Big-O and Summations
Review Session

Ricky Parada

14 April 2023
Agenda

1. Big-O Motivation
2. Big-O Definitions/Examples
3. Summation Relations
4. Summation Examples

Please ask questions as they come up!
Why Asymptotics?

1. Concise description of runtime/space complexity
2. Abstracts away implementation details
3. Answers the guiding question:

   How does my runtime grow as the input size grows?
What Big-O is, and what it isn’t

- Lower bound
- Describes short term behavior
- Worst-case analysis

- Upper bound
- Characterize long term behavior
- Growth rate
Agenda

1. Big-O Motivation
2. Big-O Definitions/Examples
3. Summation Relations
4. Summation Examples
Big-O Notation

What do we mean when we say “T(n) is O(f(n))”?

**In English**

T(n) = O(f(n)) if and only if T(n) is eventually upper bounded by a constant multiple of f(n)

**In Pictures**

- Graph showing runtime vs input size with lines for T(n), c·f(n), and f(n)
- n₀ is marked on the graph

**In Math**

T(n) = O(f(n)) if and only if there exists positive constants c and n₀ such that for all \( n \geq n₀ \)

\[ T(n) \leq c \cdot f(n) \]
Big-O Notation
What do we mean when we say “T(n) is O(f(n))”?

**In English**

T(n) = O(f(n)) if and only if T(n) is eventually upper bounded by a constant multiple of f(n)

**In Pictures**

![Graph showing runtime vs input size with T(n), c·f(n), and f(n) curves]

**n₀**

**In Math**

\[ T(n) = O(f(n)) \iff \exists c, n₀ > 0 \text{ s.t. } \forall n \geq n₀, T(n) \leq c \cdot f(n) \]
Limit sufficient Big-O Condition

What do we mean when we say “$T(n)$ is $O(f(n))$”?

\[
\lim_{n \to \infty} \frac{T(n)}{f(n)} \neq \infty \\
\Rightarrow \\
T(n) = O(f(n))
\]

(provided this limit exists!)
Prove that \( n = O(n^2) \).

Proof: Let \( c=1, n_0=1 \). \( \forall \ n \geq n_0 \), we have

\[
\begin{align*}
n & \geq 1 \\
n^2 & \geq n \\
n & \leq n^2
\end{align*}
\]

Alternatively, we have

\[
\lim_{n \to \infty} \frac{n}{n^2} = \lim_{n \to \infty} \frac{1}{n} = 0,
\]

which is not \( \infty \).
Prove that $\log n = O(n)$.

Proof: Define $f(x) = \log n$, $g(n) = n$, and let $c = 1$, $n_0 = 2$.

Noting that $f(2) = \log_2 2 = 1 \leq g(2) = 2$, and $f'(n) = \frac{1}{\ln 2 \cdot n} \leq g'(n) = 1$ for all $n \geq 2$, it follows that $\log n \leq n$, which is what we wanted to show.
Big-$\Omega$ Notation

What do we mean when we say “$T(n)$ is $\Omega(f(n))$”?

**In English**

$T(n) = \Omega(f(n))$ if and only if $T(n)$ is eventually **lower bounded** by a constant multiple of $f(n)$.

**In Pictures**

![Graph showing runtime (ms) vs. input size (n)]

**In Math**

$$T(n) = \Omega(f(n)) \iff \exists \ c, n_0 > 0 \text{ s.t. } \forall \ n \geq n_0, \ T(n) \geq c \cdot f(n)$$

Inequality switched directions!
Limit sufficient Big-Ω Condition

What do we mean when we say “T(n) is Ω(f(n))”?

\[ \lim_{n \to \infty} \frac{T(n)}{f(n)} \neq 0 \]

\[ \Rightarrow \]

T(n) = Ω(f(n))

(provided this limit exists!)
Big-$\Theta$ Notation

We say “$T(n)$ is $\Theta(f(n))$” if and only if both

$$T(n) = O(f(n)) \text{ and } T(n) = \Omega(f(n))$$

$$
\begin{align*}
T(n) = \Theta(f(n)) \\
\iff \\
\exists c_1, c_2, n_0 > 0 \text{ s.t. } \forall n \geq n_0,
\end{align*}
$$

$$c_1 \cdot f(n) \leq T(n) \leq c_2 \cdot f(n)$$
Limit sufficient Big-$\Theta$ Condition

What do we mean when we say “$T(n)$ is $\Theta(f(n))$”? 

$$\lim_{n \to \infty} \frac{T(n)}{f(n)} \neq \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{T(n)}{f(n)} \neq 0$$

or in other words $\lim_{n \to \infty} \frac{T(n)}{f(n)} = c$, for some constant $c$

$$\Rightarrow$$

$T(n) = \Theta(f(n))$

(provided this limit exists!)
f(n) = 3^n, g(n) = n^3. Is f(n) = O, Ω, or Θ(g(n))?  
Claim: f(n) = Ω(g(n)). Proof: By induction on n. Let c=1, n_0=4.
Let P(n) be the statement that 3^n ≥ n^3.
Base case (n = n_0 = 4): 3^4 = 81 ≥ 4^3 = 64.
Inductive step: Assume P(j): 3^j ≥ j^3 holds for some 4 ≤ j ≤ k. Showing P(k+1), we have 3^{k+1} = 3 \cdot 3^k ≥ 3k^3 = k^3 + 2k^3 ≥ k^3 + 8k^2 = k^3 + 3k^2 + 5k^2 ≥ k^3 + 3k^2 + 20k = k^3 + 3k^2 + 3k + 17k ≥ k^3 + 3k^2 + 3k + 68 = k^3 + 3k^2 + 3k + 1 + 67 = (k+1)^3 + 67 ≥ (k+1)^3. Induction complete!
f(n) = 3^n, g(n) = n^3. Is f(n) = O, Ω, or Θ(g(n))?  

We’ve narrowed it down to f(n) = Ω(g(n)), or f(n) = Θ(g(n)).  

We now claim: f(n) ≠ O(g(n)), meaning f(n) ≠ Θ(g(n)). Proof:  

Assume for the sake of contradiction that f(n) = O(g(n)). Then there exists constants c, n_0 > 0 such that for all n ≥ n_0, 3^n ≤ c·n^3.  

Logging both sides: n ≤ log_3(cn^3) ⇒ n ≤ log_3c + 3log_3n.  
Taking the limit of the ratio, we get  
\[ \lim_{n \to \infty} \frac{n}{\log_3 c + \log_3 n} = \lim_{n \to \infty} \frac{1}{3 \cdot \frac{1}{n \cdot \ln 3}} \]
\[ = \lim_{n \to \infty} \frac{n \cdot \ln 3}{3} = \infty, \text{ which is a contradiction!} \]
Aside: little-o and little-ω notation

**little-o**

**In Math**

\[ T(n) = o(f(n)) \iff \forall c, \exists n_0 > 0 \text{ s.t. } \forall n \geq n_0, T(n) < c \cdot f(n) \]

"for all"

strict!

**little-ω**

**In Math**

\[ T(n) = \omega(f(n)) \iff \forall c, \exists n_0 > 0 \text{ s.t. } \forall n \geq n_0, T(n) > c \cdot f(n) \]

inequality switched directions!
Limit sufficient little-o, little-ω Conditions

\[
\lim_{n \to \infty} \frac{T(n)}{f(n)} = 0 \quad \Rightarrow \quad T(n) = o(f(n))
\]

\[
\lim_{n \to \infty} \frac{T(n)}{f(n)} = \infty \quad \Rightarrow \quad T(n) = \omega(f(n))
\]

(provided this limit exists!) (provided this limit exists!)
Agenda

1. Big-O Motivation
2. Big-O Definitions/Examples
3. Summation Relations
4. Summation Examples
Summation Formulas

**Geometric series**
\[ \sum_{k=0}^{\infty} ar^k = \frac{a}{1 - r} \quad \text{for } |r| < 1 \]

**Geometric sum**
\[ \sum_{k=0}^{n-1} ar^k = a \frac{1 - r^n}{1 - r} \quad \text{for } r \neq 1 \]

\[ \sum_{k=1}^{n} k = \frac{n(n + 1)}{2} \]

\[ \sum_{k=0}^{n} \binom{n}{k} = 2^n \]
Agenda

1. Big-O Motivation
2. Big-O Definitions/Examples
3. Summation Relations
4. Summation Examples
Summation Practice!

What is \( \sum_{k=1}^{\infty} \left( \frac{5}{8} \right)^k \)?

\[
\sum_{k=1}^{\infty} \left( \frac{5}{8} \right)^k = -1 + \sum_{k=0}^{\infty} \left( \frac{5}{8} \right)^k = -1 + \frac{1}{1 - \frac{5}{8}} = -1 + \frac{1}{\frac{3}{8}} = -1 + \frac{8}{3} = \frac{5}{3}.
\]
Summation Practice!

What is $\sum_{k=0}^{9} 3 \cdot 1.2^k$?

$$\sum_{k=0}^{9} 3 \cdot 1.2^k = \sum_{k=0}^{9} 3 \cdot \left(\frac{6}{5}\right)^k = 3 \cdot \frac{1 - \left(\frac{6}{5}\right)^{10}}{1 - \frac{6}{5}} = -15(1 - 1.2^{10}) \approx 77.88$$
\[ f(n) = \sum_{i=1}^{n} i \log i, \quad g(n) = n^2 \log n. \text{ Is } f(n) = O, \Omega, \text{ or } \Theta(n)? \]

To prove Big-O, begin by inspecting the summation:

\[
\sum_{i=1}^{n} i \log i = 1 \log 1 + 2 \log 2 + 3 \log 3 + \ldots + n \log n
\]

\[
\sum_{i=1}^{n} i \log i \leq 1 \log n + 2 \log n + 3 \log n + \ldots + n \log n
\]

\[
= \log n (1 + 2 + 3 + \ldots + n)
\]

\[
= \frac{n(n+1)}{2} \log n
\]
$$f(n) = \sum_{i=1}^{n} i \log i, \ g(n) = n^2 \log n. \text{ Is } f(n) = O, \Omega, \text{ or } \Theta(n)?$$

\[
\frac{n(n+1)}{2} \log n \leq cn^2 \log n
\]

\[
\frac{n(n+1)}{2} \leq cn^2
\]

\[
\frac{1}{2}n^2 + \frac{1}{2}n \leq cn^2
\]

\[
n^2(c - \frac{1}{2}) \geq \frac{1}{2}n
\]

So we can pick \( c = 1, \ n_0 = 1 \).
\[ f(n) = \sum_{i=1}^{n} i \log i, \ g(n) = n^2 \log n. \text{ Is } f(n) = O, \Omega, \text{ or } \Theta(n)? \]

This is also Big-\(\Omega\), so overall \(f(n) = \Theta(g(n))\)! (Similar analysis to big-\(O\))

In order to prove Big-Omega, inspect summation again.

\[
\sum_{i=1}^{n} i \log i = 1 \log 1 + 2 \log 2 + 3 \log 3 + \ldots + \frac{n}{2} \log \left(\frac{n}{2}\right) + \ldots + n \log n
\]

\[
\sum_{i=1}^{n} i \log i \geq \left(\frac{n}{2} + 1\right) \log \left(\frac{n}{2}\right) + \left(\frac{n}{2} + 2\right) \log \left(\frac{n}{2}\right) + \ldots + n \log \left(\frac{n}{2}\right)
\]

\[
= \log \left(\frac{n}{2}\right)\left(\frac{n}{2} + 1\right) + \left(\frac{n}{2} + 2\right) + \ldots + n
\]

\[
= \log \left(\frac{n}{2}\right)\left(\frac{n}{2}\left(\frac{n}{2}\right) + \frac{n}{3} \left(\frac{n}{2} + 1\right)\right)
\]

\[
= \log \left(\frac{n}{2}\right)\left(\frac{3}{2} \left(\frac{n}{2}\right)^2 + \frac{n}{4}\right)
\]

(Fill in the rest!)
Give an example of $f, g$ such that $f \neq O(g)$ and $g \neq O(f)$.

$$f(n) = n, \ g(n) = 2^n \sin n$$
Takeaways

We have many “tools” to prove asymptotic bounds:

- Algebraic manipulation
- Limit analysis
- Racetrack principle
- Induction

To *disprove* an asymptotic bound, use proof by contradiction!
Questions?