Lecture 6
Binary Search Trees and Red-Black Trees
Sorting out the past couple of weeks:

We’ve seen a bunch of sorting algorithms
- **InsertionSort** - $\Theta(n^2)$
- **MergeSort** - $\Theta(n \log n)$
- **QuickSort** - $\Theta(n \log n)$ expected; $\Theta(n^2)$ worst case
- **RadixSort** - $\Theta((n + r) \log_r M)$

And a couple of silly ones:
- **BogoSort** - $\Theta(n!)$ expected; $\infty$ worst case
- **StickSort** - $\Theta(1)$
They all looked sort of like this:

Input: 6 4 3 8 1 5 2 7

SortingAlgorithm

Output: 1 2 3 4 5 6 7 8
But what happens if your input doesn’t arrive all at once?

**Input**: 6 4 3 8 1 5 2 7

**SortingAlgorithm**

**Output**: 1 2 3 4 5 6 7 8
And then some inputs also leave...?

**Input**

6 4 3 8 1 5 2 7

**SortingAlgorithm**

**Output**

1 2 3 4 5 6 7 8
Actually, we already saw something similar... but it was too slow ($\Theta(n^2)$)

InsertionSort(A):
  for i in [1:n]
    current ← A[i]
    j ← i-1
    while j >= 0 and A[j] > current:
      j ← j-1
    A[j+1] ← current

In the worst case, about $n$ iterations of this inner loop
What do we mean by “sorted”?

- Sorted arrays?
  
- Sorted linked lists?

- Sorted sticks?
Today

- Begin a brief foray into **data structures**!
  - See CS 166 for more!
- Binary search trees
  - You may remember these from CS 106B
  - They are better when they’re balanced.

**this will lead us to…**

- Self-Balancing Binary Search Trees
  - **Red-Black** trees.
Why are we studying self-balancing BSTs?

1. The punchline is important:
   • A data structure with $O(\log(n))$ INSERT/DELETE/SEARCH

2. The idea behind Red-Black Trees is clever
   • It’s good to be exposed to clever ideas.
   • Also it’s just aesthetically pleasing.
Some data structures for storing objects like 5 (aka, nodes with keys)

- (Sorted) arrays:

```
1 2 3 4 5 7 8
```

- (Sorted) linked lists:

```
HEAD 1 → 2 → 3 → 4 → 5 → 7 → 8
```

- Some basic operations:
  - INSERT, DELETE, SEARCH
Sorted Arrays

- **O(log(n))** SEARCH:
  
  eg, Binary search to see if 3 is in A.

- **O(n)** INSERT/DELETE:
Sorted linked lists

- **O(1) INSERT/DELETE:**
  - (assuming we have a pointer to the location of the insert/delete)

- **O(n) SEARCH:**
## Motivation for Binary Search Trees

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Binary tree terminology

Each node has two children

This node is the root

This is a node. It has a key (7).

Each node has a pointer to its left child, right child, and parent.

For today all keys are distinct.

The left child of 3 is 2

The right child of 3 is 4

Both children of 1 are NIL

These nodes are leaves.
Binary Search Trees

• It’s a **binary tree** so that:
  • Every LEFT descendant of a node has key less than that node.
  • Every RIGHT descendant of a node has key larger than that node.

• Example of building a binary search tree:

```
3                4
  
8                5
  
1                7
  
2
```
Binary Search Trees

• It’s a **binary tree** so that:
  • Every LEFT descendant of a node has key less than that node.
  • Every RIGHT descendant of a node has key larger than that node.
• Example of building a binary search tree:
Binary Search Trees

• It’s a binary tree so that:
  • Every LEFT descendant of a node has key less than that node.
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• Example of building a binary search tree:
Binary Search Trees

• It’s a **binary tree** so that:
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Binary Search Trees

• It’s a **binary tree** so that:
  • Every LEFT descendant of a node has key less than that node.
  • Every RIGHT descendant of a node has key larger than that node.

• Example of building a binary search tree:

Q: Is this the only binary search tree I could possibly build with these values?

A: **No.** I made choices about which nodes to choose when. Any choices would have been fine.
Aside: this should look familiar
kinda like QuickSort
Binary Search Trees

- It’s a binary tree so that:
  - Every LEFT descendant of a node has key less than that node.
  - Every RIGHT descendant of a node has key larger than that node.

Which of these is a BST?

- Binary Search Tree
- NOT a Binary Search Tree
Remember the goal

Fast **SEARCH**/**INSERT**/**DELETE**

Can we do these?
SEARCH in a Binary Search Tree
definition by example

EXAMPLE: Search for 4.

EXAMPLE: Search for 4.5
- It turns out it will be convenient to return 4 in this case
- (that is, return the last node before we went off the tree)

How long does this take?
$O(\text{length of longest path}) = O(\text{height})$

Write pseudocode (or actual code) to implement this!
EXAMPLE: Insert 4.5

- **INSERT**(key):
  - \( x = \text{SEARCH}(\text{key}) \)
  - **Insert** a new node with desired key at \( x \)...
**INSERT in a Binary Search Tree**

**EXAMPLE:** Insert 4.5

- **INSERT**(key):
  - x = **SEARCH**(key)
  - **if** key > x.key:
    - Make a new node with the correct key, and put it as the right child of x.
  - **if** key < x.key:
    - Make a new node with the correct key, and put it as the left child of x.
  - **if** x.key == key:
    - return
DELETE in a Binary Search Tree

EXAMPLE: Delete 2

- DELETE(key):
  - x = SEARCH(key)
  - if x.key == key:
    - ....delete x....
DELETE in a Binary Search Tree
several cases (by example)
say we want to delete 3

Case 1: if 3 is a leaf, just delete it.

Case 2: if 3 has just one child, move that up.

Write pseudocode for all of these!
DELETE in a Binary Search Tree

ctd.

**Case 3:** if 3 has two children, replace 3 with its immediate successor. (aka, next biggest thing after 3)

- Does this maintain the BST property?
  - Yes.
- How do we find the immediate successor?
  - SEARCH for 3 in the subtree under 3.right
- How do we remove it when we find it?
  - If [3.1] has 0 or 1 children, do one of the previous cases.
- What if [3.1] has two children?
  - It doesn’t.
How long do these operations take?

Think-Pair-Share!
How long do these operations take?

• **SEARCH** is the big one.
  • Everything else just calls **SEARCH** and then does some small $O(1)$-time operation.

Time = $O$(height of tree)

Trees have depth $O$(log(n)). **Done!**
Wait...

- This is a valid binary search tree.
- The version with n nodes has depth n, **not** $O(\log(n))$.

**Could such a tree show up?**
In what order would I have to insert the nodes?

Inserting in the order 2,3,4,5,6,7,8 would do it.

So this **could** happen.
What to do?

• Goal: Fast **SEARCH/INSERT/DELETE**
• All these things take time \(O(\text{height})\)
• And the height might be big!!! 😞

• Idea 0:
  • Keep track of how deep the tree is getting.
  • If it gets too tall, re-do everything from scratch.
    • At least \(\Omega(n)\) every so often....

• Turns out that’s not a great idea. Instead we turn to...
Self-Balancing Binary Search Trees
Idea 1: Rotations

- Maintain Binary Search Tree (BST) property, while moving stuff around.

**Claim:**
No matter what lives underneath A, B, C, this takes time $O(1)$. (Why?)

Note: A, B, C, X, Y are variable names, not the contents of the nodes.
This seems helpful
Does this work?

• Whenever something seems unbalanced, do rotations until it’s okay again.

Even for me this is pretty vague. What do we mean by “seems unbalanced”? What’s “okay”?

Lucky the Lackadaisical Lemur
Idea 2: have some proxy for balance

• Maintaining perfect balance is too hard.
• Instead, come up with some proxy for balance:
  • If the tree satisfies [SOME PROPERTY], then it’s pretty balanced.
  • We can maintain [SOME PROPERTY] using rotations.

There are actually several ways to do this, but today we’ll see...
Red-Black Trees

- A Binary Search Tree that balances itself!
- No more time-consuming by-hand balancing!
- Be the envy of your friends and neighbors with the time-saving...

Red-Black tree!

Maintain balance by stipulating that black nodes are balanced, and that there aren’t too many red nodes.

It’s just good sense!
Red-Black Trees
these rules are the proxy for balance

- Every node is colored **red** or **black**.
- The root node is a **black node**.
- NIL children count as **black nodes**.
- Children of a **red node** are **black nodes**.
- For all nodes \( x \):
  - all paths from \( x \) to NIL’s have the same number of **black nodes** on them.

I’m not going to draw the NIL children in the future, but they are treated as black nodes.
Examples(?)

• Every node is colored red or black.
• The root node is a black node.
• NIL children count as black nodes.
• Children of a red node are black nodes.
• For all nodes x:
  • all paths from x to NIL’s have the same number of black nodes on them.
Why???????

• This is pretty balanced.
  • The black nodes are balanced
  • The red nodes are “spread out” so they don’t mess things up too much.

• We can maintain this property as we insert/delete nodes, by using rotations.

This is the really clever idea!
This Red-Black structure is a proxy for balance.
It’s just a smidge weaker than perfect balance, but we can actually maintain it!
This is “pretty balanced”

• To see why, intuitively, let’s try to build a Red-Black Tree that’s unbalanced.

Conjecture: the height of a red-black tree is at most $2 \log(n)$
That turns out to be basically right.

[proof sketch]

- Say there are $b(x)$ black nodes in any path from $x$ to NIL.
  - (excluding $x$, including NIL).

- **Claim:**
  - Then there are at least $2^{b(x)} - 1$ non-NIL nodes in the subtree underneath $x$. (Including $x$).

- [Proof by induction — on board if time]

Then:

$$n \geq 2^{b(root)} - 1$$

using the **Claim**

$$\geq 2^{\text{height}/2} - 1$$

$b(root) \geq \text{height}/2$ because of RBTree rules.

Rearranging:

$$n + 1 \geq 2^{\text{height}/2} \implies \text{height} \leq 2\log(n + 1)$$
Okay, so it’s balanced...

...but can we maintain it?

• Yes!

• For the rest of lecture:
  • sketch of how we’d do this.

• See CLRS for more details.

• (You are not responsible for the details for this class – but you should understand the main ideas).
Many cases

• Suppose we want to insert here.
  • eg, want to insert 0.

• And then there are 9 more cases for all of the various symmetries of these 3 cases...
Inserting into a Red-Black Tree

- Make a new red node.
- Insert it as you would normally.

What if it looks like this?

Example: insert 0
Many cases

• Suppose we want to insert here.
  • eg, want to insert 0.

• And then there are 9 more cases for all of the various symmetries of these 3 cases...
Inserting into a Red-Black Tree

- Make a new red node.
- Insert it as you would normally.
- Fix things up if needed.

What if it looks like this?

Example: insert 0

No!
Inserting into a Red-Black Tree

• Make a new red node.
• Insert it as you would normally.
• Fix things up if needed.

Example: insert 0

Can’t we just insert 0 as a black node?

No!
We need a bit more context

What if it looks like this?

Example: insert 0
We need a bit more context

- Add 0 as a red node.

What if it looks like this?

Example: insert 0
We need a bit more context

- Add 0 as a red node.
- **Claim:** RB-Tree properties still hold.

Example: insert 0

What if it looks like this?

Flip colors!
But what if that was red?

What if it looks like this?

Example: insert 0

And we wanted to flip colors...
More context...

What if it looks like this?

Example: insert 0
More context...

What if it looks like this?

Example: insert 0

Now we’re basically inserting 6 into some smaller tree. Recurse!
Many cases

• Suppose we want to insert here.
  • eg, want to insert 0.

• And then there are 9 more cases for all of the various symmetries of these 3 cases...
Inserting into a Red-Black Tree

- Make a new red node.
- Insert it as you would normally.
- Fix things up if needed.

Example: Insert 0.
- Actually, this can’t happen?
  - 6-3 path has one black node
  - 6-7-... has at least two
- It might happen that we just turned 0 red from the previous step.
- Or it could happen if 7 is actually NIL.
Recall Rotations

• Maintain Binary Search Tree (BST) property, while moving stuff around.

CLAIM: this still has BST property.
Inserting into a Red-Black Tree

• Make a new red node.
• Insert it as you would normally.
• Fix things up if needed.

What if it looks like this?

YOINK!

Need to argue that if RB-Tree property held before, it still does.
Many cases

- Suppose we want to insert here.
  - eg, want to insert 0.

- And then there are 9 more cases for all of the various symmetries of these 3 cases...
Deleting from a Red-Black tree

Fun exercise!

Ollie the over-achieving ostrich
That’s a lot of cases

• You are **not responsible** for the nitty-gritty details of Red-Black Trees. (For this class)
  • Though implementing them is a great exercise!

• You should know:
  • What are the properties of an RB tree?
  • And (more important) why does that guarantee that they are balanced?
What was the point again?

• Red-Black Trees *always* have height at most $2\log(n+1)$.
• As with general [Binary Search Trees](#), all operations are $O(\text{height})$
  
• So all operations are $O(\log(n))$. 
## Conclusion: The best of both worlds

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Aside: many ways to balance trees!

AVL trees:
\[
balance = \text{height(left)} - \text{height(right)} \in \{-1,0,1\}
\]

(2,3) trees:
* root-leaf distance same for all leaves
* some nodes may have 3 children
Today

• Begin a brief foray into data structures!
  • See CS 166 for more!

• Binary search trees
  • You may remember these from CS 106B
  • They are better when they’re balanced.

this will lead us to...

• Self-Balancing Binary Search Trees
  • Red-Black trees.
Recap

- **Balanced binary trees** are the best of both worlds!
- But we need to **keep them balanced**.
- **Red-Black Trees** do that for us.
  - We get $O(\log(n))$-time INSERT/DELETE/SEARCH
  - Clever idea: have a proxy for balance
Next time

• **Midterm 1**
  
  *in CEMEX Auditorium*

Before next time

• Review session:
  Sunday 10/14, 4-6 PM,
  Main Quad 420-040

• Study (“work smart”)
• Get some rest the night before
Midterm 1 plan

1. A few short questions (20 pts ~ 15 min)
2. Design+analyze an algorithm (40 pts ~ 30 min)
3. Design+analyze another algorithm (40 pts ~ 30 min)
4. Bonus question (10 pts ~ only if you have time)

Study resources:

- HW3
- Review session
- Piazza
- Textbooks
- Office Hours
- HW, Sections, Lecture notes
- Classmates

(This is just my guess!)