2. First-Order Logic
3. First-Order Theories
4. Induction
5. Program Correctness: Mechanics
Inductive assertion method, Ranking function method

7. Quantified Linear Arithmetic
Quantifier elimination for integers and rationals
8. Quantifier-Free Linear Arithmetic
Linear programming for rationals
9. Quantifier-Free Equality and Data Structures
10. Combining Decision Procedures
Nelson-Oppen combination method
11. Arrays
More than quantifier-free fragment

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Motivation

Motivation I

Decision Procedures are algorithms to decide formulae.
These formulae can arise

- ▶ in software verification.
- ▶ in hardware verification

Consider the following program:

```

for
  @  $\ell \leq i \leq u \wedge (rv \leftrightarrow \exists j. \ell \leq j < i \wedge a[j] = e)$ 
  (int  $i := \ell$ ;  $i \leq u$ ;  $i := i + 1$ ) {
    if ( $a[i] = e$ )  $rv := \text{true}$ ;
  }
  
```

How can we decide whether the **formula** is a loop invariant?

Motivation II

Prove:

```

assume  $\ell \leq i \leq u \wedge (rv \leftrightarrow \exists j. \ell \leq j < i \wedge a[j] = e)$ 
assume  $i \leq u$ 
assume  $a[i] = e$ 
 $rv := \text{true}$ ;
 $i := i + 1$ 
@  $\ell \leq i \leq u \wedge (rv \leftrightarrow \exists j. \ell \leq j < i \wedge a[j] = e)$ 
  
```

Motivation III

```
assume  $\ell \leq i \leq u \wedge (rv \leftrightarrow \exists j. \ell \leq j < i \wedge a[j] = e)$ 
assume  $i \leq u$ 
assume  $a[i] \neq e$ 
 $i := i + 1$ 
@  $\ell \leq i \leq u \wedge (rv \leftrightarrow \exists j. \ell \leq j < i \wedge a[j] = e)$ 
```

A Hoare triple $\{P\} S \{Q\}$ holds, iff

$$P \rightarrow wp(S, Q)$$

(wp denotes “weakest precondition”)

Motivation IV

For assignments wp is computed by substitution:

```
assume  $\ell \leq i \leq u \wedge (rv \leftrightarrow \exists j. \ell \leq j < i \wedge a[j] = e)$ 
assume  $i \leq u$ 
assume  $a[i] = e$ 
 $rv := \text{true};$ 
 $i := i + 1$ 
@  $\ell \leq i \leq u \wedge (rv \leftrightarrow \exists j. \ell \leq j < i \wedge a[j] = e)$ 
```

Substituting $\top$ for rv and $i + 1$ for i , the postcondition (denoted by the @ symbol) holds if and only if:

$$\begin{aligned} & \ell \leq i \leq u \wedge (rv \leftrightarrow \exists j. \ell \leq j < i \wedge a[j] = e) \wedge i \leq u \wedge a[i] = e \\ \rightarrow & \ell \leq i + 1 \leq u \wedge (\top \leftrightarrow \exists j. \ell \leq j < i + 1 \wedge a[j] = e) \end{aligned}$$

Motivation V

We need an algorithm that decides whether this formula holds. If the formula does not hold, the algorithm should give a counterexample; e.g.,

$$\ell = 0, i = 1, u = 1, rv = \text{false}, a[0] = 0, a[1] = 1, e = 1.$$

We will discuss such algorithms in later lectures.

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Chapter 1: Propositional Logic (PL)

Propositional Logic (PL)

PL Syntax

- Atom truth symbols $\top$ ("true") and $\perp$ ("false")
- propositional variables $P, Q, R, P_1, Q_1, R_1, \dots$
- Literal atom α or its negation $\neg\alpha$
- Formula literal or application of a logical connective to formulae F, F_1, F_2
 - $\neg F$ "not" (negation)
 - $F_1 \wedge F_2$ "and" (conjunction)
 - $F_1 \vee F_2$ "or" (disjunction)
 - $F_1 \rightarrow F_2$ "implies" (implication)
 - $F_1 \leftrightarrow F_2$ "if and only if" (iff)

PL Semantics (meaning of PL)

Formula F + Interpretation I = Truth value
(true, false)

Interpretation

$I : \{P \mapsto \text{true}, Q \mapsto \text{false}, \dots\}$

Evaluation of F under I :

$F \parallel \neg F$		where 0 corresponds to value false 1 corresponds to value true				
0	1					
1	0					

F_1	F_2	$F_1 \wedge F_2$	$F_1 \vee F_2$	$F_1 \rightarrow F_2$	$F_1 \leftrightarrow F_2$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

Example:

formula $F : (P \wedge Q) \rightarrow (\top \vee \neg Q)$
atoms: $P, Q, \top$
literals: $P, Q, \top, \neg Q$
subformulae: $P, Q, \top, \neg Q, P \wedge Q, \top \vee \neg Q, F$
abbreviation
 $F : P \wedge Q \rightarrow \top \vee \neg Q$

Example:

$F : P \wedge Q \rightarrow P \vee \neg Q$
 $I : \{P \mapsto \text{true}, Q \mapsto \text{false}\}$ i.e., $I[P] = \text{true}, I[Q] = \text{false}$

P	Q	$\neg Q$	$P \wedge Q$	$P \vee \neg Q$	F
1	0	1	0	1	1

1 = true 0 = false

F evaluates to true under I ; i.e., $I[F] = \text{true}$.

Inductive Definition of PL's Semantics

$I \models F$ if F evaluates to true under I
 $I \not\models F$ false

Base Case:

$I \models \top$ $I \not\models \perp$

$I \models P$ iff $I[P] = \text{true}$; i.e., P is true under I

$I \not\models P$ iff $I[P] = \text{false}$

Inductive Case:

$I \models \neg F$ iff $I \not\models F$

$I \models F_1 \wedge F_2$ iff $I \models F_1$ and $I \models F_2$

$I \models F_1 \vee F_2$ iff $I \models F_1$ or $I \models F_2$ (or both)

$I \models F_1 \rightarrow F_2$ iff $I \models F_1$ implies $I \models F_2$

$I \models F_1 \leftrightarrow F_2$ iff, $I \models F_1$ and $I \models F_2$,
 or $I \not\models F_1$ and $I \not\models F_2$

Note:

$I \models F_1 \rightarrow F_2$ iff $I \not\models F_1$ or $I \models F_2$.

$I \not\models F_1 \rightarrow F_2$ iff $I \models F_1$ and $I \not\models F_2$.

$I \not\models F_1 \vee F_2$ iff $I \not\models F_1$ and $I \not\models F_2$.

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Example of Inductive Reasoning:

$F: P \wedge Q \rightarrow P \vee \neg Q$

$I: \{P \mapsto \text{true}, Q \mapsto \text{false}\}$

1. $I \models P$ since $I[P] = \text{true}$

2. $I \not\models Q$ since $I[Q] = \text{false}$

3. $I \models \neg Q$ by 2 and $\neg$

4. $I \models P \wedge Q$ by 2 and $\wedge$

5. $I \models P \vee \neg Q$ by 1 and $\vee$

6. $I \models F$ by 4 and $\rightarrow$ Why?

Thus, F is true under I .

Note: steps 1, 3, and 5 are nonessential.

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Satisfiability and Validity

F satisfiable iff there exists an interpretation I such that $I \models F$.

F valid iff for all interpretations I , $I \models F$.

F is valid iff $\neg F$ is unsatisfiable

Goal: devise an algorithm to decide validity or unsatisfiability of formula F .

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Method 1: Truth Tables

Example $F: P \wedge Q \rightarrow P \vee \neg Q$

P	Q	$P \wedge Q$	$\neg Q$	$P \vee \neg Q$	F
0	0	0	1	1	1
0	1	0	0	0	1
1	0	0	1	1	1
1	1	1	0	1	1

Thus F is valid.

Example $F: P \vee Q \rightarrow P \wedge Q$

P	Q	$P \vee Q$	$P \wedge Q$	F
0	0	0	0	1
0	1	1	0	0
1	0	1	0	0
1	1	1	1	1

← satisfying I

← falsifying I

Thus F is satisfiable, but invalid.

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- ▶ Assume F is not valid and I a falsifying interpretation:
 $I \not\models F$
- ▶ Apply proof rules.
- ▶ If no contradiction reached and no more rules applicable,
 F is invalid.
- ▶ If in every branch of proof a contradiction reached,
 F is valid.

$$\frac{I \models \neg F}{I \not\models F}$$
$$\frac{I \models F \wedge G}{I \models F \quad I \models G} \leftarrow \text{and}$$
$$\frac{I \models F \vee G}{I \models F \mid I \models G}$$

$$\frac{I \not\models \neg F}{I \models F}$$
$$\frac{I \not\models F \wedge G}{I \not\models F \mid I \not\models G} \leftarrow \text{or}$$
$$\frac{I \not\models F \vee G}{I \not\models F}$$

Example: Prove

$F : P \wedge Q \rightarrow P \vee \neg Q$ is valid.

Let's assume that F is not valid and that I is a falsifying interpretation.

$$\frac{I \models F \rightarrow G}{I \not\models F \mid I \models G}$$
$$\frac{I \models F \leftrightarrow G}{I \models F \wedge G \mid I \not\models F \vee G}$$
$$\frac{I \models F}{I \not\models F} \rightarrow I \models \perp$$

$$\frac{I \not\models F \rightarrow G}{I \models F} \rightarrow I \not\models G$$
$$\frac{I \not\models F \leftrightarrow G}{I \models F \wedge \neg G \mid I \models \neg F \wedge G}$$

1. $I \not\models P \wedge Q \rightarrow P \vee \neg Q$

2. $I \models P \wedge Q$

3. $I \not\models P \vee \neg Q$

4. $I \models P$

5. $I \not\models P$

6. $I \models \perp$
- assumption

1 and $\rightarrow$

1 and $\rightarrow$

2 and $\wedge$

3 and $\vee$

4 and 5 are contradictory

Thus F is valid.

Example: Prove

$F: (P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R)$ is valid.

Let's assume that F is not valid.

- | | | |
|----|--|---------------------|
| 1. | $I \not\models F$ | assumption |
| 2. | $I \models (P \rightarrow Q) \wedge (Q \rightarrow R)$ | 1 and $\rightarrow$ |
| 3. | $I \not\models P \rightarrow R$ | 1 and $\rightarrow$ |
| 4. | $I \models P$ | 3 and $\rightarrow$ |
| 5. | $I \not\models R$ | 3 and $\rightarrow$ |
| 6. | $I \models P \rightarrow Q$ | 2 and $\wedge$ |
| 7. | $I \models Q \rightarrow R$ | 2 and $\wedge$ |

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Example 3: Is

$F: P \vee Q \rightarrow P \wedge Q$

valid? Assume F is not valid:

- | | | |
|------|---|------------------------|
| 1. | $I \not\models P \vee Q \rightarrow P \wedge Q$ | assumption |
| 2. | $I \models P \vee Q$ | 1 and $\rightarrow$ |
| 3. | $I \not\models P \wedge Q$ | 1 and $\rightarrow$ |
| 4a. | $I \models P$ | 2, $\vee$ (case a) |
| 5aa. | $I \not\models P$ | 3, $\vee$ (subcase aa) |
| 6aa. | $I \models \perp$ | 4a, 5aa |
| 5ab. | $I \not\models Q$ | 3, $\vee$ (subcase ab) |
| 6ab. | ? | |
| 5a. | ? | |

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- | | | |
|-------|-----------------------------|----------------------------------|
| 6. | $I \models P \rightarrow Q$ | 2 and $\wedge$ |
| 7. | $I \models Q \rightarrow R$ | 2 and $\wedge$ |
| 8a. | $I \not\models P$ | 6 and $\rightarrow$ (case a) |
| 9a. | $I \models \perp$ | 4 and 8 |
| 8b. | $I \models Q$ | 6 and $\rightarrow$ (case b) |
| 9ba. | $I \not\models Q$ | 7 and $\rightarrow$ (subcase ba) |
| 10ba. | $I \models \perp$ | 8b and 9ba |
| 9bb. | $I \models R$ | 7 and $\rightarrow$ (subcase bb) |
| 10bb. | $I \models \perp$ | 5 and 9bb |
| 9b. | $I \models \perp$ | 10ba and 10bb |
| 8. | $I \models \perp$ | 9a and 9b |

Our assumption is contradictory in all cases, so F is valid.

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- | | | |
|------|-------------------|------------------------|
| 4b. | $I \models Q$ | 2, $\vee$ (case b) |
| 5ba. | $I \not\models P$ | 3, $\vee$ (subcase ba) |
| 6ba. | ? | |
| 5bb. | $I \not\models Q$ | 3, $\vee$ (subcase bb) |
| 6bb. | $I \models \perp$ | 4b, 5bb |
| 5b. | ? | |
| 5. | ? | |

We cannot derive a contradiction in both cases (4a and 4b), so we cannot prove that F is valid. To demonstrate that F is not valid, however, we must find a falsifying interpretation (here are two):

$I_1: \{P \mapsto \text{true}, Q \mapsto \text{false}\}$ $I_2: \{Q \mapsto \text{true}, P \mapsto \text{false}\}$

Note: we have to derive a contradiction in all cases for F to be valid!

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Equivalence

F_1 and F_2 are equivalent ($F_1 \Leftrightarrow F_2$)
iff for all interpretations $I, I \models F_1 \leftrightarrow F_2$

To prove $F_1 \Leftrightarrow F_2$, show $F_1 \leftrightarrow F_2$ is valid, that is,
both $F_1 \rightarrow F_2$ and $F_2 \rightarrow F_1$ are valid.

F_1 entails F_2 ($F_1 \Rightarrow F_2$)
iff for all interpretations $I, I \models F_1 \rightarrow F_2$

Note: $F_1 \Leftrightarrow F_2$ and $F_1 \Rightarrow F_2$ are not formulae!!

Example: Show

$$P \rightarrow Q \Leftrightarrow \neg P \vee Q$$

i.e.

$$F : (P \rightarrow Q) \leftrightarrow (\neg P \vee Q) \text{ is valid.}$$

Assume F is not valid, then we have two cases:

Case a: $I \not\models \neg P \vee Q,$
 $I \models P \rightarrow Q$

Case b: $I \models \neg P \vee Q,$
 $I \not\models P \rightarrow Q$

Derive contradictions in both cases.

Normal Forms

1. Negation Normal Form (NNF)

$\neg, \wedge, \vee$ are the only boolean connectives allowed.
Negations may occur only in literals of the form $\neg P$.
To transform F into equivalent F' in NNF, apply the following
template equivalences recursively (and left-to-right):

$$\begin{aligned} \neg\neg F_1 &\Leftrightarrow F_1 & \neg\top &\Leftrightarrow \perp & \neg\perp &\Leftrightarrow \top \\ \neg(F_1 \wedge F_2) &\Leftrightarrow \neg F_1 \vee \neg F_2 \\ \neg(F_1 \vee F_2) &\Leftrightarrow \neg F_1 \wedge \neg F_2 \\ F_1 \rightarrow F_2 &\Leftrightarrow \neg F_1 \vee F_2 \\ F_1 \leftrightarrow F_2 &\Leftrightarrow (F_1 \rightarrow F_2) \wedge (F_2 \rightarrow F_1) \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{De Morgan's Law}$$

“Complete” syntactic restriction: every F has an equivalent
 F' in NNF.

Example: Convert

$$F : \neg(P \rightarrow \neg(P \wedge Q))$$

to NNF.

$$\begin{aligned} F' : \neg(\neg P \vee \neg(P \wedge Q)) &\rightarrow \\ F'' : \neg\neg P \wedge \neg\neg(P \wedge Q) &\text{De Morgan's Law} \\ F''' : P \wedge P \wedge Q &\neg\neg \end{aligned}$$

F''' is equivalent to F ($F''' \Leftrightarrow F$) and is in NNF.

2. Disjunctive Normal Form (DNF)

Disjunction of conjunctions of literals

$$\bigvee_i \bigwedge_j \ell_{i,j} \quad \text{for literals } \ell_{i,j}$$

To convert F into equivalent F' in DNF,
transform F into NNF and then
use the following template equivalences (left-to-right):

$$\left. \begin{aligned} (F_1 \vee F_2) \wedge F_3 &\Leftrightarrow (F_1 \wedge F_3) \vee (F_2 \wedge F_3) \\ F_1 \wedge (F_2 \vee F_3) &\Leftrightarrow (F_1 \wedge F_2) \vee (F_1 \wedge F_3) \end{aligned} \right\} \text{dist}$$

Note: formulae can grow exponentially as the distributivity laws are applied.

3. Conjunctive Normal Form (CNF)

Conjunction of disjunctions of literals

$$\bigwedge_i \bigvee_j \ell_{i,j} \quad \text{for literals } \ell_{i,j}$$

To convert F into equivalent F' in CNF,
transform F into NNF and then
use the following template equivalences (left-to-right):

$$\left. \begin{aligned} (F_1 \wedge F_2) \vee F_3 &\Leftrightarrow (F_1 \vee F_3) \wedge (F_2 \vee F_3) \\ F_1 \vee (F_2 \wedge F_3) &\Leftrightarrow (F_1 \vee F_2) \wedge (F_1 \vee F_3) \end{aligned} \right\}$$

A disjunction of literals is called a clause.

Example: Convert

$$F : (Q_1 \vee \neg Q_2) \wedge (\neg R_1 \rightarrow R_2)$$

into equivalent DNF

$$F' : (Q_1 \vee Q_2) \wedge (R_1 \vee R_2) \quad \text{in NNF}$$

$$F'' : (Q_1 \wedge (R_1 \vee R_2)) \vee (Q_2 \wedge (R_1 \vee R_2)) \quad \text{dist}$$

$$F''' : (Q_1 \wedge R_1) \vee (Q_1 \wedge R_2) \vee (Q_2 \wedge R_1) \vee (Q_2 \wedge R_2) \quad \text{dist}$$

F''' is equivalent to F ($F''' \Leftrightarrow F$) and is in DNF.

Example: Convert

$$F : P \leftrightarrow (Q \rightarrow R)$$

to an equivalent formula F' in CNF.

First get rid of $\leftrightarrow$:

$$F_1 : (P \rightarrow (Q \rightarrow R)) \wedge ((Q \rightarrow R) \rightarrow P)$$

Now replace $\rightarrow$ with $\vee$:

$$F_2 : (\neg P \vee \neg(Q \vee R)) \wedge (\neg(\neg Q \vee R) \vee P)$$

Drop unnecessary parentheses and apply De Morgan's Law:

$$F_3 : (\neg P \vee \neg Q \vee \neg R) \wedge ((\neg\neg Q \wedge \neg R) \vee P)$$

Simplify double negation (now in NNF):

$$F_4 : (\neg P \vee \neg Q \vee \neg R) \wedge ((Q \wedge \neg R) \vee P)$$

Distribute disjunction over conjunction (now in CNF):

$$F' : (\neg P \vee \neg Q \vee \neg R) \wedge (Q \vee P) \wedge (\neg R \vee P)$$

Equisatisfiability

Definition

F and F' are *equisatisfiable*, iff

F is satisfiable if and only if F' is satisfiable

Every formula is equisatisfiable to either $\top$ or $\perp$.

Goal: Decide satisfiability of PL formula F

Step 1: Convert F to equisatisfiable formula F' in CNF

Step 2: Decide satisfiability of formula F' in CNF

Step 1: Convert F to equisatisfiable formula F' in CNF I

There is an *efficient conversion* of F to F' where

- ▶ F' is in CNF and
- ▶ F and F' are equisatisfiable

Note: efficient means polynomial in the size of F .

Basic Idea:

- ▶ Introduce a new variable P_G for every subformula G of F , unless G is already an atom.

Step 1: Convert F to equisatisfiable formula F' in CNF II

Step 1: Convert F to equisatisfiable formula F' in CNF III

- ▶ For each subformula

$$G : G_1 \circ G_2,$$

produce a small formula

$$P_G \leftrightarrow P_{G_1} \circ P_{G_2}.$$

Here $\circ$ denotes an arbitrary connective ($\neg, \vee, \wedge, \rightarrow, \leftrightarrow$); if the connective is $\neg$, G_1 should be ignored.

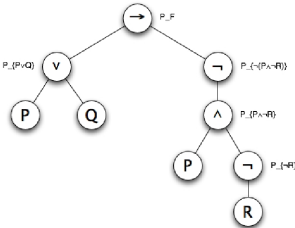


Figure: Parse tree for $F : P \vee Q \rightarrow \neg(P \wedge \neg R)$

Step 1: Convert F to equisatisfiable formula F' in CNF IV

Example: CNF I

- Convert each of these (small) formulae separately to an equivalent CNF formula

$$\text{CNF}(P_G \leftrightarrow P_{G_1} \circ P_{G_2}) .$$

Let S_F be the set of all non-atom subformulae G of F (including F itself). The formula

$$P_F \wedge \bigwedge_{G \in S_F} \text{CNF}(P_G \leftrightarrow P_{G_1} \circ P_{G_2})$$

is equisatisfiable to F . (Why?)

The number of subformulae is linear in the size of F .
The time to convert one small formula is constant!

Convert

$$F : P \vee Q \rightarrow P \wedge \neg R$$

to an equisatisfiable formula in CNF.

Introduce new variables: $P_F, P_{P \vee Q}, P_{P \wedge \neg R}, P_{\neg R}$.

Create new formulae and convert them to equivalent formulae in CNF separately:

- $F_1 = \text{CNF}(P_F \leftrightarrow (P_{P \vee Q} \rightarrow P_{P \wedge \neg R})):$

$$(\neg P_F \vee \neg P_{P \vee Q} \vee P_{P \wedge \neg R}) \wedge (P_F \vee P_{P \vee Q}) \wedge (P_F \vee \neg P_{P \wedge \neg R})$$

- $F_2 = \text{CNF}(P_{P \vee Q} \leftrightarrow P \vee Q):$

$$(\neg P_{P \vee Q} \vee P \vee Q) \wedge (P_{P \vee Q} \vee \neg P) \wedge (P_{P \vee Q} \vee \neg Q)$$

Example: CNF II

- $F_3 = \text{CNF}(P_{P \wedge \neg R} \leftrightarrow P \wedge \neg R):$

$$(\neg P_{P \wedge \neg R} \vee P) \wedge (\neg P_{P \wedge \neg R} \vee \neg R) \wedge (P_{P \wedge \neg R} \vee \neg P \vee \neg \neg R)$$

- $F_4 = \text{CNF}(P_{\neg R} \leftrightarrow \neg R):$

$$(\neg P_{\neg R} \vee \neg R) \wedge (P_{\neg R} \vee R)$$

$P_F \wedge F_1 \wedge F_2 \wedge F_3 \wedge F_4$ is in CNF and equisatisfiable to F .

Step 2: Decide the satisfiability of PL formula F' in CNF

Boolean Constraint Propagation (BCP)

If a clause contains one literal ℓ ,

Set ℓ to $\top$:

$$\dots \wedge \overset{\top}{\ell} \wedge \dots$$

Remove all clauses containing ℓ :

$$\dots \wedge (\dots \vee \overset{\top}{\ell} \vee \dots) \wedge \dots$$

Remove $\neg \ell$ in all clauses:

$$\dots \wedge (\dots \vee \neg \overset{\top}{\ell} \vee \dots) \wedge \dots$$

based on the unit resolution

$$\frac{\ell \quad \neg \ell \vee C}{C} \leftarrow \text{clause}$$

Pure Literal Propagation (PLP)

If P occurs only positive (without negation), set it to $\top$.

If P occurs only negative set it to $\perp$.

Then do the simplifications as in Boolean Constraint Propagation

Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Decides the satisfiability of PL formulae in CNF

Decision Procedure DPLL: Given F in CNF

```
let rec DPLL  $F$  =  
  let  $F' = \text{BCP } F$  in  
  let  $F'' = \text{PLP } F'$  in  
  if  $F'' = \top$  then true  
  else if  $F'' = \perp$  then false  
  else  
    let  $P = \text{CHOOSE vars}(F'')$  in  
    (DPLL  $F''\{P \mapsto \top\}$ )  $\vee$  (DPLL  $F''\{P \mapsto \perp\}$ )
```

Example I

Consider

$$F : (\neg P \vee Q \vee R) \wedge (\neg Q \vee R) \wedge (\neg Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R).$$

Branching on Q

On the first branch, we have

$$F\{Q \mapsto \top\} : (R) \wedge (\neg R) \wedge (P \vee \neg R).$$

By unit resolution,

$$\frac{R \quad (\neg R)}{\perp}$$

so $F\{Q \mapsto \top\} = \perp \Rightarrow \text{false}$.

Simplification

Simplify according to the template equivalences (left-to-right)
[exercise 1.2]

$\neg \perp \Leftrightarrow \top$	$\neg \top \Leftrightarrow \perp$	$\neg \neg F \Leftrightarrow F$
$F \wedge \top \Leftrightarrow F$	$F \wedge \perp \Leftrightarrow \perp$	$\dots$
$F \vee \top \Leftrightarrow \top$	$F \vee \perp \Leftrightarrow F$	$\dots$

Example II

Recall

$$F : (\neg P \vee Q \vee R) \wedge (\neg Q \vee R) \wedge (\neg Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R).$$

On the other branch, we have

$$F\{Q \mapsto \perp\} : (\neg P \vee R).$$

Furthermore, by PLP,

$$F\{Q \mapsto \perp, R \mapsto \top, P \mapsto \perp\} = \top \Rightarrow \text{true}$$

Thus F is satisfiable with satisfying interpretation

$$I : \{P \mapsto \text{false}, Q \mapsto \text{false}, R \mapsto \text{true}\}.$$

Example

$$F : (\neg P \vee Q \vee R) \wedge (\neg Q \vee R) \wedge (\neg Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R)$$

