Lecture 11

More dynamic programming!

Longest Common Subsequences, Knapsack, and (if time) independent sets in trees.
Last time

- Not coding in an action movie.
Last time

• Dynamic programming is an algorithm design paradigm.

• Basic idea:
  • Identify **optimal sub-structure**
    • Optimum to the big problem is built out of optima of small sub-problems
  • Take advantage of **overlapping sub-problems**
    • Only solve each sub-problem once, then use it again and again
  • Keep track of the solutions to sub-problems in a table as you build to the final solution.
Floyd Warshall

Sub-problem(k-1):
For all pairs, u,v, find the cost of the shortest path from u to v, so that all the internal vertices on that path are in \{1,...,k-1\}.

Let \(D^{(k-1)}[u,v]\) be the solution to Sub-problem(k-1).

Our DP algorithm will fill in the n-by-n arrays \(D^{(0)}, D^{(1)}, ..., D^{(n)}\) iteratively and then we’ll be done.

This is the shortest path from u to v through the blue set. It has length \(D^{(k-1)}[u,v]\)
Floyd Warshall

**Sub-problem(k-1):**
For all pairs, u,v, find the cost of the shortest path from u to v, so that all the internal vertices on that path are in \( \{1, ..., k-1\} \).

Let \( D^{(k-1)}[u,v] \) be the solution to Sub-problem(k-1).

**Question:** How can we find \( D^{(k)}[u,v] \) using \( D^{(k-1)} \)?

Our DP algorithm will fill in the n-by-n arrays \( D^{(0)}, D^{(1)}, ..., D^{(n)} \) iteratively and then we’ll be done.

Label the vertices 1,2,...,n (We omit some edges in the picture below).

This is the shortest path from u to v through the blue set. It has length \( D^{(k-1)}[u,v] \).
How can we find $D^{(k)}[u,v]$ using $D^{(k-1)}$?

$D^{(k)}[u,v]$ is the cost of the shortest path from $u$ to $v$ so that all internal vertices on that path are in $\{1, \ldots, k\}$. 

Diagram:
- Vertices 1, ..., $k$
- Vertices 1, ..., $k-1$
- $u$
- $v$
- $n$
- $k$
- $k+1$
How can we find $D^{(k)}[u,v]$ using $D^{(k-1)}$?

$D^{(k)}[u,v]$ is the cost of the shortest path from $u$ to $v$ so that all internal vertices on that path are in \{1, ..., $k$\}.

**Case 1:** we don’t need vertex $k$.

$$D^{(k)}[u,v] = D^{(k-1)}[u,v]$$
How can we find $D^{(k)}[u,v]$ using $D^{(k-1)}$?

$D^{(k)}[u,v]$ is the cost of the shortest path from $u$ to $v$ so that all internal vertices on that path are in $\{1, \ldots, k\}$.

**Case 2:** we need vertex $k$. 
Case 2 continued

- Suppose there are **no negative cycles**.
  - Then WLOG the shortest path from $u$ to $v$ through $\{1,\ldots,k\}$ is **simple**.
- If **that path** passes through $k$, it must look like this:
- **This path** is the shortest path from $u$ to $k$ through $\{1,\ldots,k-1\}$.
  - sub-paths of shortest paths are shortest paths
- Same for **this path**.

$$D^{(k)}[u,v] = D^{(k-1)}[u,k] + D^{(k-1)}[k,v]$$
Today

• Examples of dynamic programming:
  1. Longest common subsequence
  2. Knapsack problem
     • Two versions!
  3. Independent sets in trees
     • If we have time...
     • (If not the slides will be there as a reference)
The goal of this lecture

• For you to get really bored of dynamic programming
Longest Common Subsequence

• How similar are these two species?

DNA:
AGCCCTAAGGGCTACCTAGCTT

DNA:
GACAGCCTACAAGCGTTAGCTTG

You saw this in HW 4!
Longest Common Subsequence

• How similar are these two species?

DNA: AGCCCTAAAGGGCTACCTAGCTT
DNA: GACAGCCTACAAGCGTTAGCTTTG

• Pretty similar, their DNA has a long common subsequence: AGCCTAAGGCTTAGCTTT
Longest Common Subsequence

• Subsequence:
  - BDFH is a subsequence of ABCDEFGH

• If X and Y are sequences, a common subsequence is a sequence which is a subsequence of both.
  - BDFH is a common subsequence of ABCDEFGH and of ABDFGHI

• A longest common subsequence...
  - ...is a common subsequence that is longest.
  - The longest common subsequence of ABCDEFGH and ABDFGHI is ABDFGH.
We sometimes want to find these

- Applications in bioinformatics
- The unix command `diff`
- Merging in version control
  - `svn`, `git`, etc...

“Sometimes” is a joke!
Recipe for applying Dynamic Programming

• **Step 1:** Identify **optimal substructure**.

• **Step 2:** Find a **recursive formulation** for the length of the longest common subsequence.

• **Step 3:** Use **dynamic programming** to find the length of the longest common subsequence.

• **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can **find the actual LCS**.
Step 1: Optimal substructure

Prefixes:

<table>
<thead>
<tr>
<th>X</th>
<th>A</th>
<th>C</th>
<th>G</th>
<th>G</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>A</td>
<td>C</td>
<td>G</td>
<td>C</td>
<td>T</td>
</tr>
</tbody>
</table>

Notation: denote this prefix ACGC by $Y_4$

- Our sub-problems will be finding LCS’s of prefixes to $X$ and $Y$.
- Let $C[i,j] = \text{length\_of\_LCS}( X_i, Y_j )$
Optimal substructure ctd.

• Subproblem:
  • finding LCS’s of prefixes of X and Y.

• Why is this a good choice?
  • There’s some relationship between LCS’s of prefixes and LCS’s of the whole things.
  • These subproblems overlap a lot.

To see this formally, on to...
Recipe for applying Dynamic Programming

• **Step 1:** Identify optimal substructure.

• **Step 2:** Find a recursive formulation for the length of the longest common subsequence.

• **Step 3:** Use dynamic programming to find the length of the longest common subsequence.

• **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can **find the actual LCS.**
Two cases

Case 1: \( X[i] = Y[j] \)

- Our sub-problems will be finding LCS’s of prefixes to \( X \) and \( Y \).
- Let \( C[i,j] = \text{length}\_of\_LCS( X_i, Y_j ) \)

Then \( C[i,j] = 1 + C[i-1,j-1] \).
- because \( \text{LCS}(X_i, Y_j) = \text{LCS}(X_{i-1}, Y_{j-1}) \) followed by \( A \)
Two cases

Case 2: $X[i] \neq Y[j]$

- Our sub-problems will be finding LCS’s of prefixes to $X$ and $Y$.
- Let $C[i,j] = \text{length_of_LCS}(X_i, Y_j)$

\[
\begin{align*}
X_i & \quad A \quad C \quad G \quad G \quad T \\
Y_j & \quad A \quad C \quad G \quad C \quad T \quad T \quad A
\end{align*}
\]

- Then $C[i,j] = \max\{ C[i-1,j], C[i,j-1] \}$.
  - either $\text{LCS}(X_i,Y_j) = \text{LCS}(X_{i-1},Y_j)$ and $T$ is not involved,
  - or $\text{LCS}(X_i,Y_j) = \text{LCS}(X_{i},Y_{j-1})$ and $A$ is not involved,
  - (maybe both are not involved, that’s covered by the “or”).
Recursive formulation of the optimal solution

\[ C_{i,j} \begin{cases} 
0 & \text{if } i = 0 \text{ or } j = 0 \\
C_{i-1,j-1} + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\
\max\{C_{i,j-1}, C_{i-1,j}\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0
\end{cases} \]
Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the length of the longest common subsequence.
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- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual LCS.
LCS DP  OMG BBQ

- **LCS**(*X*, *Y*):
  - **C**[i,0] = **C**[0,j] = 0 for all *i* = 1,...,*m*, *j*=1,...,*n*.
  - For *i* = 1,...,*m* and *j* = 1,...,*n*:
    - If **X**[*i*] = **Y**[*j*]:
      - **C**[*i*,*j*] = **C**[*i*-1,*j*-1] + 1
    - Else:
      - **C**[*i*,*j*] = max{ **C**[*i*,*j*-1], **C**[*i*-1,*j*] }

**C**[*i*,*j*] = \[
\begin{cases} 
0 & \text{if } i = 0 \text{ or } j = 0 \\
C[i-1,j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\
\text{max} \{ C[i,j-1], C[i-1,j] \} & \text{if } X[i] \not= Y[j] \text{ and } i,j > 0
\end{cases}
\]

Running time: **O**(nm)
### Example

The Needleman-Wunsch algorithm computes the similarity score between two DNA sequences using a dynamic programming approach. The score matrix is constructed as follows:

- **Initial Conditions:**
  
  \[ C[i, j] = \begin{cases} 
  0 & \text{if } i = 0 \text{ or } j = 0 \\
  C[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i, j > 0 \\
  \max\{C[i-1, j], C[i, j-1]\} & \text{if } X[i] \neq Y[j] \text{ and } i, j > 0 
  \end{cases} \]

- **Initial Scores:**
  
  \[
  C[i, j] = \begin{cases} 
  0 & \text{if } i = 0 \text{ or } j = 0 \\
  0 & \text{if } i = 0 \text{ or } j = 0 \\
  0 & \text{if } i = 0 \text{ or } j = 0 \\
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  0 & \text{if } i = 0 \text{ or } j = 0 \\
  0 & \text{if } i = 0 \text{ or } j = 0 \\
  0 & \text{if } i = 0 \text{ or } j = 0 \\
  0 & \text{if } i = 0 \text{ or } j = 0 \\
  \end{cases}
  \]

**Note:** The above matrix is illustrative and does not correspond to the given sequences. The actual score matrix entries depend on the specific sequences and the gap penalties chosen.

**Sequences: X**

\[ A \]
\[ C \]
\[ G \]
\[ G \]
\[ A \]

**Sequences: Y**

\[ A \]
\[ C \]
\[ T \]
\[ G \]

**Score Matrix:**

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>C</th>
<th>T</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
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<td>0</td>
<td>0</td>
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<tr>
<td>G</td>
<td>0</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>G</td>
<td>0</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>A</td>
<td>0</td>
<td>0</td>
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</tr>
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Example

So the LCM of X and Y has length 3.
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Example

\[ C[i,j] = \begin{cases} 
0 & \text{if } i = 0 \text{ or } j = 0 \\
C[i - 1, j - 1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\
\max\{C[i,j - 1], C[i - 1,j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0
\end{cases} \]
Example

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\]
Example

- Once we’ve filled this in, we can work backwards.

\[
C[i, j] = \begin{cases} 
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\]
### Example

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<td></td>
<td></td>
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- Once we’ve filled this in, we can work backwards.

That 3 must have come from the 3 above it.

\[
C[i,j] = \begin{cases} 
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\end{cases}
\]
Example

Let's consider the sequences:

**X** = A C G G A

**Y** = A C T G

To find the Longest Common Subsequence (LCS), we can use a dynamic programming approach. The table below represents the **C** matrix, where each cell **C[i,j]** contains the length of the LCS of **X**[1..i-1] and **Y**[1..j-1].

- **C[i,j] = 0** if **i = 0** or **j = 0**
- **C[i,j] = C[i-1,j-1] + 1** if **X[i] = Y[j]** and **i, j > 0**
- **C[i,j] = max{C[i-1,j], C[i,j-1]}** if **X[i] ≠ Y[j]** and **i, j > 0**

Once we've filled this in, we can work backwards.

A diagonal jump means that we found an element of the LCS!

This 3 came from that 2 – we found a match!
Example

\[
C[i,j] = \begin{cases} 
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\max\{C[i,j-1],C[i-1,j]\} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0 
\end{cases}
\]

- Once we’ve filled this in, we can work backwards.
- A diagonal jump means that we found an element of the LCS!

That 2 may as well have come from this other 2.
Example

\[ C[i, j] = \begin{cases} 
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• Once we’ve filled this in, we can work backwards.
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### Example

#### Dynamic Programming

<table>
<thead>
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<th></th>
<th>A</th>
<th>C</th>
<th>T</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>T</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>G</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>A</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

\[
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Example

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\end{cases} \]

Once we’ve filled this in, we can work backwards.

A diagonal jump means that we found an element of the LCS!

This is the LCS!
Another way of seeing the same thing:

\[ C[i,j] = \begin{cases} 
0 & \text{if } i = 0 \text{ or } j = 0 \\
C[i-1,j-1] + 1 & \text{if } X[i] = Y[j] \text{ and } i,j > 0 \\
\max\{ C[i,j-1], C[i-1,j] \} & \text{if } X[i] \neq Y[j] \text{ and } i,j > 0
\end{cases} \]

Finding length of shortest path vs
Actually finding the shortest path (just like your HW!)

This is the LCS!
Finding an LCS

• See lecture notes for pseudocode
• Takes time $O(mn)$ to fill the table
• Takes time $O(n + m)$ on top of that to recover the LCS
  • We walk up and left in an $n$-by-$m$ array
  • We can only do that for $n + m$ steps.
• Altogether, we can find LCS($X,Y$) in time $O(mn)$. 
Time and Space complexity

- If we are only interested in the length of the LCS:
  - Since we go across the table one-row-at-a-time, we can only keep two rows if we want.
  - If we want to recover the LCS, we need to keep the whole table.

- Can we do better than $O(mn)$ time?
  - A bit better.
    - By a log factor or so.
  - But doing much better (e.g. $O(mn^{1.9})$) is an open problem!
    - If you can do it let me know :D
What have we learned?

• We can find LCS(X,Y) in time O(nm)
  • if |Y|=n, |X|=m

• We went through the steps of coming up with a dynamic programming algorithm.
  • We kept a 2-dimensional table, breaking down the problem by decrementing the length of X and Y.
Example 2: Knapsack Problem

- We have $n$ items with weights and values:

<table>
<thead>
<tr>
<th>Item:</th>
<th>Weight:</th>
<th>Value:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turtle</td>
<td>6</td>
<td>20</td>
</tr>
<tr>
<td>Light</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>Watermelon</td>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td>Taco</td>
<td>3</td>
<td>13</td>
</tr>
<tr>
<td>Firetruck</td>
<td>11</td>
<td>35</td>
</tr>
</tbody>
</table>

- And we have a knapsack:
  - it can only carry so much weight:

  Capacity: 10
• **Unbounded Knapsack:**
  • Suppose I have *infinite copies* of all of the items.
  • What’s the *most valuable way to fill the knapsack*?

<table>
<thead>
<tr>
<th>Item</th>
<th>Weight</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>🐢</td>
<td>6</td>
<td>20</td>
</tr>
<tr>
<td>🕯️</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>🍉</td>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td>🌶️</td>
<td>3</td>
<td>13</td>
</tr>
<tr>
<td>🥙</td>
<td>11</td>
<td>35</td>
</tr>
</tbody>
</table>

  Total weight: 10
  Total value: 42

• **0/1 Knapsack:**
  • Suppose I have *only one copy* of each item.
  • What’s the *most valuable way to fill the knapsack*?

<table>
<thead>
<tr>
<th>Item</th>
<th>Weight</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>🕯️</td>
<td></td>
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<tr>
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<td>🌶️</td>
<td></td>
<td></td>
</tr>
<tr>
<td>🥙</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

  Total weight: 9
  Total value: 35
Some notation

Item:

Weight: $W_1$ $W_2$ $W_3$ $\ldots$ $W_n$

Value: $V_1$ $V_2$ $V_3$ $V_n$

Capacity: $W$
Recipe for applying Dynamic Programming

• **Step 1:** Identify optimal substructure.

• **Step 2:** Find a recursive formulation for the value of the optimal solution.

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• **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
Optimal substructure

• Sub-problems:
  • Unbounded Knapsack with a smaller knapsack.

First solve the problem for small knapsacks
Then larger knapsacks
Then larger knapsacks
Optimal substructure

• Suppose this is an optimal solution for capacity $x$:

Say that the optimal solution contains at least one copy of item $i$.

• Then this optimal for capacity $x - w_i$:

If I could do better than the second solution, then adding a turtle to that improvement would improve the first solution.
Recipe for applying Dynamic Programming

• **Step 1:** Identify **optimal substructure**.

• **Step 2:** Find a **recursive formulation** for the value of the optimal solution.

• **Step 3:** Use **dynamic programming** to find the value of the optimal solution.

• **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can **find the actual solution**.
Recursive relationship

• Let $K[x]$ be the **optimal value** for capacity $x$.

$$K[x] = \max_i \left\{ \text{Optimal way to fill the smaller knapsack} + \text{The value of item } i \right\}$$

The maximum is over all $i$ so that $w_i \leq x$.

$$K[x] = \text{Think-Pair-Share!}$$
Recursive relationship

• Let $K[x]$ be the optimal value for capacity $x$.

$$K[x] = \max_i \{ \text{The maximum is over all } i \text{ so that } w_i \leq x. \text{ Optimal way to fill the smaller knapsack} \}$$

$$K[x] = \max_i \{ K[x - w_i] + v_i \} \text{ The value of item } i.$$  

• (And $K[x] = 0$ if the maximum is empty).
  • That is, if there are no $i$ so that $w_i \leq x$
Recipe for applying Dynamic Programming

• **Step 1:** Identify optimal substructure.
• **Step 2:** Find a recursive formulation for the value of the optimal solution.
• **Step 3:** Use dynamic programming to find the value of the optimal solution.
• **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
Let’s write a bottom-up DP algorithm

• **UnboundedKnapsack**(W, n, weights, values):
  • K[0] = 0
  • for x = 1, ..., W:
    • K[x] = 0
    • for i = 1, ..., n:
      • if \( w_i \leq x \):
        • \( K[x] = \max \{ K[x], K[x - w_i] + v_i \} \)
      • return K[W]

Running time: \( O(nW) \)

Why does this work?

Because our recursive relationship makes sense.
Can we do better?

• Writing down $W$ takes $\log(W)$ bits.
• Writing down all $n$ weights takes at most $n\log(W)$ bits.
• Input size: $n\log(W)$.
  • Maybe we could have an algorithm that runs in time $O(n\log(W))$ instead of $O(nW)$?
  • Or even $O(n^{1000000} \log^{1000000}(W))$?

• Open problem!
  • (But probably the answer is no...otherwise $P = NP$)
Recipe for applying Dynamic Programming

• **Step 1:** Identify optimal substructure.

• **Step 2:** Find a recursive formulation for the value of the optimal solution.

• **Step 3:** Use dynamic programming to find the value of the optimal solution.

• **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
Let’s write a bottom-up DP algorithm

• **UnboundedKnapsack**(\(W, n, \text{weights}, \text{values}\)):
  • \(K[0] = 0\)
  • **for** \(x = 1, ..., W\):
    • \(K[x] = 0\)
    • **for** \(i = 1, ..., n\):
      • **if** \(w_i \leq x\):
        • \(K[x] = \max\{ K[x], K[x - w_i] + v_i \} \)
  • **return** \(K[W]\)

\[K[x] = \max_i \{ \text{backpack} + \text{turtle} \} = \max_i \{ K[x - w_i] + v_i \}\]
Let’s write a bottom-up DP algorithm

• UnboundedKnapsack\((W, n, \text{weights, values})\):
  • \(K[0] = 0\)
  • \(\text{ITEMS}[0] = \emptyset\)
  • \(\text{for } x = 1, \ldots, W:\)
    • \(K[x] = 0\)
    • \(\text{for } i = 1, \ldots, n:\)
      • if \(w_i \leq x\):
        • \(K[x] = \max\{ K[x], K[x - w_i] + v_i \} \)
      • If \(K[x]\) was updated:
        • \(\text{ITEMS}[x] = \text{ITEMS}[x - w_i] \cup \{ \text{item } i \} \)
  • \text{return } \text{ITEMS}[W]
Example

```
• UnboundedKnapsack(W, n, weights, values):
  • K[0] = 0
  • ITEMS[0] = ∅
  • for x = 1, ..., W:
    • K[x] = 0
    • for i = 1, ..., n:
      • if w_i ≤ x:
        • K[x] = max{ K[x], K[x - w_i] + v_i }
      • if K[x] was updated:
        • ITEMS[x] = ITEMS[x - w_i] ∪ { item i }
  • return ITEMS[W]
```

<table>
<thead>
<tr>
<th>Item</th>
<th>Weight</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turtle</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Light</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>Watermelon</td>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>

Capacity: 4
Example

UnboundedKnapsack(W, n, weights, values):

- K[0] = 0
- ITEMS[0] = ∅
- for x = 1, ..., W:
  - K[x] = 0
  - for i = 1, ..., n:
    - if \( w_i \leq x \):
      - \( K[x] = \max\{ K[x], K[x - w_i] + v_i \} \)
    - If K[x] was updated:
      - ITEMS[x] = ITEMS[x - w_i] \cup \{ item i \}

- return ITEMS[W]

<table>
<thead>
<tr>
<th>K</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>ITEMS</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

ITEMS[1] = ITEMS[0] +

Item:

<table>
<thead>
<tr>
<th>Weight</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
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<td>4</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>

Capacity: 4
Example

\[ \text{UnboundedKnapsack}(W, n, \text{weights, values}): \]

\[ \begin{align*}
\text{K}[0] & = 0 \\
\text{ITEMS}[0] & = \emptyset \\
\text{for } x & = 1, \ldots, W: \\
& \quad \text{K}[x] = 0 \\
& \quad \text{for } i = 1, \ldots, n: \\
& \quad \quad \text{if } w_i \leq x:
& \quad \quad \quad \text{K}[x] = \max\{ \text{K}[x], \text{K}[x - w_i] + v_i \} \\
& \quad \quad \text{If K}[x] was updated:
& \quad \quad \quad \text{ITEMS}[x] = \text{ITEMS}[x - w_i] \cup \{ \text{item } i \} \\
& \quad \text{return } \text{ITEMS}[W]
\end{align*} \]


\[ \begin{array}{c|c|c|c|c}
\text{K} & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 2 & \\
\text{ITEMS} & & & &
\end{array} \]
Example

- **UnboundedKnapsack** \((W, n, \text{weights}, \text{values})\):
  - \(K[0] = 0\)
  - \(\text{ITEMS}[0] = \emptyset\)
  - **for** \(x = 1, \ldots, W\):
    - \(K[x] = 0\)
    - **for** \(i = 1, \ldots, n\):
      - if \(w_i \leq x\):
        - \(K[x] = \max\{ K[x], K[x - w_i] + v_i \} \)
      - If \(K[x]\) was updated:
        - \(\text{ITEMS}[x] = \text{ITEMS}[x - w_i] \cup \{ \text{item } i \} \)
  - return \(\text{ITEMS}[W]\)

---

**ITEMS[2] = ITEMS[0] + \[ \text{Light bulb} \]**

**Item:**

- **Weight:**
  - 1
  - 2
  - 3

- **Value:**
  - 1
  - 4
  - 6

**Capacity:** 4
Example

UnboundedKnapsack(\(W, n, \text{weights, values}\)):

- \(K[0] = 0\)
- \(\text{ITEMS}[0] = \emptyset\)
- for \(x = 1, \ldots, W\):
  - \(K[x] = 0\)
  - for \(i = 1, \ldots, n\):
    - if \(w_i \leq x\):
      - \(K[x] = \max\{ K[x], K[x - w_i] + v_i \} \)
    - If \(K[x]\) was updated:
      - \(\text{ITEMS}[x] = \text{ITEMS}[x - w_i] \cup \{ \text{item } i \} \)
- return \(\text{ITEMS}[W]\)


Example:

<table>
<thead>
<tr>
<th>Item</th>
<th>Weight</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turtle</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Light</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>Watermelon</td>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>

Capacity: 4
Example

- **UnboundedKnapsack** \( (W, n, \text{weights, values}) \):
  - \( K[0] = 0 \)
  - \( \text{ITEMS}[0] = \emptyset \)
  - \( \text{for } x = 1, \ldots, W: \)
    - \( K[x] = 0 \)
    - \( \text{for } i = 1, \ldots, n: \)
      - \( \text{if } w_i \leq x: \)
        - \( K[x] = \max \{ K[x], K[x - w_i] + v_i \} \)
      - \( \text{If } K[x] \text{ was updated:} \)
        - \( \text{ITEMS}[x] = \text{ITEMS}[x - w_i] \cup \{ \text{item } i \} \)
  - \( \text{return } \text{ITEMS}[W] \)

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<th>Value:</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Lightbulb</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>Watermelon</td>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>

**Capable of:** 4

\[ \text{ITEMS}[3] = \text{ITEMS}[0] + \text{Watermelon} \]
Example

UnboundedKnapsack($W$, $n$, weights, values):

- $K[0] = 0$
- ITEMS[0] = Ø
- for $x = 1, ..., W$:
  - $K[x] = 0$
  - for $i = 1, ..., n$:
    - if $w_i \leq x$:
      - $K[x] = \max\{ K[x], K[x - w_i] + v_i \}$
    - If $K[x]$ was updated:
      - ITEMS[x] = ITEMS[x - w_i] ∪ \{ item i \}
- return ITEMS[W]

Example

UnboundedKnapsack(\(W, n, \text{weights}, \text{values}\)):

- \(K[0] = 0\)
- \(\text{ITEMS}[0] = \emptyset\)
- \(\text{for } x = 1, \ldots, W:\)
  - \(K[x] = 0\)
  - \(\text{for } i = 1, \ldots, n:\)
    - \(\text{if } w_i \leq x:\)
      - \(K[x] = \max\{K[x], K[x - w_i] + v_i\}\)
    - \(\text{If } K[x] \text{ was updated:}\)
      - \(\text{ITEMS}[x] = \text{ITEMS}[x - w_i] \cup \{ \text{item } i \}\)
- \(\text{return } \text{ITEMS}[W]\)

\(K\)
\[\begin{array}{ccccc}
0 & 1 & 2 & 3 & 4 \\
0 & 1 & 4 & 6 & 8 \\
\end{array}\]

\(\text{ITEMS}\)

\(\text{ITEMS}[4] = \text{ITEMS}[2] + \text{Item}\)

Item:

Weight:

Value:

Capacity: 4
What have we learned?

• We can solve unbounded knapsack in time $O(nW)$.
  • If there are $n$ items and our knapsack has capacity $W$.

• We again went through the steps to create DP solution:
  • We kept a one-dimensional table, creating smaller problems by making the knapsack smaller.
### Unbounded Knapsack:
- Suppose I have **infinite copies** of all of the items.
- What’s the **most valuable way to fill the knapsack**?

<table>
<thead>
<tr>
<th>Item</th>
<th>Weight</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turtle</td>
<td>6</td>
<td>20</td>
</tr>
<tr>
<td>Light</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>Watermelon</td>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td>Taco</td>
<td>3</td>
<td>13</td>
</tr>
</tbody>
</table>

Total weight: 10
Total value: 42

### 0/1 Knapsack:
- Suppose I have **only one copy** of each item.
- What’s the **most valuable way to fill the knapsack**?

<table>
<thead>
<tr>
<th>Item</th>
<th>Weight</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Light</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>Watermelon</td>
<td>4</td>
<td>13</td>
</tr>
<tr>
<td>Taco</td>
<td>3</td>
<td>13</td>
</tr>
</tbody>
</table>

Total weight: 9
Total value: 35
Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
Optimal substructure: try 1

• Sub-problems:
  • Unbounded Knapsack with a smaller knapsack.

First solve the problem for small knapsacks

Then larger knapsacks

Then larger knapsacks
This won’t quite work...

• We are only allowed **one copy of each item**.
• The sub-problem needs to “know” what items we’ve used and what we haven’t.

I can’t use any turtles...
Optimal substructure: try 2

• Sub-problems:

Think-Pair-Share!
Optimal substructure: try 2

- Sub-problems:
  - 0/1 Knapsack with fewer items.

First solve the problem with few items

Then more items

Then yet more items

We’ll still increase the size of the knapsacks.

(We’ll keep a two-dimensional table).
Our sub-problems:

- Indexed by $x$ and $j$
Two cases

• **Case 1**: Optimal solution for \( j \) items does not use item \( j \).
• **Case 2**: Optimal solution for \( j \) items does use item \( j \).
Two cases

- **Case 1**: Optimal solution for *j* items does not use item *j*.

- Then this is an optimal solution for *j*-1 items:
  - First *j* items
  - First *j*-1 items

  Capacity x
  Value V
  Use only the first *j* items

  Capacity x
  Value V
  Use only the first *j*-1 items.
Two cases

• **Case 2**: Optimal solution for *j* items uses item *j*.

• Then this is an optimal solution for *j*-1 items and a smaller knapsack:

  - Capacity *x* – *w*<sub>i</sub>
  - Value *V* – *v*<sub>i</sub>

Use only the first *j*-1 items.
Recipe for applying Dynamic Programming

• **Step 1:** Identify optimal substructure.

• **Step 2:** Find a recursive formulation for the value of the optimal solution.

• **Step 3:** Use dynamic programming to find the value of the optimal solution.

• **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
Recursive relationship

• Let $K[x,j]$ be the optimal value for:
  • capacity $x$,
  • with $j$ items.

$$K[x,j] = \max\{K[x, j-1], K[x - w_j, j-1] + v_j\}$$

- Case 1
- Case 2

• (And $K[x,0] = 0$ and $K[0,j] = 0$).
Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Step 3:** Use dynamic programming to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
Bottom-up DP algorithm

- Zero-One-Knapsack(W, n, w, v):
  - $K[x,0] = 0$ for all $x = 0,...,W$
  - $K[0,i] = 0$ for all $i = 0,...,n$
  - for $x = 1,...,W$:
    - for $j = 1,...,n$:
      - if $w_j \leq x$:
        - $K[x,j] = \max\{ K[x,j], K[x - w_j, j-1] + v_j \}$  \hspace{1cm} \text{Case 2}
      - $K[x,j] = K[x, j-1]$  \hspace{1cm} \text{Case 1}
  - return $K[W,n]$

Running time $O(nW)$
Example

Zero-One-Knapsack(W, n, w, v):
• \( K[x,0] = 0 \) for all \( x = 0, \ldots, W \)
• \( K[0,i] = 0 \) for all \( i = 0, \ldots, n \)
• for \( x = 1, \ldots, W \):
  • for \( j = 1, \ldots, n \):
    • \( K[x,j] = K[x, j-1] \)
    • if \( w_j \leq x \):
      • \( K[x,j] = \max\{ K[x,j], K[x - w_j, j-1] + v_j \} \)
• return \( K[W,n] \)

<table>
<thead>
<tr>
<th></th>
<th>x=0</th>
<th>x=1</th>
<th>x=2</th>
<th>x=3</th>
</tr>
</thead>
<tbody>
<tr>
<td>j=0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>j=1</td>
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<tr>
<td>j=2</td>
<td>0</td>
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</tr>
<tr>
<td>j=3</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Item:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight:</td>
</tr>
<tr>
<td>Value:</td>
</tr>
<tr>
<td>Capacity: 3</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>1</td>
</tr>
</tbody>
</table>
### Example

#### Zero-One-Knapsack ($W, n, w, v$):
- $K[x,0] = 0$ for all $x = 0,...,W$
- $K[0,i] = 0$ for all $i = 0,...,n$
- **for** $x = 1,...,W$:
  - **for** $j = 1,...,n$:
    - $K[x,j] = K[x, j-1]$
    - if $w_j \leq x$:
      - $K[x,j] = \max\{ K[x,j], K[x - w_j, j-1] + v_j \}$
- **return** $K[W,n]$

#### Table:

<table>
<thead>
<tr>
<th>j=0</th>
<th>x=0</th>
<th>x=1</th>
<th>x=2</th>
<th>x=3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>j=1</td>
<td></td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>j=2</td>
<td></td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>j=3</td>
<td></td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- **Item:**
  - 1: Turtle
  - 2: Lightbulb
  - 3: Watermelon

- **Weight:**
  - 1: Turtle
  - 2: Lightbulb
  - 3: Watermelon

- **Value:**
  - 1: Turtle
  - 4: Lightbulb
  - 6: Watermelon

- **Capacity:** 3
Example

```
Zero-One-Knapsack(W, n, w, v):
  K[x,0] = 0 for all x = 0,...,W
  K[0,i] = 0 for all i = 0,...,n
  for x = 1,...,W:
    for j = 1,...,n:
      if w_j ≤ x:
        K[x,j] = max{ K[x,j],
                     K[x-w_j, j-1] + v_j }
  return K[W,n]
```

Item:
- Turtle
- Lightbulb
- Watermelon

Weight:
- Turtle: 1
- Lightbulb: 2
- Watermelon: 3

Value:
- Turtle: 1
- Lightbulb: 4
- Watermelon: 6

Capacity: 3
Example

Zero-One-Knapsack(W, n, w, v):

- \( K[x,0] = 0 \) for all \( x = 0, ..., W \)
- \( K[0,i] = 0 \) for all \( i = 0, ..., n \)
- for \( x = 1, ..., W \):
  - for \( j = 1, ..., n \):
    - \( K[x,j] = K[x, j-1] \)
    - if \( w_j \leq x \):
      - \( K[x,j] = \max\{ K[x,j], K[x-w_j, j-1] + v_j \} \)
- return \( K[W,n] \)

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</tr>
<tr>
<td>j=1</td>
<td>0</td>
<td>1</td>
<td></td>
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<tr>
<td>j=2</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>j=3</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Current entry | Relevant previous entry
---|---
Item:
- Weight: 1 2 3
- Value: 1 4 6
Capacity: 3
Example

- Zero-One-Knapsack(W, n, w, v):
  - \( K[x,0] = 0 \) for all \( x = 0,\ldots,W \)
  - \( K[0,i] = 0 \) for all \( i = 0,\ldots,n \)
  - for \( x = 1,\ldots,W \):
    - for \( j = 1,\ldots,n \):
      - \( K[x,j] = K[x, j-1] \)
      - if \( w_j \leq x \):
        - \( K[x,j] = \max\{ K[x,j], K[x-w_j, j-1] + v_j \} \)
  - return \( K[W,n] \)

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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x=0 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>0</td>
<td>1</td>
<td></td>
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<td>( x=3 )</td>
<td>0</td>
<td>1</td>
<td></td>
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- Item:
  - Weight: 1 2 3
  - Value: 1 4 6

Capacity: 3
Example

Zero-One-Knapsack(W, n, w, v):

- $K[x,0] = 0$ for all $x = 0,...,W$
- $K[0,i] = 0$ for all $i = 0,...,n$
- for $x = 1,...,W$:
  - for $j = 1,...,n$:
    - $K[x,j] = K[x, j-1]$
    - if $w_j \leq x$:
      - $K[x,j] = \max\{ K[x,j], K[x - w_j, j-1] + v_j \}$
- return $K[W,n]$

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Item:
- Turtles
- Lamps
- Watermelons

Weight:
- 1
- 2
- 3

Value:
- 1
- 4
- 6

Capacity: 3
### Example

Zero-One-Knapsack($W, n, w, v$):

- $K[x,0] = 0$ for all $x = 0,...,W$
- $K[0,i] = 0$ for all $i = 0,...,n$
- for $x = 1,...,W$
  - for $j = 1,...,n$
    - $K[x,j] = K[x, j-1]$
    - if $w_j \leq x$
      - $K[x,j] = \max\{K[x,j], K[x-w_j, j-1] + v_j\}$

- return $K[W,n]$

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- **Item:**
  - Weight: 1, 2, 3
  - Value: 1, 4, 6
- **Capacity:** 3
### Example

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- **Item:**
  - Weight: 1, 2, 3
  - Value: 1, 4, 6

- **Capacity:** 3

#### Zero-One-Knapsack \(W, n, w, v\):
- \(K[x,0] = 0\) for all \(x = 0, ..., W\)
- \(K[0,i] = 0\) for all \(i = 0, ..., n\)
- **for** \(x = 1, ..., W\):
  - **for** \(j = 1, ..., n\):
    - \(K[x,j] = K[x, j-1]\)
    - **if** \(w_j \leq x\):
      - \(K[x,j] = \max\{ K[x,j], K[x - w_j, j-1] + v_j \} \)

- **return** \(K[W,n]\)
Zero-One-Knapsack($W$, $n$, $w$, $v$):

- $K[x,0] = 0$ for all $x = 0,...,W$
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- return $K[W,n]$

### Example

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**Item:**
- **Weight:** 1 2 3
- **Value:** 1 4 6

**Capacity:** 3
### Example

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- **Item:**
  - **Weight:** 1 2 3
  - **Value:** 1 4 6

- **Capacity:** 3

- **Zero-One-Knapsack(W, n, w, v):**
  - $K[x,0] = 0$ for all $x = 0,...,W$
  - $K[0,i] = 0$ for all $i = 0,...,n$
  - **for** $x = 1,...,W$:
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**Item:**
- Turtles
- Lightbulbs
- Watermelons

**Weight:**
- 1
- 2
- 3

**Value:**
- 1
- 4
- 6

**Capacity:** 3

---

- **Zero-One-Knapsack** $(W, n, w, v)$:
  - $K[x,0] = 0$ for all $x = 0,...,W$
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  - return $K[W,n]$
### Example

**Zero-One-Knapsack**($W$, $n$, $w$, $v$):
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- Item:
  - Turtle: 1
  - Light bulb: 2
  - Watermelon: 3

- Weight:
  - Turtle: 1
  - Light bulb: 2
  - Watermelon: 3

- Value:
  - Turtle: 1
  - Light bulb: 4
  - Watermelon: 6

Capacity: 3
### Example

#### Item:
- **Turtle**: 1
- **Lightbulb**: 2
- **Watermelon**: 3

#### Weight:
- **Turtle**: 1
- **Lightbulb**: 2
- **Watermelon**: 3

#### Value:
- **Turtle**: 1
- **Lightbulb**: 4
- **Watermelon**: 6

#### Capacity: 3

#### Zero-One-Knapsack(W, n, w, v):
- \( K[x,0] = 0 \) for all \( x = 0,...,W \)
- \( K[0,i] = 0 \) for all \( i = 0,...,n \)
- for \( x = 1,...,W:\)
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  - return \( K[W,n] \)
Example

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**Current entry**

- **Item:** turtle
  - **Weight:** 1
  - **Value:** 1
- **Item:** light bulb
  - **Weight:** 2
  - **Value:** 4
- **Item:** watermelon
  - **Weight:** 3
  - **Value:** 6

**Relevant previous entry**

- **Zero-One-Knapsack(W, n, w, v):**
  - \( K[x,0] = 0 \) for all \( x = 0, \ldots, W \)
  - \( K[0,i] = 0 \) for all \( i = 0, \ldots, n \)
  - **for** \( x = 1, \ldots, W: \)
    - **for** \( j = 1, \ldots, n: \)
      - \( K[x,j] = K[x, j-1] \)
      - **if** \( w_j \leq x: \)
        - \( K[x,j] = \max\{ K[x,j], K[x - w_j, j-1] + v_j \} \)
  - **return** \( K[W,n] \)
### Example

#### Zero-One-Knapsack(W, n, w, v):
- \( K[x,0] = 0 \) for all \( x = 0, \ldots, W \)
- \( K[0,i] = 0 \) for all \( i = 0, \ldots, n \)
- \textbf{for} \( x = 1, \ldots, W \):
  - \textbf{for} \( j = 1, \ldots, n \):
    - \( K[x,j] = K[x, j-1] \)
    - \textbf{if} \( w_j \leq x \):
      - \( K[x,j] = \max\{ K[x,j], K[x - w_j, j-1] + v_j \} \)
- \textbf{return} \( K[W,n] \)

#### Table

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**Capacity:** 3
### Example

#### Zero-One-Knapsack($W$, $n$, $w$, $v$):
- $K[x,0] = 0$ for all $x = 0,...,W$
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**Item:**
- Turtles: 1
- Lightbulbs: 2
- Watermelons: 3

**Weight:**
- Turtles: 1
- Lightbulbs: 2
- Watermelons: 3

**Value:**
- Turtles: 1
- Lightbulbs: 4
- Watermelons: 6

**Capacity:** 3
Example

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- **Zero-One-Knapsack** \( W, n, w, v \):
  - \( K[x,0] = 0 \) for all \( x = 0,...,W \)
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      - if \( w_j \leq x \):
        - \( K[x,j] = \max\{ K[x,j], K[x-w_j, j-1] + v_j \} \)
  - return \( K[W,n] \)

So the optimal solution is to put one watermelon in your knapsack!
Recipe for applying Dynamic Programming

• **Step 1:** Identify optimal substructure.

• **Step 2:** Find a recursive formulation for the value of the optimal solution.

• **Step 3:** Use dynamic programming to find the value of the optimal solution.

• **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.

You do this one!
(We did it on the slide in the previous example, just not in the pseudocode!)
What have we learned?

• We can solve 0/1 knapsack in time $O(nW)$.
  • If there are $n$ items and our knapsack has capacity $W$.

• We again went through the steps to create DP solution:
  • We kept a two-dimensional table, creating smaller problems by restricting the set of allowable items.
Question

• How did we know which substructure to use in which variant of knapsack?

Answer in retrospect:

This one made sense for unbounded knapsack because it doesn’t have any memory of what items have been used.

VS.

In 0/1 knapsack, we can only use each item once, so it makes sense to leave out one item at a time.

Operational Answer: try some stuff, see what works!
Example 3: Independent Set

if we still have time

An independent set is a set of vertices so that no pair has an edge between them.

• Given a graph with weights on the vertices...

• What is the independent set with the largest weight?
Actually this problem is **NP-complete**. So we are unlikely to find an efficient algorithm

- But if we also assume that the graph is a tree...

**Problem:**
find a maximal independent set in a tree (with vertex weights).
Recipe for applying Dynamic Programming

• **Step 1:** Identify optimal substructure.

• **Step 2:** Find a recursive formulation for the value of the optimal solution

• **Step 3:** Use dynamic programming to find the value of the optimal solution

• **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
Optimal substructure

- **Subtrees** are a natural candidate.
- There are **two cases**:
  1. The root of this tree is in a **not** in a maximal independent set.
  2. Or it is.
Case 1:
the root is **not** in an maximal independent set

- Use the optimal solution from **these smaller problems**.
Case 2:
the root is in an maximal independent set

• Then its children can’t be.
• Below that, use the optimal solution from these smaller subproblems.
Recipe for applying Dynamic Programming

• **Step 1:** Identify optimal substructure.

• **Step 2:** Find a recursive formulation for the value of the optimal solution.

• **Step 3:** Use dynamic programming to find the value of the optimal solution.

• **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
Recursive formulation: try 1

- Let $A[u]$ be the weight of a maximal independent set in the tree rooted at $u$.

$$A[u] = \max \left\{ \text{weight}(u) + \sum_{v \in u.\text{grandchildren}} A[v], \sum_{v \in u.\text{children}} A[v] \right\}$$

When we implement this, how do we keep track of this term?
Recursive formulation: try 2

Keep two arrays!

- Let $A[u]$ be the weight of a maximal independent set in the tree rooted at $u$.
- Let $B[u] = \sum_{v \in u.\text{children}} A[v]$
- $A[u] = \max \left\{ \text{weight}(u) + \sum_{v \in u.\text{children}} B[v], \sum_{v \in u.\text{children}} A[v] \right\}$
Recipe for applying Dynamic Programming

• **Step 1:** Identify optimal substructure.

• **Step 2:** Find a recursive formulation for the value of the optimal solution.

• **Step 3:** Use dynamic programming to find the value of the optimal solution.

• **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual solution.
A top-down DP algorithm

• MIS_subtree(u):
  • if u is a leaf:
    • A[u] = weight(u)
    • B[u] = 0
  • else:
    • for v in u.children:
      • MIS_subtree(v)
    • A[u] = max{ \( \sum_{v \in u.\text{children}} A[v] \), \( \text{weight}(u) + \sum_{v \in u.\text{children}} B[v] \) }
    • B[u] = \( \sum_{v \in u.\text{children}} A[v] \)

• MIS(T):
  • MIS_subtree(T.root)
  • return A[T.root]

Running time?
• We visit each vertex once, and at every vertex we do O(1) work:
  • Make a recursive call
  • look stuff up in tables
• Running time is O(|V|)
Why is this different from divide-and-conquer? That’s always worked for us with tree problems before...

- **MIS_subtree(u):**
  - **if** u is a leaf:
    - **return** weight(u)
  - **else**:
    - **for** v in u.children:
      - MIS_subtree(v)
    - **return** \( \max \{ \sum_{v \in u.\text{children}} \text{MIS_subtree}(v), \text{weight}(u) + \sum_{v \in u.\text{grandchildren}} \text{MIS_subtree}(v) \} \)

- **MIS(T):**
  - **return** MIS_subtree(T.root)
Why is this different from divide-and-conquer?
That’s always worked for us with tree problems before...

How often would we ask about the subtree rooted here?

Once for this node and once for this one.

But we then ask about this node twice, here and here.

This will blow up exponentially without using dynamic programming to take advantage of overlapping subproblems.
Recipe for applying Dynamic Programming

• **Step 1:** Identify *optimal substructure*.

• **Step 2:** Find a *recursive formulation* for the value of the optimal solution.

• **Step 3:** Use *dynamic programming* to find the value of the optimal solution.

• **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can **find the actual solution**.

You do this one!
What have we learned?

• We can find maximal independent sets in trees in time $O(|V|)$ using dynamic programming!

• For this example, it was natural to implement our DP algorithm in a top-down way.
Recap

• Today we saw examples of how to come up with dynamic programming algorithms.
  • Longest Common Subsequence
  • Knapsack two ways
  • (If time) maximal independent set in trees.

• There is a recipe for dynamic programming algorithms.
Recipe for applying Dynamic Programming

• **Step 1:** Identify optimal substructure.
• **Step 2:** Find a recursive formulation for the value of the optimal solution.
• **Step 3:** Use dynamic programming to find the value of the optimal solution.
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Recap

• Today we saw examples of how to come up with dynamic programming algorithms.
  • Longest Common Subsequence
  • Knapsack two ways
  • (If time) maximal independent set in trees.

• There is a **recipe** for dynamic programming algorithms.

• Sometimes coming up with the right substructure takes some creativity
Next time

• **Midterm 2**
  
in Cubberley Auditorium

**Before next time**

• Review session:
  
  Sunday 11/4, 4-6 PM,
  
  Main Quad 420-040

• Study ("work smart")
• Get some rest the night before
Midterm 2 plan

1. A few short questions (20 pts ~ 15 min)
2. Design+analyze an algorithm (40 pts ~ 30 min)
3. Design+analyze another algorithm (40 pts ~ 30 min)

Study resources:

- Practice Problems
- Review session
- Office Hours
- Textbooks
- Piazza
- HW, Sections, Lecture notes
- Classmates

(This is just my guess!)