Multiple choice warmup!

For each of the following quantities, identify all of the options that correctly describe the quantity.

(a) The function \( f(n) \), where \( f(n) = n \log(n) \). \((A), (C)\)

(b) \( T(n) \) given by \( T(n) = T(n/4) + \Theta(n^2) \) with \( T(n) = 1 \) for all \( n \leq 8 \). \((A)(B)(C)\)

(c) \( T(n) \) which is the running time of the following algorithm:

\[
\text{mysteryAlg}( n ):
\]
\[
\text{if } n < 3:
\]
\[
\text{return } 1
\]
\[
\text{return mysteryAlg}( n/2 ) + \text{mysteryAlg}( (n/2) + 1 )
\]

where above all division is integer division (so \( a/b \) means \( \lfloor a/b \rfloor \)). \((A), (C), (D)\)

\((A) \ O(n^2) \quad (B) \ \Theta(n^2) \quad (C) \ \Omega(n) \quad (D) \ O(n) \quad (E) \ O(\log^2(n)).\)
Let $G = (V, E)$ be an undirected weighted graph, and let $T$ be a minimum spanning tree in $G$. Decide whether the following statements must be true or may be false, and prove it!

(a) For any pair of distinct vertices $s, t \in V$, there is a unique path from $s$ to $t$ in $T$.

(b) For any pair of distinct vertices $s, t \in V$, the cost of a path between $s$ and $t$ in $T$ is minimal among all paths from $s$ to $t$ in $G$. 

---

(a) True

(b) False
Let $\mathcal{U}$ be a universe of size $m$, where $m$ is a prime, and consider the following two hash families which hash $\mathcal{U}$ into $n$ buckets, where $n$ is much smaller than $m$.

- First, consider $\mathcal{H}_1$, which is the set of all functions from $\mathcal{U}$ to $\{1, \ldots, n\}$:
  \[ \mathcal{H}_1 = \{ h \mid h: \mathcal{U} \rightarrow \{1, \ldots, n\} \} \]

- Second, let $p = m$ (so $p$ is prime since we assumed $m$ to be prime), and choose $\mathcal{H}_2$ to be
  \[ \mathcal{H}_2 = \{ h_{a,b} \mid a \in \{1, \ldots, p - 1\}, b \in \{0, \ldots, p - 1\} \} \]
  where $h_{a,b}(x) = (ax + b \mod p) \mod n$.

You want to implement a hash table using one of these two families. Why would you choose $\mathcal{H}_2$ over $\mathcal{H}_1$? **Choose the best answer.**

(A) $\mathcal{H}_1$ isn't a universal hash family.

(B) Storing an element of $\mathcal{H}_1$ takes a lot of space.

(C) Storing all of $\mathcal{H}_1$ takes a lot of space.
Shortest Paths

- When might you prefer breadth-first search to Dijkstra’s algorithm?
  If the graph is unweighted

- When might you prefer Floyd-Warshall to Bellman-Ford?
  If you want shortest paths between all pairs of vertices.

- When might you prefer Bellman-Ford to Dijkstra’s algorithm?
  If there are negative edge weights
Randomized algorithms

Suppose that \( b_1, \ldots, b_n \) are \( n \) distinct integers in a **uniformly random order**. Consider the following algorithm:

\[
\text{findMax}(b_1, \ldots, b_n):
\begin{align*}
\text{currentMax} &= -\text{Infinity} \\
\text{for } i = 1, \ldots, n: & \\
\text{if } b_i > \text{currentMax}: & \\
\text{currentMax} &= b_i \\
\text{return } \text{currentMax}
\end{align*}
\]

What is the expected number of times that \( \text{currentMax} \) is updated? (Asymptotic notation is fine).

\[
\mathbb{E}\{ \text{#times currentMax updated} \} \\
= \mathbb{E}\{ \sum_{i=1}^{n} \mathbb{I}\{b_i > b_1, \ldots, b_{i-1}\} \} \\
= \sum_{i=1}^{n} \mathbb{P}\{b_i > b_1, \ldots, b_{i-1}\} \\
= \sum_{i=1}^{n} \frac{1}{i} \\
= \Theta(\log(n))
\]

since \( b_1, \ldots, b_i \) are uniform - the probability that any one is largest is \( \frac{1}{i} \).
Suppose that roads in a city are laid out in an $n \times n$ grid, but some of the roads are obstructed.

For example, for $n = 3$, the city may look like this:

```
Define $M[i,j] =$ #paths from $(0,0)$ to $(i,j)$.

$M[i,j] = \begin{cases} 
1 & i = 0 \\
\sum_{l=0}^{i-1} M[l,j] + \sum_{l=0}^{j-1} M[i,l] & \text{otherwise} 
\end{cases}
```

**Algorithm:**

1. Initialize $M[0,0] \leftarrow 1$.
2. For $i = 0, \ldots, n-1$:
   - For $j = 0, \ldots, n-1$:
     - If $i > 0$ and there is a road from $(0,j)$ to $(i,j)$: $M[i,j] \leftarrow M[i,j] + M[i-1,j]$.
     - If $j > 0$ and there is a road from $(0,i)$ to $(i,j)$: $M[i,j] \leftarrow M[i,j] + M[i,j-1]$.

where we have only drawn the roads that are not blocked. You want to count the number of ways to get from $(0,0)$ to $(n-1, n-1)$, using paths that only go up and to the right. In the example above, the number of paths is 3.

Design a DP algorithm to solve this problem.
Divide and Conquer!

- Given an array $A$ of length $n$, we say that an array $B$ is a circular shift of $A$ if there is an integer $k$ between 1 and $n$ (inclusive) so that


where $+$ denotes concatenation.

- For example, if $A = [2, 5, 6, 8, 9]$, then $B = [6, 8, 9, 2, 5]$ is a circular shift of $A$ (with $k = 2$). The sorted array $A$ itself is also a circular shift of $A$ (with $k = 1$).

- Design a $O(\log(n))$-time algorithm that takes as input an array $B$ which is a circular shift of a sorted array which contains distinct positive integers, and returns the value of the largest element in $B$. For example, give $B$ as above, your algorithm should return 9.

Solution on next page.
def findMax(B):
    n = len(B)
    if B[0] <= B[n-1]:  # case 1
        return B[n-1]
    mid = ⌊n/2⌋ + 1
    if B[mid] > B[0]:  # case 2
        return findMax(B[mid:n])
    else:  # case 3
        return findMax(B[mid+1:])

Idea:

1. In case 1, the situation looks like

2. In case 2, it looks like

3. In case 3, it looks like

So the max is on the right side and we recurse on B[mid:]

So the max is on the left side and we recurse on B[mid+1]
Greedy Algorithms!

There are $n$ final exams on Dec. 13 at Stanford; exam $i$ is scheduled to begin at time $a_i$ and end at time $b_i$. Two exams which overlap cannot be administered in the same classroom; two exams $i$ and $j$ are defined to be overlapping if $[a_i, b_i] \cap [a_j, b_j] \neq \emptyset$ (including if $b_i = a_j$, so one starts exactly at the time that the other ends). Design an algorithm which solves the following problem.

- **Input:** Arrays $A$ and $B$ of length $n$ so that $A[i] = a_i$ and $B[i] = b_i$.
- **Output:** The smallest number of classrooms necessary to schedule all of the exams, and an optimal assignment of exams to classrooms.
- **Running time:** $O(n \log(n) + nk)$, where $k$ is the minimum number of classrooms needed.
- **For example:** Suppose there are three exams, with start and finish times as given below:

<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_i$</td>
<td>12pm</td>
<td>4pm</td>
<td>2pm</td>
</tr>
<tr>
<td>$b_i$</td>
<td>3pm</td>
<td>6pm</td>
<td>5pm</td>
</tr>
</tbody>
</table>

Then the exams can be scheduled in two rooms; Exam 1 and Exam 2 can be scheduled in Room 1 and Exam 3 can be scheduled in Room 2.
def scheduleRooms(A, B):
    n ← len(A)
    C = \{ (A[i], j) \mid i = 0, \ldots, n-1 \}
    sort C // increasing order by start time.
    rooms = [] // list of rooms
    endTimes = []
    for i = 0, \ldots, n-1:
        C[i][1] ≥ endTimes[i]
        rooms + C[i][1]
        endTimes[i] + B[C[i][1]]
        Else: // did not break
        rooms + C[i][1]
        endTimes + B[C[i][1]]
    Return rooms.

Correctness by induction

Inductive Hypothesis: After adding the ith exam, there is an optimal schedule that extends the current solution.

Base Case: After adding 0 exams, there is an optimal all extending times.

Inductive Step: Suppose the inductive hypothesis holds for i-1. Let S be the optimal schedule that extends it.

If S puts exam i where we would put it (say, room r), then we are done, so suppose that S puts exam i in room r.

Let j > i be the next exam scheduled in room r. Then a_j ≥ a_i, since a_i had the smallest start time of all exams not yet picked. So consider the schedule S' where we swap the room of exam i and the rest of room r.

This is still a valid schedule, and uses the same number of rooms as S, so it is also optimal. And it puts exam i in room r, so we're done.

Conclusion: At the end of the algo, there's still an optimal solution extending the current one to the current one is optimal.
Universal Hash Families

- Definition: A hash family $\mathcal{H}$ (mapping $\mathcal{U}$ into $n$ buckets) is **2-universal** if for all $x \neq y \in \mathcal{U}$ and for all $a, b \in \{1, \ldots, n\}$,

$$\mathbb{P}((h(x), h(y)) = (a, b)) = \frac{1}{n^2}.$$  

(a) Show that if $\mathcal{H}$ is 2-universal, then it is universal.

(b) Show that the converse is not true. That is, there is a universal family that’s not 2-universal.

(a) Suppose that $\mathcal{H}$ is 2-universal. Then $\forall x \neq y \in \mathcal{U}$,

$$\mathbb{P}_{h \in \mathcal{H}}(h(x) = h(y)) = \sum_{t \in \{1, \ldots, n\}} \mathbb{P}_{(a, b) \in \{1, \ldots, n\}^2}(h(x), h(y)) = (a, b) = \frac{1}{n}.$$  

So by definition $\mathcal{H}$ is universal.

(b) Consider $\mathcal{U} = \{x, y\}$, $\mathcal{H} = \{h_1, h_2\}$, where:

- $h_1(x) = 0$, $h_1(y) = 1$,
- $h_2(x) = 1$, $h_2(y) = 0$.

Then $\mathbb{P}_{h \in \mathcal{H}}(h(x) = h(y)) = \frac{1}{2}$.

But $\mathbb{P}_{h \in \mathcal{H}}((h(x), h(y)) = (0, 0)) = \frac{1}{2}$, not $\frac{1}{4}$.
More universal hash families

Say that $\mathcal{H}$ is a universal hash family, containing functions $h : \mathcal{U} \rightarrow \{1, \ldots, n\}$. Consider the following game.

- You choose $h \in \mathcal{H}$ uniformly at random and keep it secret.
- A bad guy chooses $x \in \mathcal{U}$, and asks you for $h(x)$. (You give it to them).
- The bad guy chooses $y \in \mathcal{U} \setminus \{x\}$, and tries to get $h(y) = h(x)$.
- If $h(x) = h(y)$, the bad guy wins. Otherwise, you win.

One of the following two is true.

1. There is a universal hash family $\mathcal{H}$ so that the bad guy wins with probability 1.
2. For any universal hash family $\mathcal{H}$, the probability that the bad guy wins is at most $1/n$.

Which is true and why?
Which of the following can be colored as a red-black tree? Either give a coloring or explain why not.

This one can’t be colored.

This path from the root to NIL can contain at most 3 black nodes (including NIL).

But THIS one must have at least 4: The root, NIL, and then at least two internal ones.