Lecture 13

Greedy algorithms!
Announcements

• I am not Prof. Rubinstein
  • He will be back next week
  • My name is Mary Wootters

• New HW posted today!
MIDTERM 1
Data structures
MIDTERM 2
Graphs!

More detailed schedule on the website!
This week

• Greedy algorithms!
Greedy algorithms

• Make choices one-at-a-time.
• Never look back.
• Hope for the best.
Today

• One example of a greedy algorithm that does not work:
  • Knapsack again

• Three examples of greedy algorithms that do work:
  • Activity Selection
  • Job Scheduling
  • Huffman Coding
Non-example

• Unbounded Knapsack.
• Unbounded Knapsack:
  • Suppose I have infinite copies of all of the items.
  • What’s the most valuable way to fill the knapsack?

<table>
<thead>
<tr>
<th>Item</th>
<th>Weight</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tacos</td>
<td>6</td>
<td>20</td>
</tr>
<tr>
<td>Light bulb</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>Watermelon</td>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td>Tacos</td>
<td>3</td>
<td>13</td>
</tr>
<tr>
<td>Total weight: 10</td>
<td>Total value: 42</td>
<td></td>
</tr>
</tbody>
</table>

• “Greedy” algorithm for unbounded knapsack:
  • Tacos have the best Value/Weight ratio!
  • Keep grabbing tacos!

<table>
<thead>
<tr>
<th>Item</th>
<th>Weight</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tacos</td>
<td>11</td>
<td>35</td>
</tr>
<tr>
<td>Total weight: 9</td>
<td>Total value: 39</td>
<td></td>
</tr>
</tbody>
</table>
Example where greedy works

Activity selection

You can only do one activity at a time, and you want to maximize the number of activities that you do.

What to choose?
Activity selection

• Input:
  • Activities $a_1, a_2, ..., a_n$
  • Start times $s_1, s_2, ..., s_n$
  • Finish times $f_1, f_2, ..., f_n$

• Output:
  • How many activities can you do today?

Think-pair-share!

What greedy algorithm would you use?
Greedy Algorithm

- Pick activity you can add with the smallest finish time.
- Repeat.
Greedy Algorithm

- Pick activity you can add with the smallest finish time.
- Repeat.
Greedy Algorithm

- Pick activity you can add with the smallest finish time.
- Repeat.
Greedy Algorithm

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Greedy Algorithm

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Greedy Algorithm

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Greedy Algorithm

- Pick activity you can add with the smallest finish time.
- Repeat.
Greedy Algorithm

- Pick activity you can add with the smallest finish time.
- Repeat.
At least it’s fast

• Running time:
  • $O(n)$ if the activities are already sorted by finish time.
  • Otherwise $O(n \log(n))$ if you have to sort them first.
What makes it greedy?

• At each step in the algorithm, make a choice.
  • Hey, I can increase my activity set by one,
  • And leave lots of room for future choices,
  • Let’s do that and hope for the best!!!

• **Hope** that at the end of the day, this results in a globally optimal solution.
Three Questions

1. Does this greedy algorithm for activity selection work?
   • Yes. (We will see why in a moment...)

2. In general, when are greedy algorithms a good idea?
   • When the problem exhibits especially nice optimal substructure.

3. The “greedy” approach is often the first you’d think of...
   • Why are we getting to it now, in Week 8?
     • Proving that greedy algorithms work is often not so easy...
Back to Activity Selection

- Pick activity you can add with the smallest finish time.
- Repeat.
Why does it work?

• Whenever we make a choice, we don’t rule out an optimal solution.

Our next choice would be this one:

There’s some optimal solution that contains our next choice.
Assuming we can prove that

- **We never rule out an optimal solution**
- At the end of the algorithm, we’ve got some solution.
- So it must be optimal.

Lucky the Lackadaisical Lemur
To prove we never rule out an optimal solution:

Optimal Substructure

• Subproblem i :
  • Solve the activity selection problem after Activity i finishes.
  • A[i] = Number of activities you can do after Activity i finishes.
Claim

- Let $a_k$ have the smallest finish time among activities do-able after $a_i$ finishes.

A[k]: how many activities can I do here?

A[i]: how many activities can I do here?
Proof

• Clearly $A[i] \geq A[k] + 1$
  • Since we have a solution with $A[k] + 1$ activities.

• Let $a_k$ have the smallest finish time among activities do-able after $a_i$ finishes.
Proof

• Now we’ll show $A[i] \leq A[k] + 1$.
• Say there’s an optimal solution to subproblem(i) that doesn’t use $a_k$
• Say $a_j$ ends first after $a_i$ in that better solution.
• Remove $a_j$ and add $a_k$ from the better solution.
• Now you have a solution of the same size...
  but it includes $a_k$ so it must have size $\leq A[k] + 1$.

• Let $a_k$ have the smallest finish time among activities do-able after $a_i$ finishes.
Proof

• Clearly $A[i] \geq A[k] + 1$
  • Since we have a solution with $A[k] + 1$ activities.
• And $A[i] \leq A[k] + 1$
  • We just showed that.
• So $A[i] = A[k] + 1$

Let $a_k$ have the smallest finish time among activities do-able after $a_i$ finishes.
We never rule out an optimal solution

- Suppose that after adding $a_i$, there is an optimal solution:
  - $X$ activities done and $A[i]$ activities left.
  - $X + A[i]$ activities total.
- Then we add the next thing, $a_k$
- $A[k] = A[i] - 1$ (by the claim)
- Now there is some solution extending this one so that:
  - $X+1$ activities done and $A[i] - 1$ activities left.
  - $X + A[i]$ activities total.
  - Still optimal.
So the algorithm is correct

• We never rule out an optimal solution
• At the end of the algorithm, we’ve got some solution.
• So it must be optimal.
So the algorithm is correct

• Inductive Hypothesis:
  • After adding the t’th thing, there is an optimal solution that extends the current solution.

• Base case:
  • After adding zero activities, there is an optimal solution extending that.

• Inductive step:  
  • TO DO

• Conclusion:
  • After adding the last activity, there is an optimal solution that extends the current solution.
  • The current solution is the only solution that extends the current solution.
  • So the current solution is optimal.
Need to show in the inductive step

• Suppose that after adding the t’th thing \((a_i)\), there is an optimal solution extending the current solution.

• Then after adding the next thing \((a_k)\), there is still an optimal solution extending the current solution.

• (Aka, we don’t rule out an optimal solution).

We just did that!
So the algorithm is correct

• Inductive Hypothesis:
  • After adding the t’th thing, there is an optimal solution that extends the current solution.

• Base case:
  • After adding zero activities, there is an optimal solution extending that.

• Inductive step:
  • TO DO

• Conclusion:
  • After adding the last activity, there is an optimal solution that extends the current solution.
  • The current solution is the only solution that extends the current solution.
  • So the current solution is optimal.
Common strategy for greedy algorithms

• Make a **series of choices**.

• Show that, at each step, our choice **won’t rule out an optimal solution** at the end of the day.

• After we’ve made all our choices, we haven’t ruled out an optimal solution, **so we must have found one**.
Common strategy (formally) for greedy algorithms

• Inductive Hypothesis:
  • After greedy choice $t$, you haven’t ruled out success.

• Base case:
  • Success is possible before you make any choices.

• Inductive step:
  • TODO

• Conclusion:
  • If you reach the end of the algorithm and haven’t ruled out success then you must have succeeded.
Relationship to other sorts of algorithms

- This algorithm is most naturally viewed as a greedy algorithm.
  - Make greedy choices
  - Never rule out success

- We could also view it as a divide-and-conquer algorithm...
  - Without the “dividing” part.

- We could also view it as a DP algorithm...
  - Take advantage of optimal sub-structure and fill in a table.
  - See hidden slides for more detail...
More detail: Greedy vs. DP

- Just for pedagogy!
- (This isn’t the best way to think about activity selection).
Recipe for applying Dynamic Programming

• **Step 1:** Identify **optimal substructure**.

• **Step 2:** Find a **recursive formulation** for the value of the optimal solution.

• **Step 3:** Use **dynamic programming** to find the value of the optimal solution.

• **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can **find the actual solution**.

• **Step 5:** If needed, **code this up like a reasonable person**.
Optimal substructure

• Subproblem \( i \):
  • \( A[i] \) = number of activities you can do after Activity \( i \) finishes.
Recipe for applying Dynamic Programming

• **Step 1:** Identify optimal substructure.

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• **Step 5:** If needed, code this up like a reasonable person.
We did that already

• Let $a_k$ have the smallest finish time among activities do-able after $a_i$ finishes.

A[k]: how many activities can I do here?

A[i]: how many activities can I do here?
Recipe for applying Dynamic Programming

• **Step 1:** Identify optimal substructure.

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• **Step 5:** If needed, code this up like a reasonable person.
Top-down DP

• Initialize a global array A to [None,...,None]
• Make a “dummy” activity that ends at time -1.

• **def** findNumActivities(i):
  • If A[i] != None:
    • **Return** A[i]
  • Let Activity k be the activity I can fit in my schedule after Activity i with the smallest finish time.
  • **If** there is no such activity k, set A[i] = 0
  • **Else**, A[i] = findNumActivities(k) + 1
  • **Return** A[i]

• **Return** findNumActivities(0)

This is a terrible way to write this!
The only thing that matters here is that the highlighted lines are our recursive relationship.
Recipe for applying Dynamic Programming

- **Step 1:** Identify **optimal substructure**.
- **Step 2:** Find a **recursive formulation** for the value of the optimal solution.
- **Step 3:** Use **dynamic programming** to find the value of the optimal solution.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can **find the actual solution**.
- **Step 5:** If needed, **code this up like a reasonable person**.
Top-down DP

- Initialize a global array $A$ to [None, ..., None]
- Initialize a global array $\text{Next}$ to [None, ..., None]
- Make a “dummy” activity that ends at time -1.
- \textbf{def} findNumActivities($i$):
  - If $A[i] \neq \text{None}$:
    - \textbf{Return} $A[i]$
  - Let Activity $k$ be the activity I can fit in my schedule after Activity $i$ with the smallest finish time.
  - \textbf{If} there is no such activity $k$, set $A[i] = 0$
  - \textbf{Else}, $A[i] = \text{findNumActivities}(k) + 1$ and $\text{Next}[i] = k$
  - \textbf{Return} $A[i]$
- findNumActivities(0)
- Step through “Next” array to get schedule.

This is a terrible way to write this!
The only thing that matters here is that the highlighted lines are our recursive relationship.
Let’s step through it.

- Start with the activity with the smallest finish time.
Let’s step through it

- Now find the next activity still do-able with the smallest finish time, and recurse after that.
Let’s step through it

- Now find the next activity still do-able with the smallest finish time, and recurse after that.
Let’s step through it

- Now find the next activity still do-able with the smallest finish time, and recurse after that.
Let’s step through it

• Ta-da!

It’s exactly the same* as the greedy solution!

*if you implement the top-down DP solution appropriately.
Sub-problem graph view

• Divide-and-conquer:
Sub-problem graph view

• Dynamic Programming:
Sub-problem graph view

• Greedy algorithms:
Sub-problem graph view

• Greedy algorithms:

  - Not only is there **optimal sub-structure**: 
    - optimal solutions to a problem are made up from optimal solutions of sub-problems
  
  - but each problem **depends on only one sub-problem**.
Three Questions

1. Does this greedy algorithm for activity selection work?
   • Yes.

2. In general, when are greedy algorithms a good idea?
   • When they exhibit especially nice optimal substructure.

3. The “greedy” approach is often the first you’d think of...
   • Why are we getting to it now, in Week 8?
     • Proving that greedy algorithms work is often not so easy.
Let’s see a few more examples
Another example:
Scheduling

- CS161 HW!
- Call your parents!
- Math HW!
- Administrative stuff for your student club!
- Econ HW!
- Do laundry!
- Meditate!
- Practice musical instrument!
- Read CLRS!
- Have a social life!
- Sleep!

Overcommitted Stanford Student
Scheduling

- n tasks
- Task i takes $t_i$ hours
- **Everything is already late!**
  - For every hour that passes until task i is done, pay $c_i$

- CS161 HW, then Sleep: \[\text{costs } 10 \cdot 2 + (10 + 8) \cdot 3 = 74 \text{ units}\]
- Sleep, then CS161 HW: \[\text{costs } 8 \cdot 3 + (10 + 8) \cdot 2 = 60 \text{ units}\]
Optimal substructure

• This problem breaks up nicely into sub-problems:

Suppose this is the optimal schedule:

Job A  Job B  Job C  Job D

Then this must be the optimal schedule on just jobs B,C,D.

Think-pair-share
Optimal substructure

• Seems amenable to a greedy algorithm:

Take the best job first

Job A

Take the best job first

Job B

Then solve this problem

Job C

Job D

Take the best job first

Job C

Take the best job first

Job B

Then solve this problem

Job D

(That one’s easy 😊)
What does “best” mean?

• Of these two jobs, which should we do first?

![](成本: \(z\) 单位每小时直到完成。

Cost: \(w\) 单位每小时直到完成。

What matters is the ratio:

\[
\frac{\text{cost of delay}}{\text{time it takes}}
\]

“Best” means biggest ratio.

\[
\begin{align*}
\text{Cost}(\ A \text{ then } B\ ) & = x \cdot z + (x + y) \cdot w \\
\text{Cost}(\ B \text{ then } A\ ) & = y \cdot w + (x + y) \cdot z
\end{align*}
\]

A then B is better than B then A when:

\[
\begin{align*}
xz + (x + y)w & \leq yw + (x + y)z \\
xz + xw + yw & \leq yw + xz + yz \\
xz + xw + yw & \leq yw + xz + yz
\end{align*}
\]

\[
\begin{align*}
wz & \leq yz \\
w & \leq z \\
y & \leq x
\end{align*}
\]
Lemma

• Given jobs so that Job \( i \) takes time \( t_i \) with cost \( c_i \),
• There is an optimal schedule so that the first job is the one that maximizes the ratio \( c_i/t_i \)

• Proof:
  • Say Job B maximizes this ratio, and it’s not first:
    • Switch A and B! Nothing else will change, and we showed on the previous slide that the cost won’t increase.

\[
\begin{align*}
\text{Job C} & \quad \text{Job A} & \quad \text{Job B} & \quad \text{Job D} \\
\frac{c_A}{t_A} & \geq \frac{c_B}{t_B}
\end{align*}
\]

• Repeat until B is first.
Choose greedily:
Biggest cost/time ratio first

• **Job i** takes time $t_i$ with cost $c_i$

• There is an optimal schedule so that the first job is the one that maximizes the ratio $c_i/t_i$

• So if we choose jobs greedily according to $c_i/t_i$, we never rule out success!
Greedy Scheduling Solution

• scheduleJobs( JOBS ):
  • Sort JOBS by the ratio:
    • \( r_i = \frac{c_i}{t_i} = \frac{\text{cost of delaying job } i}{\text{time job } i \text{ takes to complete}} \)
  • Say that \textbf{sorted_JOBS}[i] is the job with the i’th biggest \( r_i \)
  • Return \textbf{sorted_JOBS}

The running time is \( O(n\log(n)) \)

Now you can go about your schedule peacefully, in the optimal way.
Formally, use induction!

- **Inductive hypothesis:**
  - There is an optimal ordering so that the first t jobs are `sorted_JOBS[:t]`.

- **Base case:**
  - When t=0, this reads: “There is an optimal ordering so that the first 0 jobs are []”
  - That’s true.

- **Inductive Step:**
  - Boils down to: there is an optimal ordering on `sorted_JOBS[t:]` so that `sorted_JOBS[t]` is first.
  - This follows from the Lemma.

- **Conclusion:**
  - When t=n, this reads: “There is an optimal ordering so that the first n jobs are `sorted_JOBS`.”
  - aka, what we returned is an optimal ordering.
What have we learned?

• A greedy algorithm works for scheduling

• This followed the same outline as the previous example:
  • Identify **optimal substructure**:
    • Find a way to make choices that **won’t rule out an optimal solution**.
      • largest ratios first.
One more example

Huffman coding

• everyday english sentence
  • 01100101 01110110 01100101 01110010 01111001 01100100 01100001 01111001 00100000 01100101 01101110 01100111 01101100 01101001 01110011 01101000 00100000 01110011 01100101 01101110 01100111 01101100 01100101 01101110 01100011 01100101

• qwertyui_opasdfg+hjklzxcv
  • 01110001 01110111 01100101 01110010 01110100 01110011 01110101 01101001 01011111 01101111 01110000 01100001 01110011 01100100 01100110 01100111 00101011 01101000 01101010 01101011 01101100 01111010 01111000 01100011 01110110
One more example

Huffman coding

• everyday english sentence
  • 01100101 01110110 01100101 01110010 01111001 01100100 01100001
  01111001 00100000 01100101 01101110 01100111 01101100 01101001
  01110011 01101000 00100000 01110011 01100101 01100110 01110010
  01100101 01101110 01100011 01100101

• qwertyui_opasdfg+hjklzxcv
  • 01110001 01110111 01100101 01110010 01110100 01111001 01110101
  01101001 01011111 01101111 01110000 01100001 01110011 01100100
  01100110 01100111 00101011 01101000 01101010 01101011 01101100
  01111010 01111000 01100011 01110110
Suppose we have some distribution on characters.
Suppose we have some distribution on characters

How to encode them as efficiently as possible?
Try 0 (like ASCII)

- Every letter is assigned a binary string of three bits.

**Wasteful!**
- 110 and 111 are never used.
- We should have a shorter way of representing A.
Try 1

- Every letter is assigned a **binary string** of one or two bits.
- The more frequent letters get the shorter strings.
- **Problem:**
  - Does 000 mean AAA or BA or AB?
Try 2: prefix-free coding

- Every letter is assigned a **binary string**.
- More frequent letters get shorter strings.
- No encoded string is a **prefix** of any other.

Confusingly, “prefix-free codes” are also sometimes called “prefix codes” (including in CLRS).
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Try 2: prefix-free coding

- Every letter is assigned a **binary string**.
- More frequent letters get shorter strings.
- No encoded string is a **prefix** of any other.

![Bar graph showing percentages and binary codes for letters A to F.]

<table>
<thead>
<tr>
<th>Letter</th>
<th>Percentage</th>
<th>Binary Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>45</td>
<td>01</td>
</tr>
<tr>
<td>B</td>
<td>13</td>
<td>101</td>
</tr>
<tr>
<td>C</td>
<td>12</td>
<td>110</td>
</tr>
<tr>
<td>D</td>
<td>16</td>
<td>00</td>
</tr>
<tr>
<td>E</td>
<td>9</td>
<td>111</td>
</tr>
<tr>
<td>F</td>
<td>5</td>
<td>100</td>
</tr>
</tbody>
</table>

10010101 = F
Confusingly, “prefix-free codes” are also sometimes called “prefix codes” (including in CLRS).

Try 2: prefix-free coding

- Every letter is assigned a binary string.
- More frequent letters get shorter strings.
- No encoded string is a prefix of any other.

A: 01, 45%
B: 101, 13%
C: 110, 12%
D: 00, 16%
E: 111, 9%
F: 100, 5%

10010101 FB
Try 2: prefix-free coding

- Every letter is assigned a **binary string**.
- More frequent letters get shorter strings.
- No encoded string is a **prefix** of any other.

Confusingly, “prefix-free codes” are also sometimes called “prefix codes” (including in CLRS).

**Question**: What is the most efficient way to do prefix-free coding? (This isn’t it).

```
100101101
```

FBA

<table>
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Letter

<table>
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<td>110</td>
</tr>
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<td>D</td>
<td>00</td>
</tr>
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<td>111</td>
</tr>
<tr>
<td>F</td>
<td>100</td>
</tr>
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</table>
A prefix-free code is a tree

As long as all the letters show up as leaves, this code is prefix-free.
How good is a tree?

- Imagine choosing a letter at random from the language.
- Not uniform, but according to our histogram!
- The **cost of a tree** is the expected length of the encoding of that letter.

Expected cost of encoding a letter with this tree:

\[2(0.45 + 0.16) + 3(0.05 + 0.13 + 0.12 + 0.09) = 2.39\]
Question

• Given a distribution $P$ on letters, find the lowest-cost tree, where

$$
\text{cost(tree)} = \sum_{\text{leaves } x} P(x) \cdot \text{depth}(x)
$$

- $P(x)$ is the probability of letter $x$
- The depth in the tree is the length of the encoding
Optimal sub-structure

• Suppose this is an optimal tree:

Then this is an optimal tree on fewer letters.

Otherwise, we could change this sub-tree and end up with a better overall tree.
Greedy algorithm

• Greedily build sub-trees from the bottom up.
• Greedy goal: less frequent letters should be further down the tree.
Solution
greedily build subtrees, starting with the infrequent letters
Solution

greedily build subtrees, starting with the infrequent letters

A: 45  B: 13  C: 12
D:  16  E:  9  F:  5
Solution

greedily build subtrees, starting with the infrequent letters
Solution

greedily build subtrees, starting with the infrequent letters
Solution

greedily build subtrees, starting with the infrequent letters
Solution

greedily build subtrees, starting with the infrequent letters

<table>
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Expected cost of encoding a letter:

\[ 1 \cdot 0.45 + 3 \cdot 0.41 + 4 \cdot 0.14 = 2.24 \]
What exactly was the algorithm?

- Create a node like [D: 16] for each letter/frequency
  - The key is the frequency (16 in this case)
- Let [CURRENT] be the list of all these nodes.
- while len([CURRENT]) > 1:
  - X and Y ← the nodes in [CURRENT] with the smallest keys.
  - Create a new node Z with Z.key = X.key + Y.key
  - Set Z.left = X, Z.right = Y
  - Add Z to [CURRENT] and remove X and Y
- return [CURRENT][0]
Does it work?

• Yes.

• Same strategy:
  • Show that at each step, the choices we are making won’t rule out an optimal solution.
  • Lemma:
    • Suppose that x and y are the two least-frequent letters. Then there is an optimal tree where x and y are siblings.
**Lemma**

**proof idea**

- Say that an optimal tree looks like this:

![Diagram of a tree with nodes labeled x, a, and y.](image)

- What happens to the cost if we swap $x$ for $a$?
  - the cost can’t increase; $a$ was more frequent than $x$, and we just made its encoding shorter.

- Repeat this logic until we get an optimal tree with $x$ and $y$ as siblings.
  - The cost never increased so this tree is still optimal.

If $x$ and $y$ are the two least-frequent letters, there is an optimal tree where $x$ and $y$ are siblings.

- Lowest-level sibling nodes: at least one of them is neither $x$ nor $y$
Lemma proof idea

• Say that an optimal tree looks like this:
  ![Optimal Tree Diagram]

• What happens to the cost if we swap $x$ for $a$?
  • the cost can’t increase; $a$ was more frequent than $x$, and we just made its encoding shorter.

• Repeat this logic until we get an optimal tree with $x$ and $y$ as siblings.
  • The cost never increased so this tree is still optimal.
Proof strategy
just like before

• Show that at each step, the choices we are making \textbf{won't rule out} an optimal solution.

• Lemma:
  • Suppose that \(x\) and \(y\) are the two least-frequent letters. Then there is an optimal tree where \(x\) and \(y\) are siblings.
Proof strategy
just like before

• Show that at each step, the choices we are making won’t rule out an optimal solution.

• Lemma:
  • Suppose that x and y are the two least-frequent letters. Then there is an optimal tree where x and y are siblings.

That’s enough to show that we don’t rule out optimality after the first step.
What about once we start grouping stuff?

<table>
<thead>
<tr>
<th></th>
<th>A: 45</th>
<th>B:13</th>
<th>C:12</th>
<th>D: 16</th>
<th>E:9</th>
<th>F:5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>16</td>
<td>13</td>
<td>12</td>
<td>16</td>
<td>9</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>30</td>
<td>25</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>14</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Lemma 2
this distinction doesn’t really matter

The first thing is an optimal tree on \{A,B,C,D,E,F\}
if and only if
the second thing is an optimal tree on \{A,G,H\}
Lemma 2
this distinction doesn’t really matter

• For a proof:
  • See CLRS, Lemma 16.3
    • Rigorous although presented in a slightly different way
  • See the Lecture Notes
    • A bit sketchier, but presented in the same way as here
  • Prove it yourself!
    • This is the best!

Getting all the details isn’t that important, but you should convince yourself that this is true.

Siggi the Studious Stork
Together

• Lemma 1:
  • Suppose that \( x \) and \( y \) are the two least-frequent letters. Then there is an optimal tree where \( x \) and \( y \) are siblings.

• Lemma 2:
  • We may as well imagine that CURRENT contains only leaves.

• These imply:
  • At each step, our choice doesn’t rule out an optimal tree.

Write this out formally as a proof by induction! (See hidden slides for a starting point).

Siggi the Studious Stork
The whole argument

After the t’th step, we’ve got a bunch of current sub-trees:

• Inductive hypothesis:
  • after the t’th step,
    • there is an optimal tree containing the current subtrees as “leaves”

• Base case:
  • after the 0’th step,
    • there is an optimal tree containing all the characters.

• Inductive step:
  • TO DO

• Conclusion:
  • after the last step,
    • there is an optimal tree containing this whole tree as a subtree.
  • aka,
    • after the last step the tree we’ve constructed is optimal.
**Inductive step**

• Suppose that the inductive hypothesis holds for $t-1$
  • After $t-1$ steps, there is an optimal tree containing all the current sub-trees as “leaves.”

• Want to show:
  • After $t$ steps, there is an optimal tree containing all the current sub-trees as leaves.
Inductive step

• Suppose that the inductive hypothesis holds for $t-1$
  • After $t-1$ steps, there is an optimal tree containing all the current sub-trees as “leaves.”

• By Lemma 2, may as well treat $a$ as 

We’ve got a bunch of current sub-trees: say that $x$ and $y$ are the two smallest.
Inductive step

• Suppose that the inductive hypothesis holds for t-1
  • After t-1 steps, there is an optimal tree containing all the current sub-trees as “leaves.”

• By Lemma 2, may as well treat \( a \) as \( a \)

• In particular, optimal trees on this new alphabet correspond to optimal trees on the original alphabet.
Inductive step

• Suppose that the inductive hypothesis holds for \( t-1 \)
  • After \( t-1 \) steps, there is an optimal tree containing all the current sub-trees as “leaves.”

• Our algorithm would do this at level \( t \):

We’ve got a bunch of current sub-trees:

\[ w \quad z \quad x \quad y \]

say that \( x \) and \( y \) are the two smallest.

\[ a = x + y \]
Inductive step

• Suppose that the inductive hypothesis holds for t-1
  • After t-1 steps, there is an optimal tree containing all the current sub-trees as “leaves.”

Lemma 1 implies that there’s an optimal sub-tree that looks like this; aka, what our algorithm did okay.

• Our algorithm would do this at level t:
Inductive step

• Suppose that the inductive hypothesis holds for t-1
  • After t-1 steps, there is an optimal tree containing all the current sub-trees as “leaves.”

• Our algorithm would do this at level t:

  Lemma 2 again says that there’s an optimal tree that looks like this.
Inductive step

- Suppose that the inductive hypothesis holds for \( t-1 \)
  - After \( t-1 \) steps, there is an optimal tree containing all the current sub-trees as “leaves.”

- Our algorithm would do this at level \( t \):
  - We’ve got a bunch of current sub-trees:
  - Lemma 2 again says that there’s an optimal tree that looks like this
  - aka, there is an optimal tree containing all the level-\( t \) sub-trees as “leaves.”
  - This is what we wanted to show for the inductive step.
Inductive outline:

After the t’th step, we’ve got a bunch of current sub-trees:

• Inductive hypothesis:
  • after the t’th step,
    • there is an optimal tree containing the current subtrees as “leaves”

• Base case:
  • after the 0’th step,
    • there is an optimal tree containing all the vertices.

• Inductive step:
  • TO DO

• Conclusion:
  • after the last step,
    • there is an optimal tree containing this whole tree as a subtree.
  • aka,
    • after the last step the tree we’ve constructed is optimal.
What have we learned?

• ASCII isn’t an optimal way* to encode English, since the distribution on letters isn’t uniform.
• Huffman Coding is an optimal way!
• To come up with an optimal scheme for any language efficiently, we can use a greedy algorithm.

• To come up with a greedy algorithm:
  • Identify optimal substructure
  • Find a way to make choices that won’t rule out an optimal solution.
    • Create subtrees out of the smallest two current subtrees.

*If all we care about is number of bits.
Recap I

• Greedy algorithms!

• Three examples:
  • Activity Selection
  • Scheduling Jobs
  • Huffman Coding
Recap II

- Greedy algorithms!
- Often easy to write down
  - But may be hard to come up with and hard to justify
- The natural greedy algorithm may not always be correct.
- A problem is a good candidate for a greedy algorithm if:
  - it has optimal substructure
  - that optimal substructure is **REALLY NICE**
    - solutions depend on just one other sub-problem.
Next time

• Greedy algorithms for Minimum Spanning Tree!

Before next time

• Go to section!