

CS156: The Calculus of Computation

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Chapter 7: Quantified Linear Arithmetic

Quantifier Elimination (QE)

Algorithm for elimination of all quantifiers of formula F until quantifier-free formula (qff) G that is equivalent to F remains

Note: Could be enough if F is equisatisfiable to G , that is F is satisfiable iff G is satisfiable

A theory T admits quantifier elimination iff

there is an algorithm that given Σ -formula F returns a quantifier-free Σ -formula G that is T -equivalent to F .

Example: $\exists x. 2x = y$

For $\Sigma_{\mathbb{Q}}$ -formula

$$F : \exists x. 2x = y,$$

quantifier-free $T_{\mathbb{Q}}$ -equivalent $\Sigma_{\mathbb{Q}}$ -formula is

$$G : \top$$

For $\Sigma_{\mathbb{Z}}$ -formula

$$F : \exists x. 2x = y,$$

there is no quantifier-free $T_{\mathbb{Z}}$ -equivalent $\Sigma_{\mathbb{Z}}$ -formula.

Let $\widehat{T}_{\mathbb{Z}}$ be $T_{\mathbb{Z}}$ with divisibility predicates $|$.

For $\widehat{\Sigma}_{\mathbb{Z}}$ -formula

$$F : \exists x. 2x = y,$$

a quantifier-free $\widehat{T}_{\mathbb{Z}}$ -equivalent $\widehat{\Sigma}_{\mathbb{Z}}$ -formula is

$$G : 2 \mid y.$$

About QE Algorithm

In developing a QE algorithm for theory T , we need only consider formulae of the form

$$\exists x. F$$

for quantifier-free F .

Example: For Σ -formula

$$G_1 : \exists x. \forall y. \underbrace{\exists z. F_1[x, y, z]}_{F_2[x, y]}$$

$$G_2 : \exists x. \forall y. F_2[x, y]$$

$$G_3 : \exists x. \underbrace{\neg \exists y. \neg F_2[x, y]}_{F_3[x]}$$

$$G_4 : \underbrace{\exists x. \neg F_3[x]}_{F_4}$$

$$G_5 : F_4$$

G_5 is quantifier-free and T -equivalent to G_1

Quantifier Elimination for $T_{\mathbb{Z}}$

$\Sigma_{\mathbb{Z}} : \{\dots, -2, -1, 0, 1, 2, \dots, -3\cdot, -2\cdot, 2\cdot, 3\cdot, \dots, +, -, =, <\}$

Lemma:

Given quantifier-free $\Sigma_{\mathbb{Z}}$ -formula $F[y]$ s.t. $\text{free}(F[y]) = \{y\}$.

S represents the set of integers

$$S : \{n \in \mathbb{Z} : F[n] \text{ is } T_{\mathbb{Z}}\text{-valid}\} .$$

Either $S \cap \mathbb{Z}^+$ or $\mathbb{Z}^+ \setminus S$ is finite.

Note: $\mathbb{Z}^+$ is the set of positive integers.

Example: $\Sigma_{\mathbb{Z}}$ -formula $F[y] : \exists x. 2x = y$

S : even integers

$S \cap \mathbb{Z}^+$: positive even integers — infinite

$\mathbb{Z}^+ \setminus S$: positive odd integers — infinite

Therefore, by the lemma, there is no quantifier-free $T_{\mathbb{Z}}$ -formula that is $T_{\mathbb{Z}}$ -equivalent to $F[y]$.

Thus, $T_{\mathbb{Z}}$ does not admit QE.

Augmented theory $\widehat{T}_{\mathbb{Z}}$

$\widehat{\Sigma}_{\mathbb{Z}}$: $\Sigma_{\mathbb{Z}}$ with countable number of unary divisibility predicates
 $k \mid \cdot$ for $k \in \mathbb{Z}^+$

Intended interpretations:

$k \mid x$ holds iff k divides x without any remainder

Example:

$$x > 1 \wedge y > 1 \wedge 2 \mid x + y$$

is satisfiable (choose $x = 2, y = 2$).

$$\neg(2 \mid x) \wedge 4 \mid x$$

is not satisfiable.

Axioms of $\widehat{T}_{\mathbb{Z}}$: axioms of $T_{\mathbb{Z}}$ with additional countable set of axioms

$$\forall x. k \mid x \leftrightarrow \exists y. x = ky \quad \text{for } k \in \mathbb{Z}^+$$

$\widehat{T}_{\mathbb{Z}}$ admits QE (Cooper's method)

Algorithm: Given $\widehat{\Sigma}_{\mathbb{Z}}$ -formula

$$\exists x. F[x] ,$$

where F is quantifier-free, construct quantifier-free $\widehat{\Sigma}_{\mathbb{Z}}$ -formula that is equivalent to $\exists x. F[x]$.

1. Put $F[x]$ into Negation Normal Form (NNF).
2. Normalize literals: $s < t$, $k|t$, or $\neg(k|t)$.
3. Put x in $s < t$ on one side: $hx < t$ or $s < hx$.
4. Replace hx with x' without a factor.
5. Replace $F[x']$ by $\bigvee F[j]$ for finitely many j .

Cooper's Method: Step 1

Put $F[x]$ in Negation Normal Form (NNF) $F_1[x]$, so that $\exists x. F_1[x]$

- ▶ has negations only in literals (only $\wedge, \vee$)
- ▶ is $\widehat{T}_{\mathbb{Z}}$ -equivalent to $\exists x. F[x]$

Example:

$$\exists x. \neg(x - 6 < z - x \wedge 4 \mid 5x + 1 \rightarrow 3x < y)$$

is equivalent to

$$\exists x. x - 6 < z - x \wedge 4 \mid 5x + 1 \wedge \neg(3x < y)$$

Note:

$$\neg(A \wedge B \rightarrow C) \Leftrightarrow (A \wedge B \wedge \neg C)$$

Cooper's Method: Step 2

Replace (left to right)

$$s = t \Leftrightarrow s < t + 1 \wedge t < s + 1$$

$$\neg(s = t) \Leftrightarrow s < t \vee t < s$$

$$\neg(s < t) \Leftrightarrow t < s + 1$$

The output $\exists x. F_2[x]$ contains only literals of form

$$s < t, \quad k \mid t, \quad \text{or} \quad \neg(k \mid t),$$

where s, t are $\widehat{T}_{\mathbb{Z}}$ -terms and $k \in \mathbb{Z}^+$.

Example:

$$\neg(x < y) \wedge \neg(x = y + 3)$$

$$\Downarrow$$

$$y < x + 1 \wedge (x < y + 3 \vee y + 3 < x)$$

Cooper's Method: Step 3

Collect terms containing x so that literals have the form

$$hx < t, \quad t < hx, \quad k \mid hx + t, \quad \text{or} \quad \neg(k \mid hx + t),$$

where t is a term (does not contain x) and $h, k \in \mathbb{Z}^+$. The output is the formula $\exists x. F_3[x]$, which is $\widehat{T}_{\mathbb{Z}}$ -equivalent to $\exists x. F[x]$.

Example:

$$x + x + y < z + 3z + 2y - 4x$$

$$\Downarrow$$

$$6x < 4z + y$$

$$5 \mid -7x + t$$

$$\Downarrow$$

$$5 \mid 7x - t$$

Cooper's Method: Step 4 I

Let

$$\delta' = \text{lcm}\{h : h \text{ is a coefficient of } x \text{ in } F_3[x]\},$$

where lcm is the least common multiple. Multiply atoms in $F_3[x]$ by constants so that δ' is the coefficient of x everywhere:

$$hx < t \Leftrightarrow \delta'x < h't \quad \text{where } h'h = \delta'$$

$$t < hx \Leftrightarrow h't < \delta'x \quad \text{where } h'h = \delta'$$

$$k \mid hx + t \Leftrightarrow h'k \mid \delta'x + h't \quad \text{where } h'h = \delta'$$

$$\neg(k \mid hx + t) \Leftrightarrow \neg(h'k \mid \delta'x + h't) \quad \text{where } h'h = \delta'$$

The result $\exists x. F'_3[x]$, in which all occurrences of x in $F'_3[x]$ are in terms $\delta'x$.

Replace $\delta'x$ terms in F'_3 with a fresh variable x' to form

$$F''_3 : F_3\{\delta'x \mapsto x'\}$$

Cooper's Method: Step 4 II

Finally, construct

$$\exists x'. \underbrace{F_3''[x'] \wedge \delta' \mid x'}_{F_4[x']}$$

$\exists x'. F_4[x']$ is equivalent to $\exists x. F[x]$ and each literal of $F_4[x']$ has one of the forms:

(A) $x' < t$

(B) $t < x'$

(C) $k \mid x' + t$

(D) $\neg(k \mid x' + t)$

where t is a term that does not contain x , and $k \in \mathbb{Z}^+$.

Cooper's Method: Step 4 III

Example: $\widehat{T}_{\mathbb{Z}}$ -formula

$$\exists x. \underbrace{3x + 1 > y \wedge 2x - 6 < z \wedge 4 \mid 5x + 1}_{F[x]}$$

After step 3:

$$\exists x. \underbrace{2x < z + 6 \wedge y - 1 < 3x \wedge 4 \mid 5x + 1}_{F_3[x]}$$

Collecting coefficients of x (step 4):

$$\delta' = \text{lcm}(2, 3, 5) = 30$$

Multiply when necessary:

$$\exists x. 30x < 15z + 90 \wedge 10y - 10 < 30x \wedge 24 \mid 30x + 6$$

Cooper's Method: Step 4 IV

Multiply when necessary:

$$\exists x. 30x < 15z + 90 \wedge 10y - 10 < 30x \wedge 24 \mid 30x + 6$$

Replacing $30x$ with fresh x' and adding divisibility conjunct:

$$\exists x'. \underbrace{x' < 15z + 90 \wedge 10y - 10 < x' \wedge 24 \mid x' + 6 \wedge 30 \mid x'}_{F_4[x']}$$

$\exists x'. F_4[x']$ is equivalent to $\exists x. F[x]$.

Cooper's Method: Step 5

Construct left infinite projection $F_{-\infty}[x']$ of $F_4[x']$ by

(A) replacing literals $x' < t$ by $\top$

(B) replacing literals $t < x'$ by $\perp$

Idea: very small numbers satisfy (A) literals but not (B) literals

Let

$$\delta = \text{lcm} \left\{ \begin{array}{l} k \text{ of (C) literals } k \mid x' + t \\ k \text{ of (D) literals } \neg(k \mid x' + t) \end{array} \right\}$$

and B be the set of terms t appearing in (B) literals of $F_4[x']$.

Construct

$$F_5 : \bigvee_{j=1}^{\delta} F_{-\infty}[j] \vee \bigvee_{j=1}^{\delta} \bigvee_{t \in B} F_4[t + j] .$$

F_5 is quantifier-free and $\widehat{T}_{\mathbb{Z}}$ -equivalent to $\exists x. F[x]$.

Intuition of Step 5 I

Property (Periodicity)

if $m \mid \delta$

then $m \mid n$ iff $m \mid n + \lambda\delta$ for all $\lambda \in \mathbb{Z}$

That is, $m \mid \cdot$ cannot distinguish between $m \mid n$ and $m \mid n + \lambda\delta$.

By the choice of δ (lcm of the k 's) — no $\mid$ literal in F_5 can distinguish between n and $n + \lambda\delta$, for any $\lambda \in \mathbb{Z}$.

$$F_5 : \bigvee_{j=1}^{\delta} F_{-\infty}[j] \vee \bigvee_{j=1}^{\delta} \bigvee_{t \in B} F_4[t + j]$$

Intuition of Step 5 II

- ▶ left disjunct $\bigvee_{j=1}^{\delta} F_{-\infty}[j]$:

Contains only | literals

Asserts: no least $n \in \mathbb{Z}$ s.t. $F_4[n]$.

For if there exists n satisfying $F_{-\infty}$,
then every $n - \lambda\delta$, for $\lambda \in \mathbb{Z}^+$, also satisfies $F_{-\infty}$

- ▶ right disjunct $\bigvee_{j=1}^{\delta} \bigvee_{t \in B} F_4[t + j]$:

Asserts: There is least $n \in \mathbb{Z}$ s.t. $F_4[n]$.

For let $t^* = \{\text{largest } t \mid t < x' \text{ in } (B)\}$.

If $n \in \mathbb{Z}$ is s.t. $F_4[n]$, then

$$\exists j(1 \leq j \leq \delta). t^* + j \leq n \wedge F_4[t^* + j]$$

In other words,

if there is a solution,

then one must appear in δ interval to the right of t^*

Example of Step 5 I

$$\exists x. \underbrace{3x + 1 > y \wedge 2x - 6 < z \wedge 4 \mid 5x + 1}_{F[x]}$$

$\Downarrow$

$$\exists x'. \underbrace{x' < 15z + 90 \wedge 10y - 10 < x' \wedge 24 \mid x' + 6 \wedge 30 \mid x'}_{F_4[x']}$$

By step 5,

$$F_{-\infty}[x'] : \top \wedge \perp \wedge 24 \mid x' + 6 \wedge 30 \mid x',$$

which simplifies to $\perp$.

Example of Step 5 II

Compute

$$\delta = \text{lcm}\{24, 30\} = 120 \quad \text{and} \quad B = \{10y - 10\}.$$

Then replacing x' by $10y - 10 + j$ in $F_4[x']$ produces

$$F_5 : \bigvee_{j=1}^{120} \left[\begin{array}{l} 10y - 10 + j < 15z + 90 \wedge 10y - 10 < 10y - 10 + j \\ \wedge 24 \mid 10y - 10 + j + 6 \wedge 30 \mid 10y - 10 + j \end{array} \right]$$

which simplifies to

$$F_5 : \bigvee_{j=1}^{120} \left[\begin{array}{l} 10y + j < 15z + 100 \wedge 0 \leq j \\ \wedge 24 \mid 10y + j - 4 \wedge 30 \mid 10y - 10 + j \end{array} \right].$$

F_5 is quantifier-free and $\widehat{T}_{\mathbb{Z}}$ -equivalent to $\exists x. F[x]$.

Cooper's Method: Example I

$$\underbrace{\exists x. (3x + 1 < 10 \vee 7x - 6 > 7) \wedge 2 \mid x}_{F[x]}$$

Isolate x terms

$$\exists x. (3x < 9 \vee 13 < 7x) \wedge 2 \mid x ,$$

so

$$\delta' = \text{lcm}\{3, 7\} = 21 .$$

After multiplying coefficients by proper constants,

$$\exists x. (21x < 63 \vee 39 < 21x) \wedge 42 \mid 21x ,$$

we replace $21x$ by x' :

$$\exists x'. \underbrace{(x' < 63 \vee 39 < x') \wedge 42 \mid x' \wedge 21 \mid x'}_{F_4[x']} .$$

Cooper's Method: Example II

Then

$$F_{-\infty}[x'] : (\top \vee \perp) \wedge 42 \mid x' \wedge 21 \mid x' ,$$

or, simplifying,

$$F_{-\infty}[x'] : 42 \mid x' \wedge 21 \mid x' .$$

Finally,

$$\delta = \text{lcm}\{21, 42\} = 42 \quad \text{and} \quad B = \{39\} ,$$

so F_5 :

$$\bigvee_{j=1}^{42} (42 \mid j \wedge 21 \mid j) \vee \bigvee_{j=1}^{42} ((39 + j < 63 \vee 39 < 39 + j) \wedge 42 \mid 39 + j \wedge 21 \mid 39 + j) .$$

Since $42 \mid 42$ and $21 \mid 42$, the left main disjunct simplifies to $\top$, so that F_5 is $\widehat{T}_{\mathbb{Z}}$ -equivalent to $\top$. Thus, $\exists x. F[x]$ is $\widehat{T}_{\mathbb{Z}}$ -valid.

Cooper's Method: Example I

$$\exists x. \underbrace{2x = y}_{F[x]}$$

Rewriting

$$\exists x. \underbrace{2x < y + 1 \wedge y - 1 < 2x}_{F_3[x]}$$

Then

$$\delta' = \text{lcm}\{2, 2\} = 2 ,$$

so by Step 4

$$\exists x'. \underbrace{x' < y + 1 \wedge y - 1 < x' \wedge 2 \mid x'}_{F_4[x']}$$

$F_{-\infty}$ produces $\perp$.

Cooper's Method: Example II

However,

$$\delta = \text{lcm}\{2\} = 2 \quad \text{and} \quad B = \{y - 1\} ,$$

so

$$F_5 : \bigvee_{j=1}^2 (y - 1 + j < y + 1 \wedge y - 1 < y - 1 + j \wedge 2 \mid y - 1 + j)$$

Simplifying,

$$F_5 : \bigvee_{j=1}^2 (j < 2 \wedge 0 < j \wedge 2 \mid y - 1 + j)$$

and then

$$F_5 : 2 \mid y ,$$

which is quantifier-free and $\widehat{T}_{\mathbb{Z}}$ -equivalent to $\exists x. F[x]$.

Improvement: Symmetric Elimination

In step 5, if there are fewer

(A) literals $x' < t$

than

(B) literals $t < x'$,

construct the right infinite projection $F_{+\infty}[x']$ from $F_4[x']$ by replacing

(A) literal $x' < t$ by $\perp$

than

(B) literal $t < x'$ by $\top$

Then right elimination.

$$F_5 : \bigvee_{j=1}^{\delta} F_{+\infty}[-j] \vee \bigvee_{j=1}^{\delta} \bigvee_{t \in A} F_4[t - j] .$$

Improvement: Eliminating Blocks of Quantifiers I

Given

$$\exists x_1. \dots \exists x_n. F[x_1, \dots, x_n]$$

where F quantifier-free.

Eliminating x_n (left elimination) produces

$$G_1 : \exists x_1. \dots \exists x_{n-1}. \bigvee_{j=1}^{\delta} F_{-\infty}[x_1, \dots, x_{n-1}, j] \vee \bigvee_{j=1}^{\delta} \bigvee_{t \in B} F_4[x_1, \dots, x_{n-1}, t + j]$$

which is equivalent to

$$G_2 : \bigvee_{j=1}^{\delta} \exists x_1. \dots \exists x_{n-1}. F_{-\infty}[x_1, \dots, x_{n-1}, j] \vee \bigvee_{j=1}^{\delta} \bigvee_{t \in B} \exists x_1. \dots \exists x_{n-1}. F_4[x_1, \dots, x_{n-1}, t + j]$$

Improvement: Eliminating Blocks of Quantifiers II

Treat j as a free variable and examine only $1 + |B|$ formulae

- ▶ $\exists x_1. \dots \exists x_{n-1}. F_{-\infty}[x_1, \dots, x_{n-1}, j]$
- ▶ $\exists x_1. \dots \exists x_{n-1}. F_4[x_1, \dots, x_{n-1}, t + j]$ for each $t \in B$

Example I

$$F : \exists y. \exists x. x < -2 \wedge 1 - 5y < x \wedge 1 + y < 13x$$

Since $\delta' = \text{lcm}\{1, 13\} = 13$

$$\exists y. \exists x. 13x < -26 \wedge 13 - 65y < 13x \wedge 1 + y < 13x$$

Then

$$\exists y. \exists x'. x' < -26 \wedge 13 - 65y < x' \wedge 1 + y < x' \wedge 13 \mid x'$$

There is one (A) literal $x' < \dots$ and two (B) literals $\dots < x'$, we use right elimination.

$$F_{+\infty} = \perp \quad \delta = \{13\} = 13 \quad A = \{-26\}$$

$$F' : \exists y. \bigvee_{j=1}^{13} \left[\begin{array}{l} -26 - j < -26 \wedge 13 - 65y < -26 - j \\ \wedge 1 + y < -26 - j \wedge 13 \mid -26 - j \end{array} \right]$$

Example II

Commute

$$G[j] : \bigvee_{j=1}^{13} \underbrace{\exists y. j > 0 \wedge 39 + j < 65y \wedge y < -27 - j \wedge 13 \mid -26 - j}_{H[j]}$$

Treating j as free variable (and removing $j > 0$), apply QE to

$$H[j] : \exists y. 39 + j < 65y \wedge y < -27 - j \wedge 13 \mid -26 - j$$

Simplify...

$$H'[j] : \bigvee_{k=1}^{65} (k < -1794 - 66j \wedge 13 \mid -26 - j \wedge 65 \mid 39 + j + k)$$

Replace $H[j]$ with $H'[j]$ in $G[j]$

Example III

$$F'' : \bigvee_{j=1}^{13} \bigvee_{k=1}^{65} (k < -1794 - 66j \wedge 13 \mid -26 - j \wedge 65 \mid 39 + j + k)$$

$\uparrow$
 $j = 13$

$\uparrow$
 $k = 13$

simplified to

$$13 < -1794 - 66 \cdot 13$$

$\perp$

This qff formula is $\widehat{T}_{\mathbb{Z}}$ -equivalent to F .

Quantifier Elimination over Rationals

$$\Sigma_{\mathbb{Q}} : \{0, 1, +, -, =, \geq\}$$

Recall: we use $>$ instead of $\geq$, as

$$x \geq y \Leftrightarrow x > y \vee x = y \quad x > y \Leftrightarrow x \geq y \wedge \neg(x = y) .$$

Ferrante & Rackoff's Method

Given a $\Sigma_{\mathbb{Q}}$ -formula $\exists x. F[x]$, where $F[x]$ is quantifier-free, generate quantifier-free formula F_4 (four steps) s.t.

$$F_4 \text{ is } \Sigma_{\mathbb{Q}}\text{-equivalent to } \exists x. F[x]$$

by

1. putting $F[x]$ in NNF,
2. replacing negated literals,
3. solving literals such that x appears isolated on one side, and
4. taking finite disjunction $\bigvee_t F[t]$.

Ferrante & Rackoff's Method: Steps 1 and 2

Step 1: Put $F[x]$ in NNF. The result is $\exists x. F_1[x]$.

Step 2: Replace literals (left to right)

$$\neg(s < t) \Leftrightarrow t < s \vee t = s$$

$$\neg(s = t) \Leftrightarrow t < s \vee t > s$$

The result $\exists x. F_2[x]$ does not contain negations.

Ferrante & Rackoff's Method: Step 3

Solve for x in each atom of $F_2[x]$, e.g.,

$$t_1 < cx + t_2 \quad \Rightarrow \quad \frac{t_1 - t_2}{c} < x$$

where $c \in \mathbb{Z} - \{0\}$.

All atoms in the result $\exists x. F_3[x]$ have form

(A) $x < t$

(B) $t < x$

(C) $x = t$

where t is a term that does not contain x .

Ferrante & Rackoff's Method: Step 4 I

Construct from $F_3[x]$

- ▶ left infinite projection $F_{-\infty}$ by replacing
 - (A) atoms $x < t$ by $\top$
 - (B) atoms $t < x$ by $\perp$
 - (C) atoms $x = t$ by $\perp$
- ▶ right infinite projection $F_{+\infty}$ by replacing
 - (A) atoms $x < t$ by $\perp$
 - (B) atoms $t < x$ by $\top$
 - (C) atoms $x = t$ by $\perp$

Let S be the set of t terms from (A), (B), (C) atoms.

Construct the final

$$F_4 : F_{-\infty} \vee F_{+\infty} \vee \bigvee_{s,t \in S} F_3 \left[\frac{s+t}{2} \right],$$

which is $T_{\mathbb{Q}}$ -equivalent to $\exists x. F[x]$.

Ferrante & Rackoff's Method: Step 4 II

- ▶ $F_{-\infty}$ captures the case when small $x \in \mathbb{Q}$ satisfy $F_3[x]$
- ▶ $F_{+\infty}$ captures the case when large $x \in \mathbb{Q}$ satisfy $F_3[x]$
- ▶ last disjunct: for $s, t \in S$
 - if $s \equiv t$, check whether $s \in S$ satisfies $F_3[s]$
 - if $s \not\equiv t$, in any $T_{\mathbb{Q}}$ -interpretation,
 - ▶ $|S| - 1$ pairs $s, t \in S$ are adjacent. For each such pair, (s, t) is an interval in which no other $s' \in S$ lies.
 - ▶ Since $\frac{s+t}{2}$ represents the whole interval (s, t) , simply check $F_3[\frac{s+t}{2}]$.

Ferrante & Rackoff's Method: Intuition

Step 4 says that four cases are possible:

1. There is a left open interval s.t. all elements satisfy $F(x)$.

$$\overline{\quad\quad\quad} \leftarrow)$$

2. There is a right open interval s.t. all elements satisfy $F(x)$.

$$\overline{\quad\quad\quad} (\longrightarrow)$$

3. Some term t satisfies $F(x)$.

$$\overline{\quad\quad\quad} \quad \dots \quad t \quad \dots$$

4. There is an open interval between two s, t terms such that every element satisfies $F(x)$.

$$\overline{\quad\quad\quad} (\longleftrightarrow)$$

$$\dots \quad s \quad \uparrow \quad t \quad \dots$$

$$\frac{s+t}{2}$$

Correctness of Step 4 I

Theorem

Let

$$F_4 : F_{-\infty} \vee F_{+\infty} \vee \bigvee_{s,t \in S} F_3 \left[\frac{s+t}{2} \right],$$

be the formula constructed from $\exists x. F_3[x]$ as in Step 4. Then $\exists x. F_3[x] \Leftrightarrow F_4$.

Proof:

$\Leftarrow$ If F_4 is true, then $F_{-\infty}$, $F_{+\infty}$ or $F_3[\frac{s+t}{2}]$ is true.

If $F_3[\frac{s+t}{2}]$ is true, then obviously $\exists x. F_3[x]$ is true.

If $F_{-\infty}$ is true, choose some small $x, x < t$ for all $t \in S$.

Then $F_3[x]$ is true.

If $F_{+\infty}$ is true, choose some big $x, x > t$ for all $t \in S$.

Then $F_3[x]$ is true.

Correctness of Step 4 II

⇒ If $I \models \exists x. F_3[x]$ then there is value v such that

$$I \models F_3[v].$$

If $v < \alpha_I[t]$ for all $t \in S$, then $I \models F_{-\infty}$.

If $v > \alpha_I[t]$ for all $t \in S$, then $I \models F_{+\infty}$.

If $v = \alpha_I[t]$ for some $t \in S$, then $I \models F[\frac{t+t}{2}]$.

Otherwise choose largest $s \in S$ with $\alpha_I[s] < v$ and smallest $t \in S$ with $\alpha_I[t] > v$.

Since no atom of F_3 can distinguish between values in interval (s, t) ,

$$I \models F_3[v] \quad \text{iff} \quad I \models F_3\left[\frac{s+t}{2}\right].$$

Hence, $I \models F[\frac{s+t}{2}]$. In all cases $I \models F_4$.

Ferrante & Rackoff's Method: Example I

$\Sigma_{\mathbb{Q}}$ -formula

$$\exists x. \underbrace{3x + 1 < 10 \wedge 7x - 6 > 7}_{F[x]}$$

Solving for x

$$\exists x. \underbrace{x < 3 \wedge x > \frac{13}{7}}_{F_3[x]}$$

$$\begin{array}{ll} \text{Step 4: } x > \frac{13}{7} \text{ in (B)} & \Rightarrow F_{-\infty} = \perp \\ x < 3 \text{ in (A)} & \Rightarrow F_{+\infty} = \perp \end{array}$$

$$F_4 : \bigvee_{s,t \in S} \underbrace{\left(\frac{s+t}{2} < 3 \wedge \frac{s+t}{2} > \frac{13}{7} \right)}_{F_3[\frac{s+t}{2}]}$$

Ferrante & Rackoff's Method: Example II

$$S = \{3, \frac{13}{7}\} \Rightarrow$$

$$F_3 \left[\frac{3+3}{2} \right] = \perp \quad F_3 \left[\frac{\frac{13}{7} + \frac{13}{7}}{2} \right] = \perp$$

$$F_3 \left[\frac{\frac{13}{7} + 3}{2} \right] : \frac{\frac{13}{7} + 3}{2} < 3 \wedge \frac{\frac{13}{7} + 3}{2} > \frac{13}{7} = \top$$

$$F_4 : \perp \vee \dots \vee \perp \vee \top = \top$$

Thus, $F_4 : \top$ is $T_{\mathbb{Q}}$ -equivalent to $\exists x. F[x]$,

so $\exists x. F[x]$ is $T_{\mathbb{Q}}$ -valid.

Example

$$\exists x. \underbrace{2x > y \wedge 3x < z}_{F[x]}$$

Solving for x

$$\exists x. \underbrace{x > \frac{y}{2} \wedge x < \frac{z}{3}}_{F_3[x]}$$

Step 4: $F_{-\infty} = \perp$, $F_{+\infty} = \perp$, $F_3[\frac{y}{2}] = \perp$ and $F_3[\frac{z}{3}] = \perp$.

$$F_4 : \frac{\frac{y}{2} + \frac{z}{3}}{2} > \frac{y}{2} \wedge \frac{\frac{y}{2} + \frac{z}{3}}{2} < \frac{z}{3}$$

which simplifies to:

$$F_4 : 2z > 3y$$

F_4 is $T_{\mathbb{Q}}$ -equivalent to $\exists x. F[x]$.