

# Welcome to CS166!

- Six handouts!
  - Four are available up front.
  - All are available online!
- Today:
  - Why study data structures?
  - The range minimum query problem.

Why Study Data Structures?

# Why Study Data Structures?

- ***Explore where theory meets practice.***
  - Some of the data structures we'll cover are used extensively in practice. Many were invented within a twenty-mile radius of us!
- ***Challenge your intuition for the limits of efficiency.***
  - You'd be amazed how many times we'll take a problem you're sure you know how to solve and then see how to solve it faster.
- ***See the beauty of theoretical computer science.***
  - We'll cover some amazingly clever theoretical techniques in the course of this class. You'll love them.
- ***Equip yourself to solve complex problems.***
  - Powerful data structures make excellent building blocks for solving seemingly difficult problems.

# Course Staff

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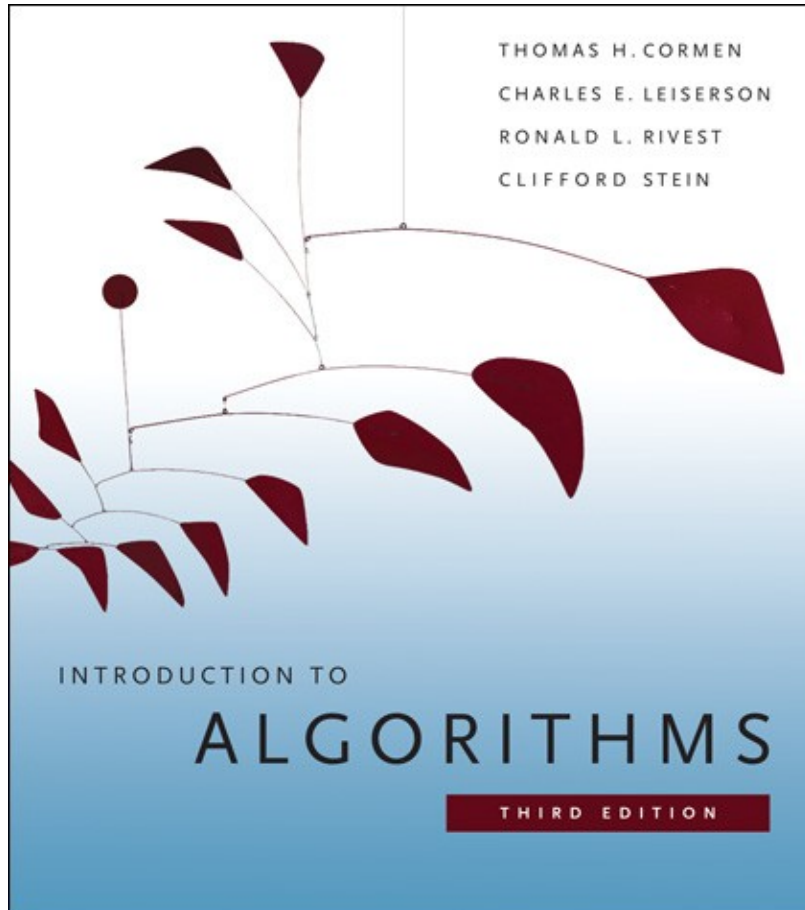
Anton de Leon  
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# The Course Website

**<http://cs166.stanford.edu>**

# *Recommended* Reading



- ***Introduction to Algorithms, Third Edition*** by Cormen, Leiserson, Rivest, and Stein.
- You'll want the third edition for this course.
- Available in the bookstore; several copies on hold at the Engineering Library.

# Prerequisites

- **CS161** (Design and Analysis of Algorithms)
  - We'll assume familiarity with asymptotic notation, correctness proofs, algorithmic strategies (e.g. divide-and-conquer, dynamic programming), classical algorithms, recurrence relations, universal hashing, etc.
- **CS107** (Computer Organization and Systems)
  - We'll assume comfort working from the command-line, designing and testing nontrivial programs, and manipulating bitwise representations of data. You should have some knowledge of the memory hierarchy. You should also know how to code in both high-level and low-level languages.

# Problem Sets

- The first problem set of the quarter, Problem Set 0, goes out today. It's due next Tuesday at 2:30PM.
- This problem set is designed as a refresher on the techniques and concepts that we'll be using over the course of this class.
- You're welcome to work in pairs or individually. See the "Problem Set Policies" handout for more details.



# Grading Policies



- 1/3 Assignments
- 1/3 Midterm
- 1/3 Final Project

Take-Home Midterm  
Goes out Tuesday, May 28<sup>th</sup>  
Comes due Thursday, May 30<sup>th</sup>

Let's Get Started!

# Range Minimum Queries

# The RMQ Problem

- The ***Range Minimum Query problem*** (***RMQ*** for short) is the following:

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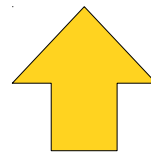
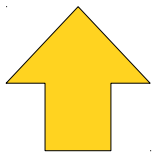
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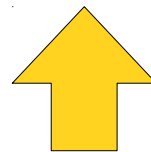
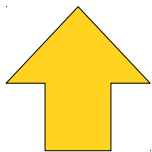
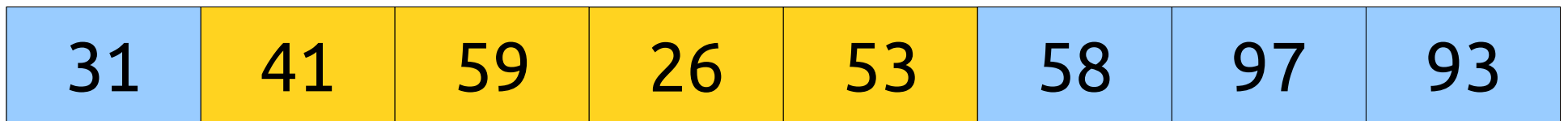
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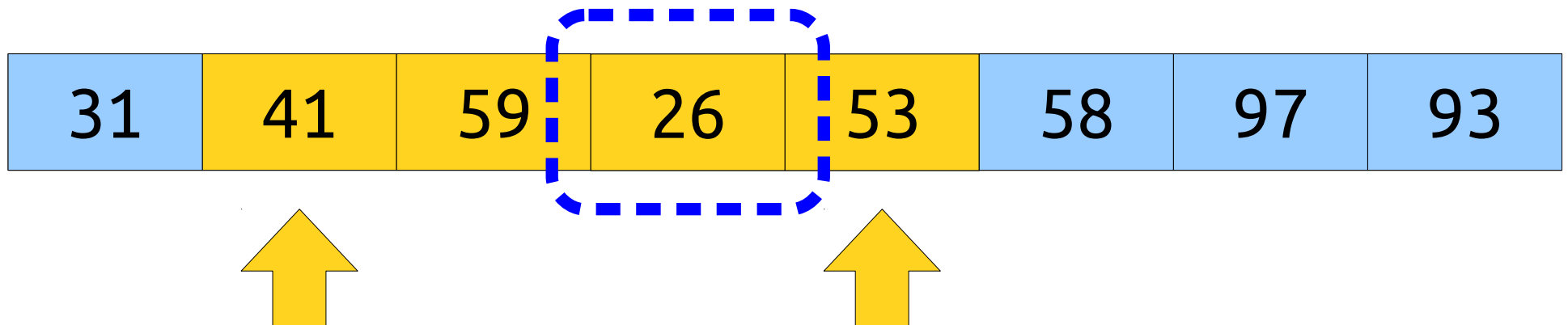
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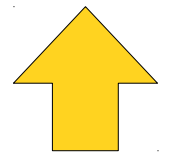
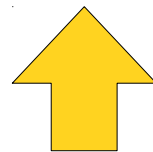


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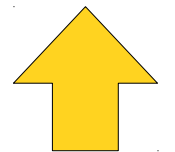
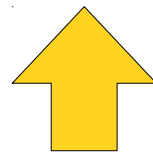


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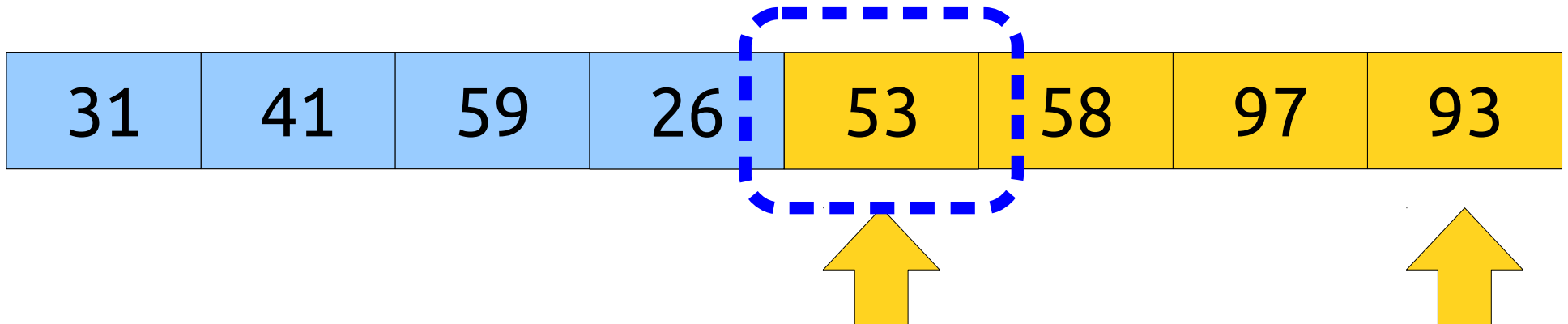
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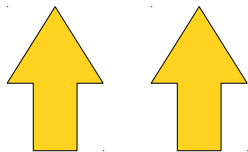


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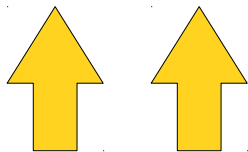


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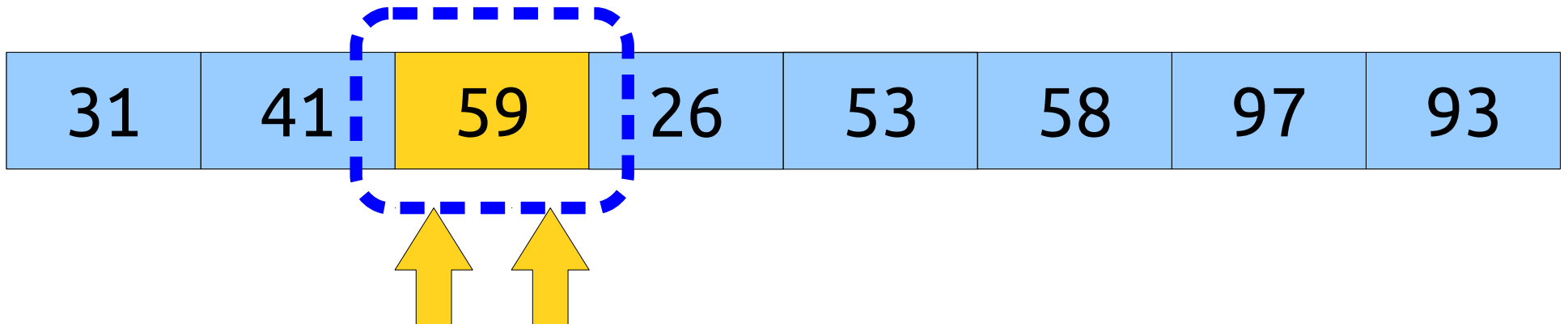
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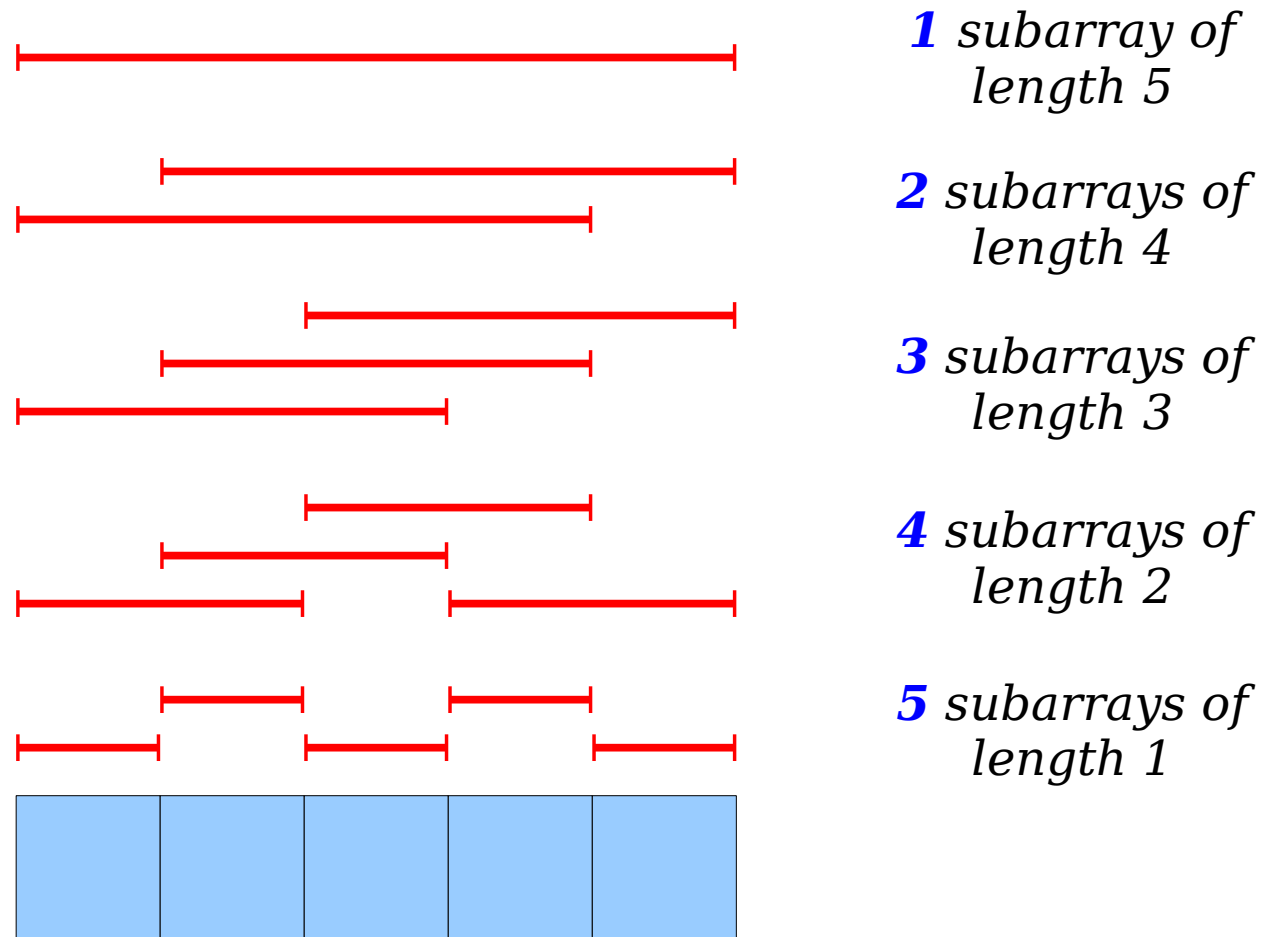
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  - Given an array  $A$  and two indices  $i \leq j$ , what is the smallest element out of  $A[i], A[i + 1], \dots, A[j - 1], A[j]$ ?
- Notation: We'll denote a range minimum query in array  $A$  between indices  $i$  and  $j$  as  **$RMQ_A(i, j)$** .
- For simplicity, let's assume 0-indexing.

# A Trivial Solution

- There's a simple  $O(n)$ -time algorithm for evaluating  $\text{RMQ}_A(i, j)$ : just iterate across the elements between  $i$  and  $j$ , inclusive, and take the minimum!
- So... why is this problem at all algorithmically interesting?
- Suppose that the array  $A$  is fixed in advance and you're told that we're going to make a number of different queries on it.
- Can we do better than the naïve algorithm?

# An Observation

- In an array of length  $n$ , there are only  $\Theta(n^2)$  possible queries.
- Why?





# A Different Approach

- There are only  $\Theta(n^2)$  possible RMQs in an array of length  $n$ .
- If we precompute all of them, we can answer RMQ in time  $O(1)$  per query.

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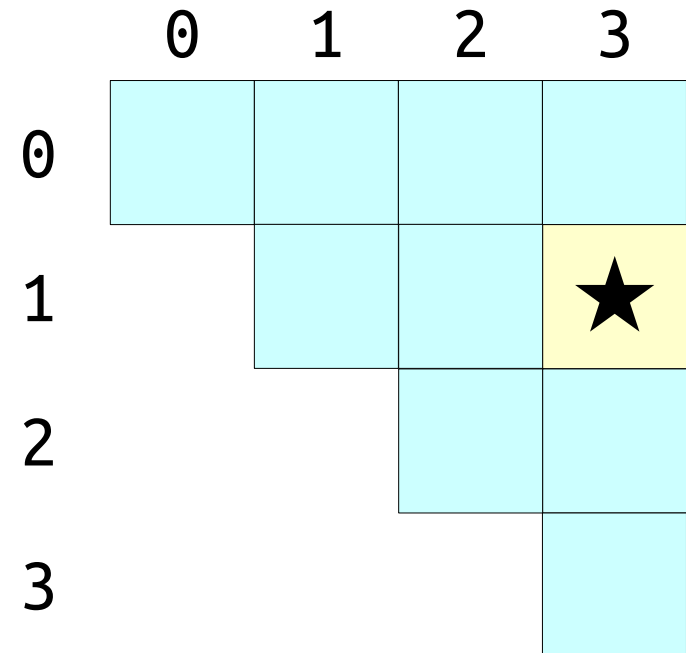
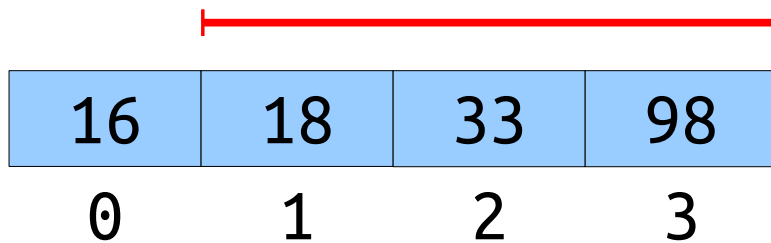
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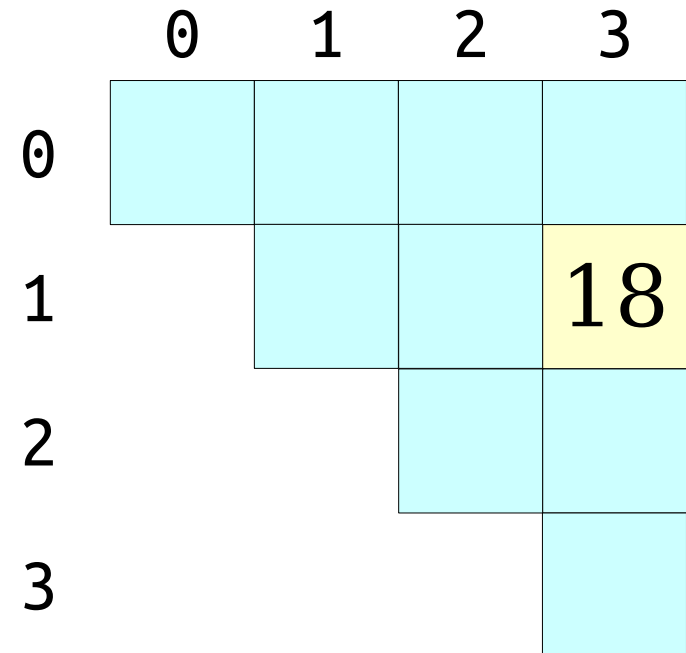
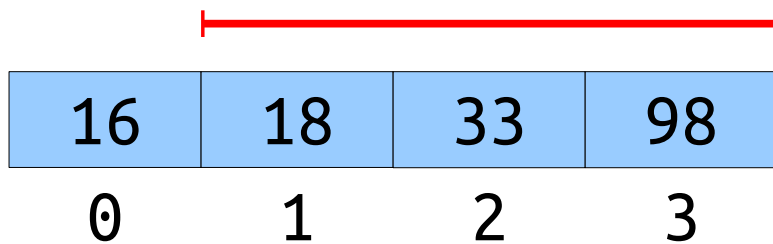
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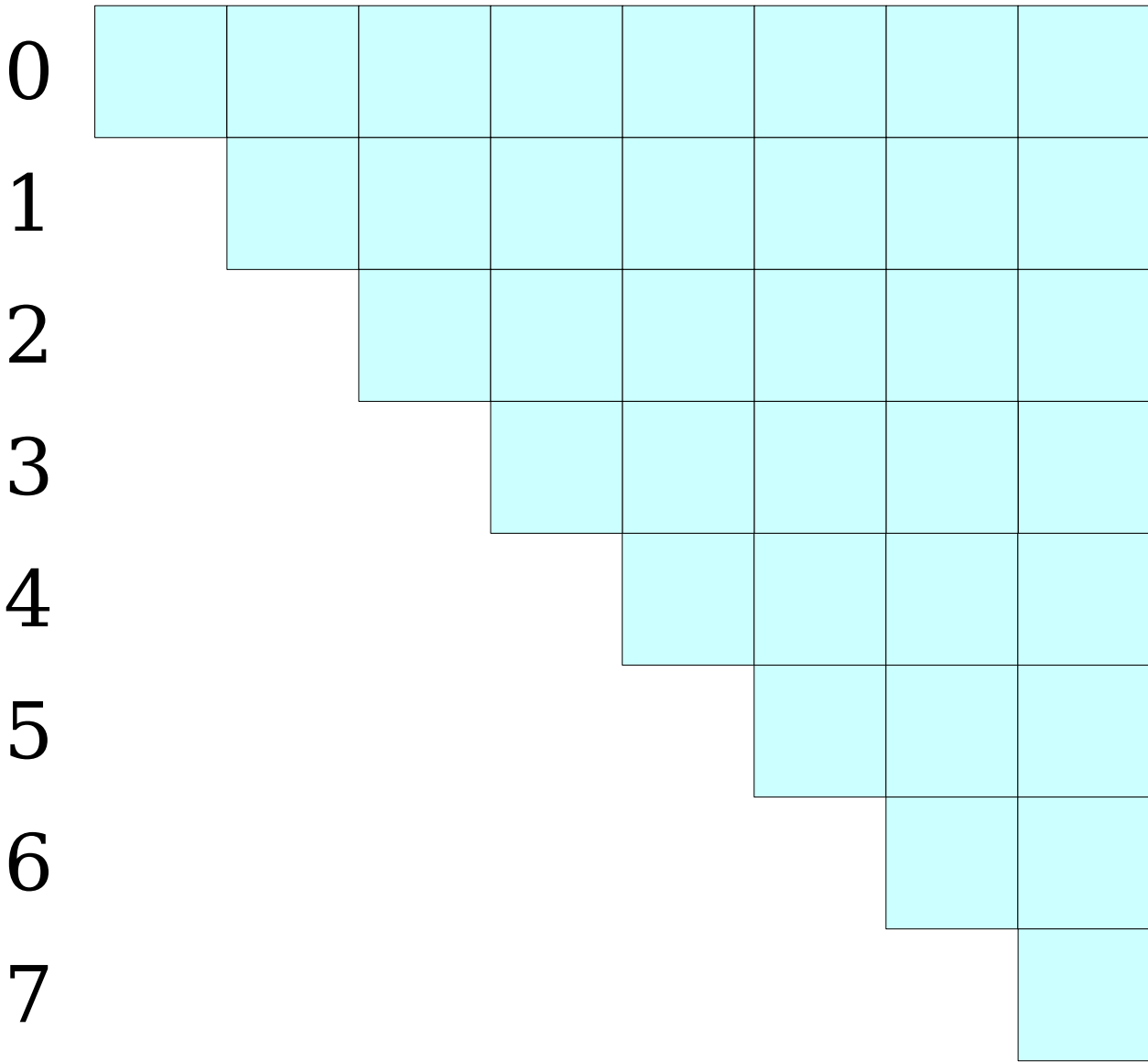
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# Building the Table

- One simple approach: for each entry in the table, iterate over the range in question and find the minimum value.
- How efficient is this?
  - Number of entries:  $\Theta(n^2)$ .
  - Time to evaluate each entry:  $O(n)$ .
  - Time required:  $O(n^3)$ .
- The runtime is  $O(n^3)$  using this approach. Is it also  $\Theta(n^3)$ ?

0 1 2 3 4 5 6 7



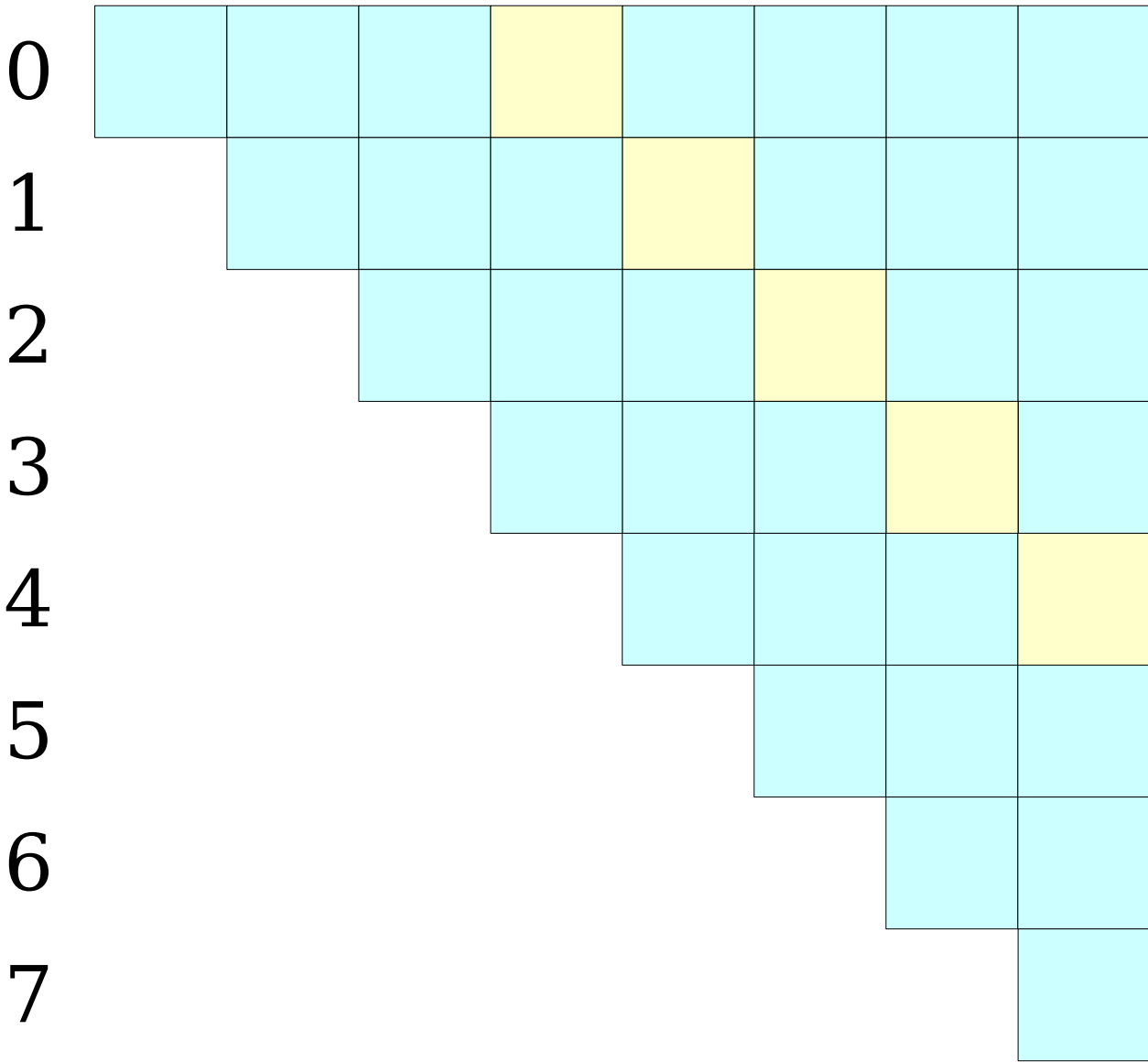




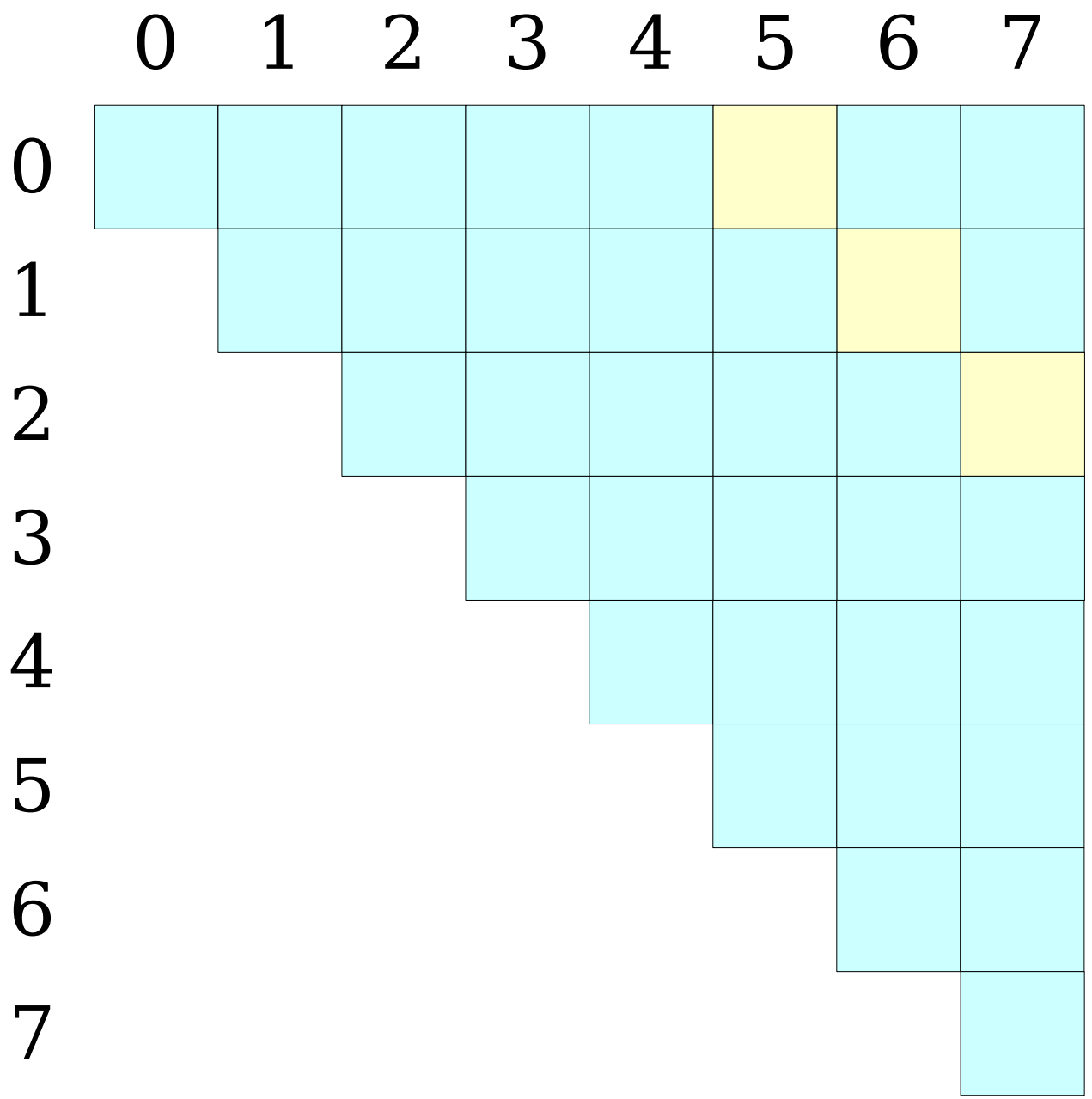


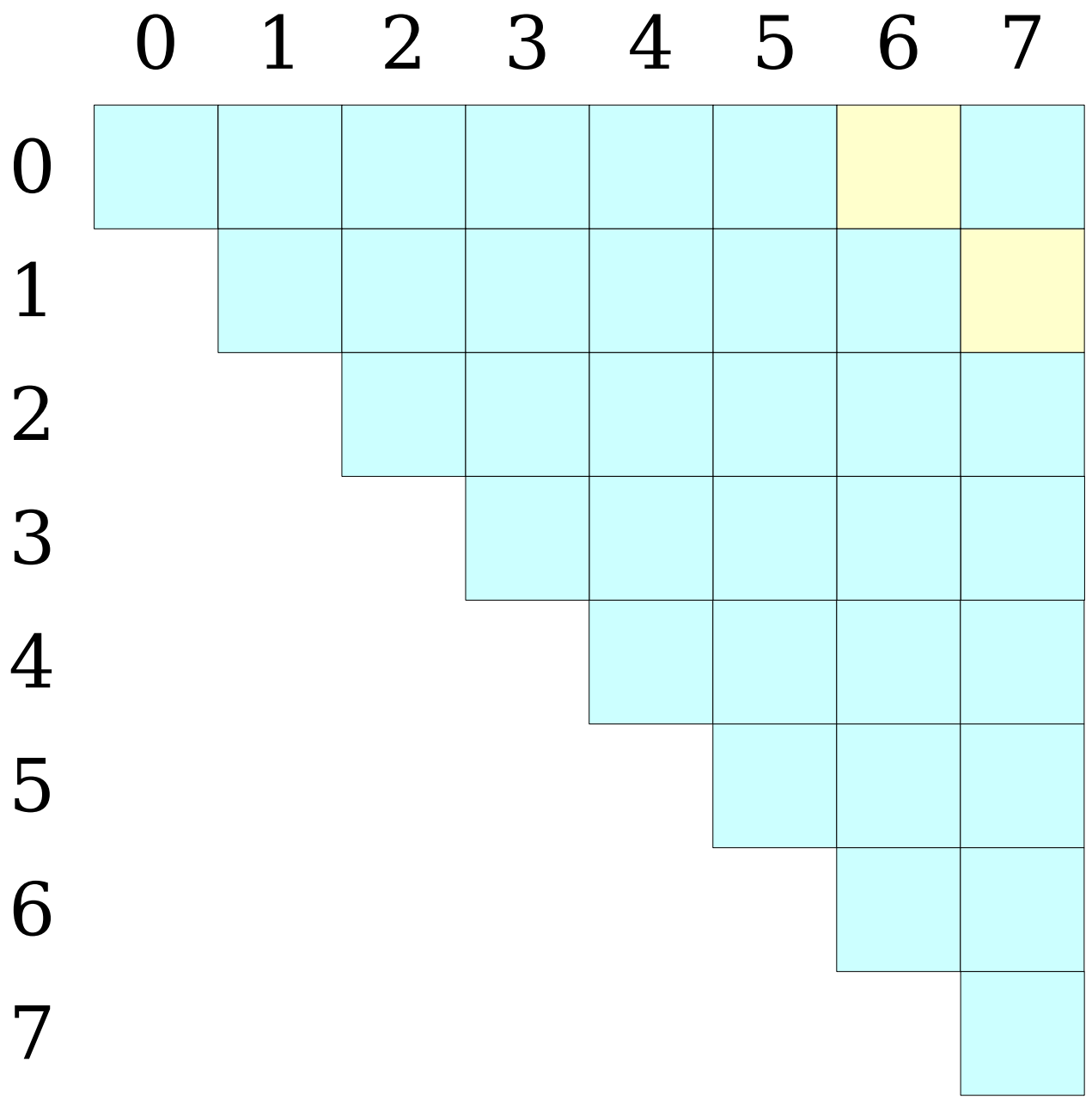


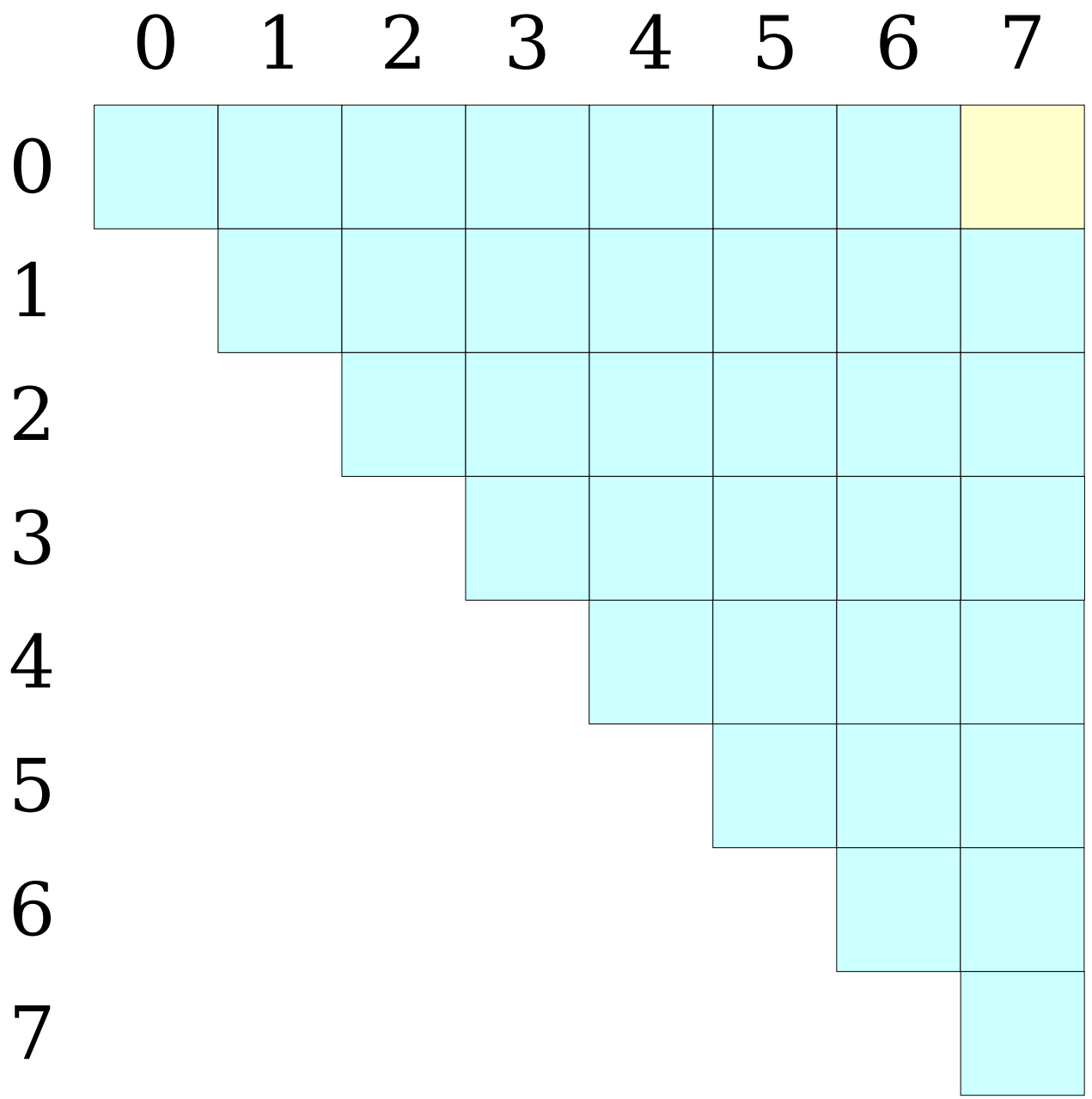
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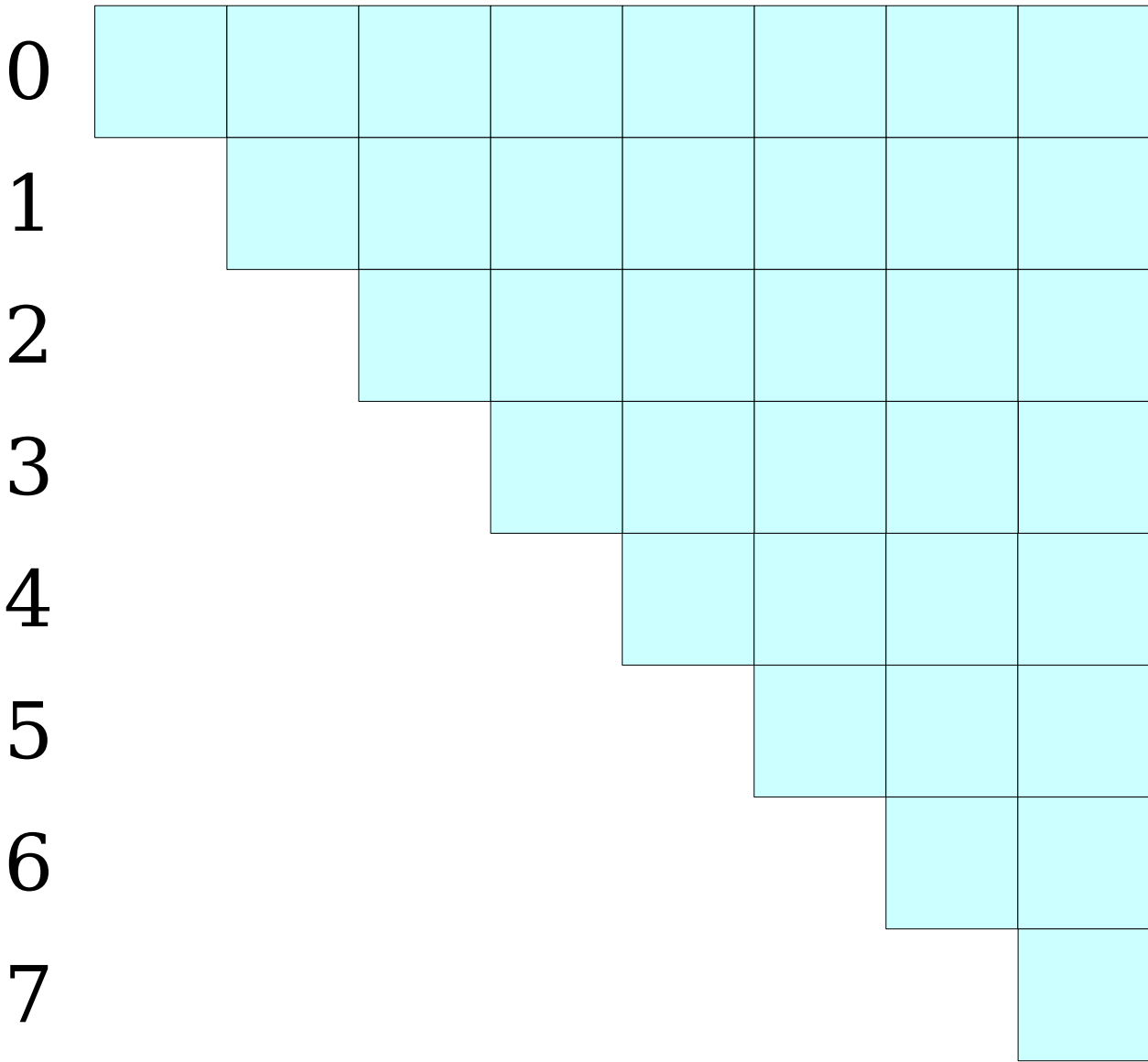




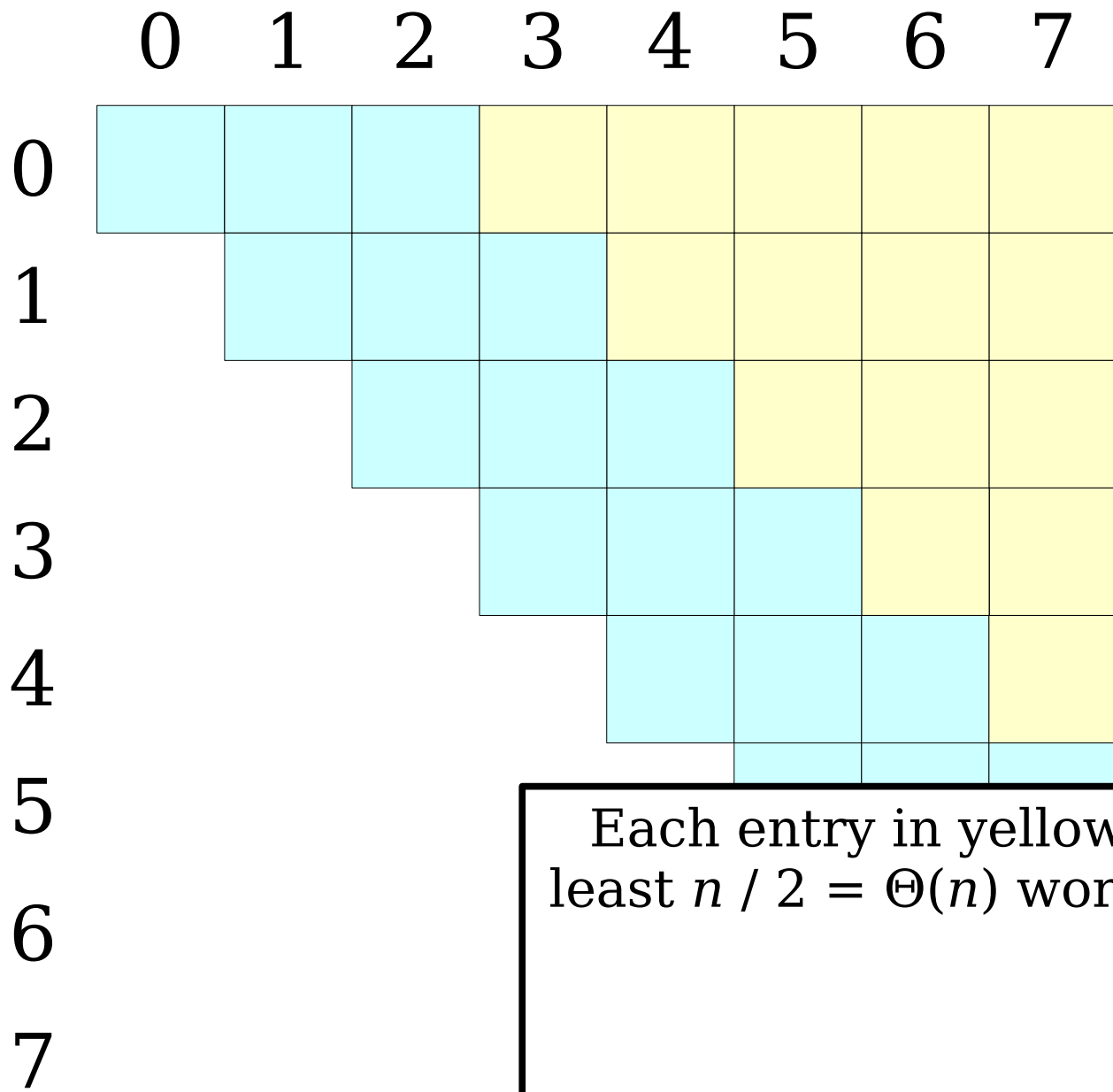




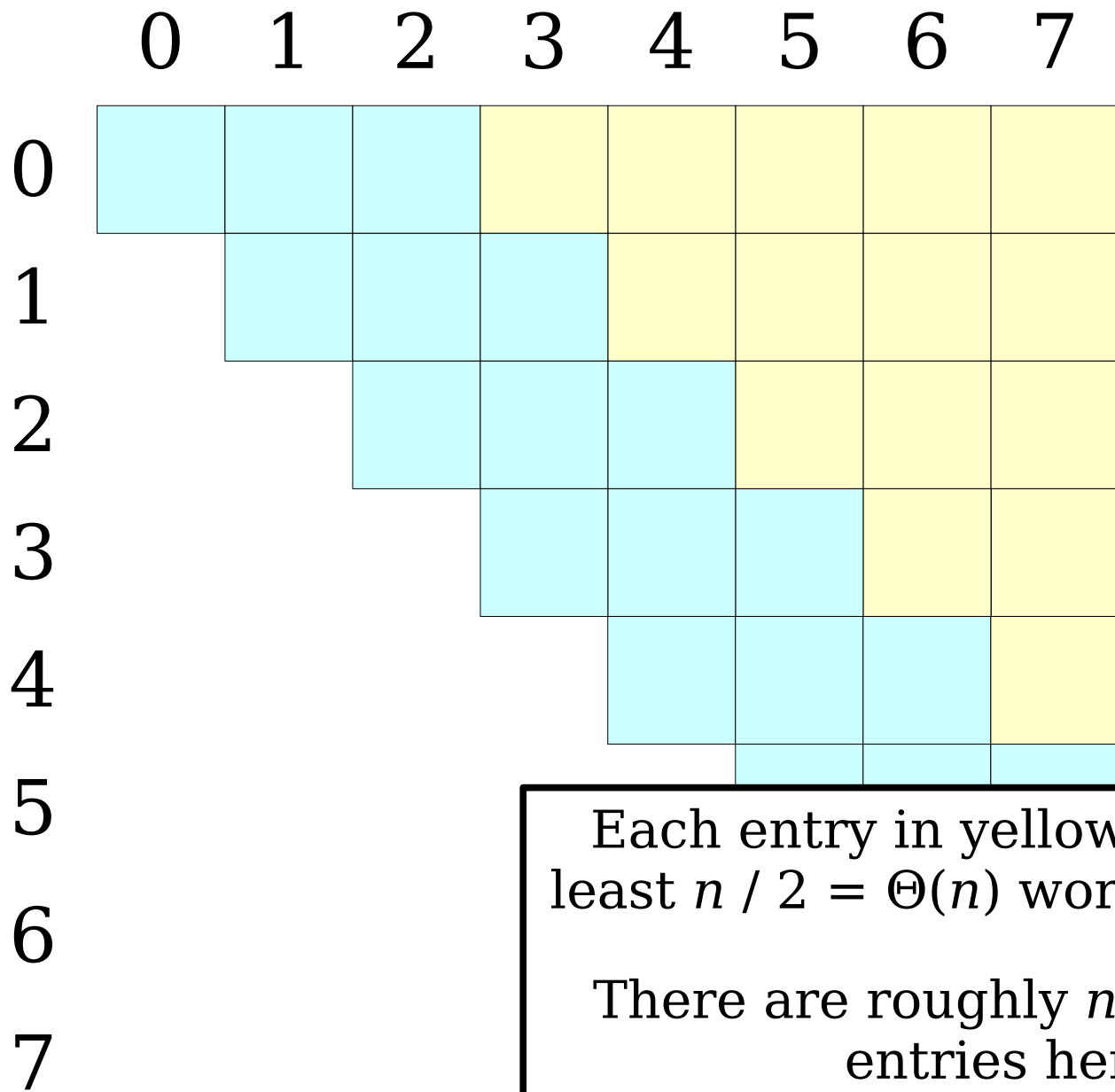
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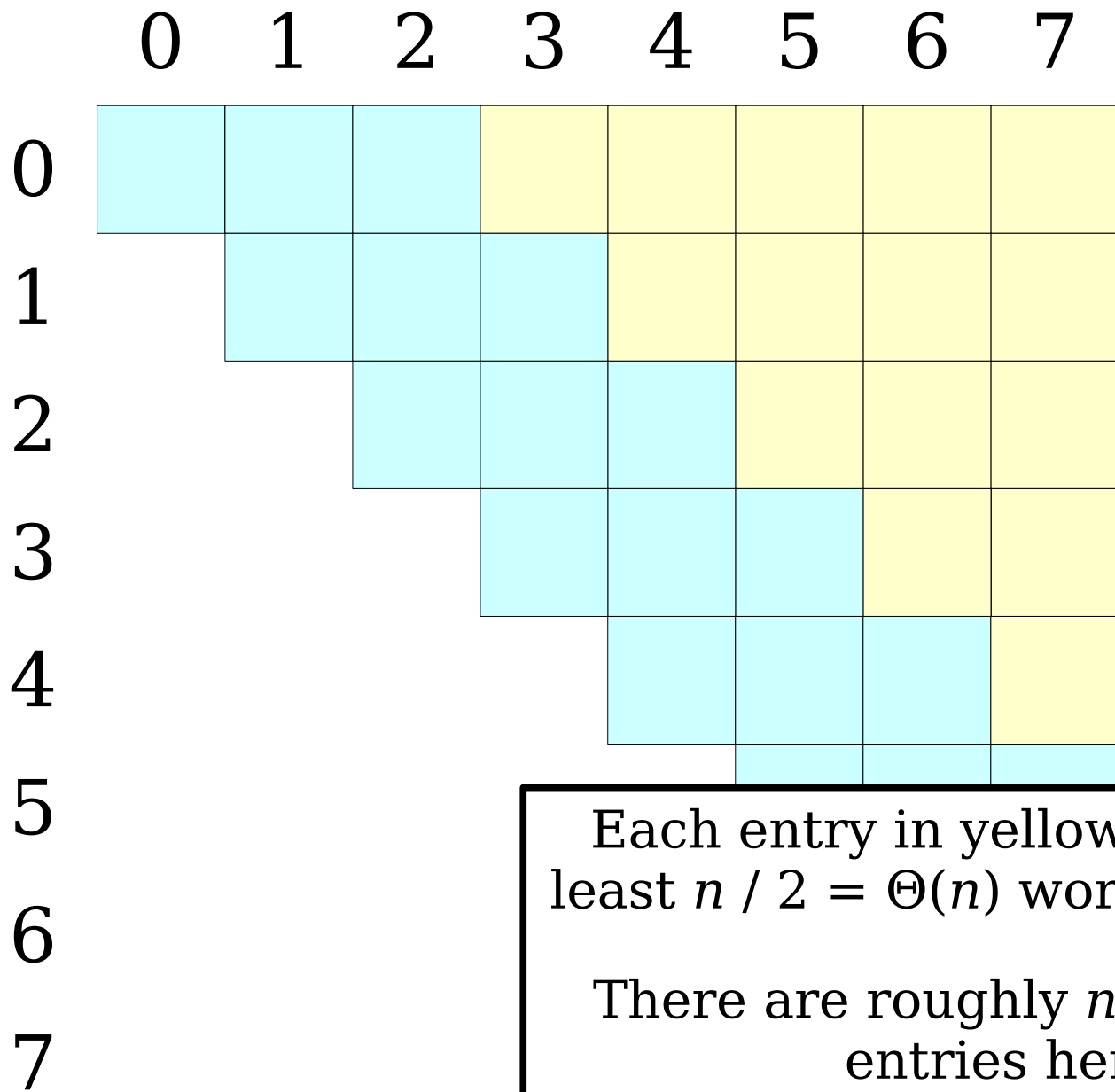


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Total work required:  $\Theta(n^3)$

# A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- **Claim:** We can precompute all subarrays in time  $\Theta(n^2)$  using dynamic programming.

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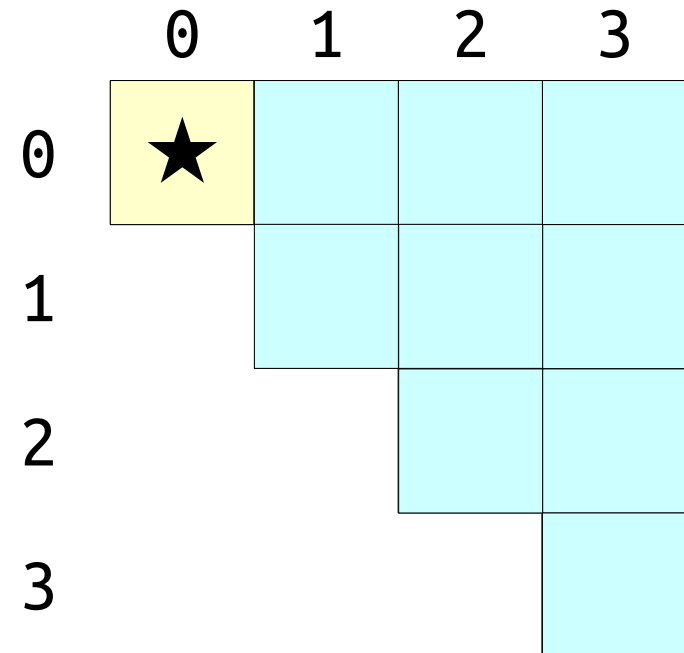
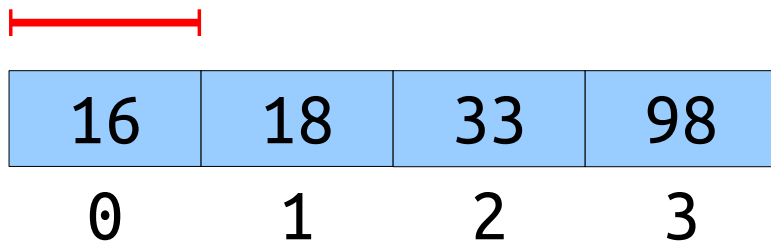
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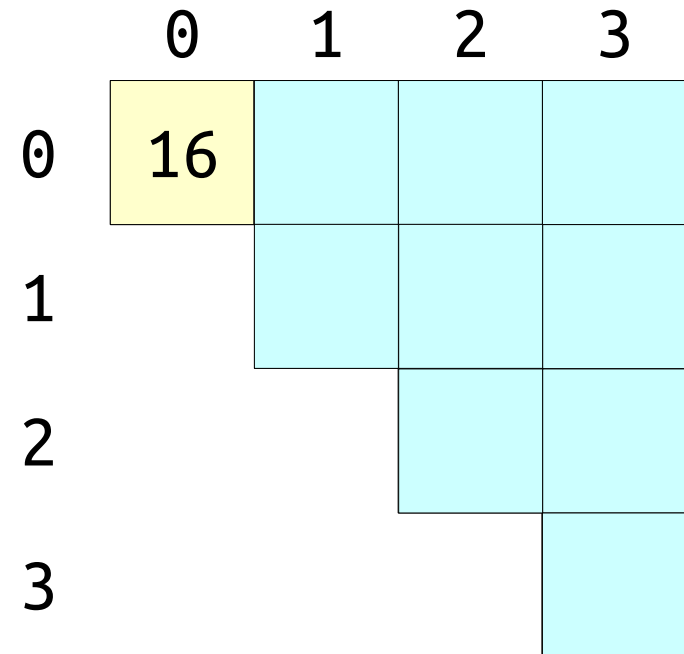
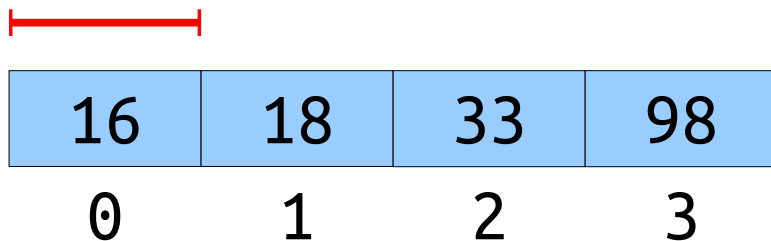
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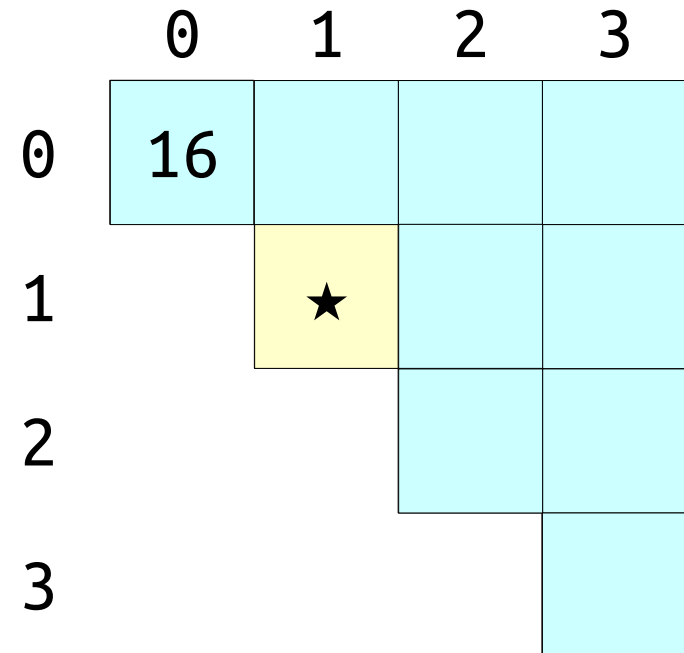
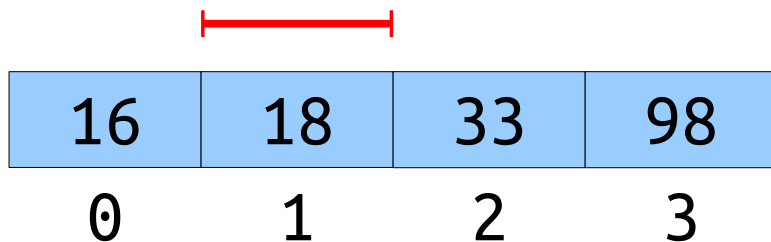
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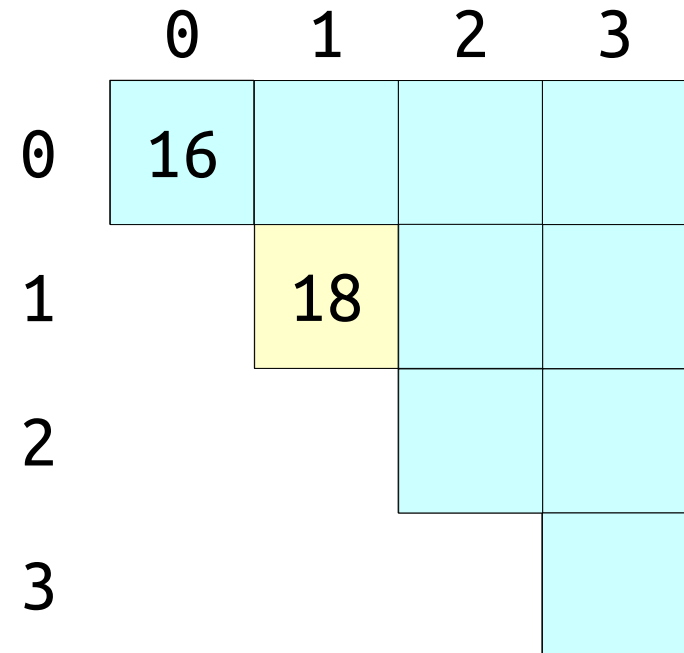
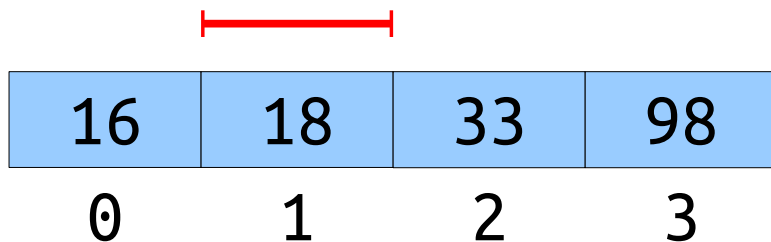
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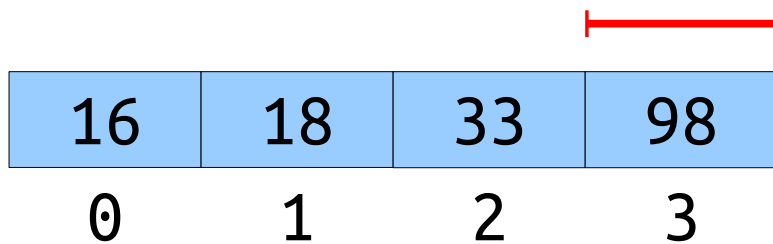
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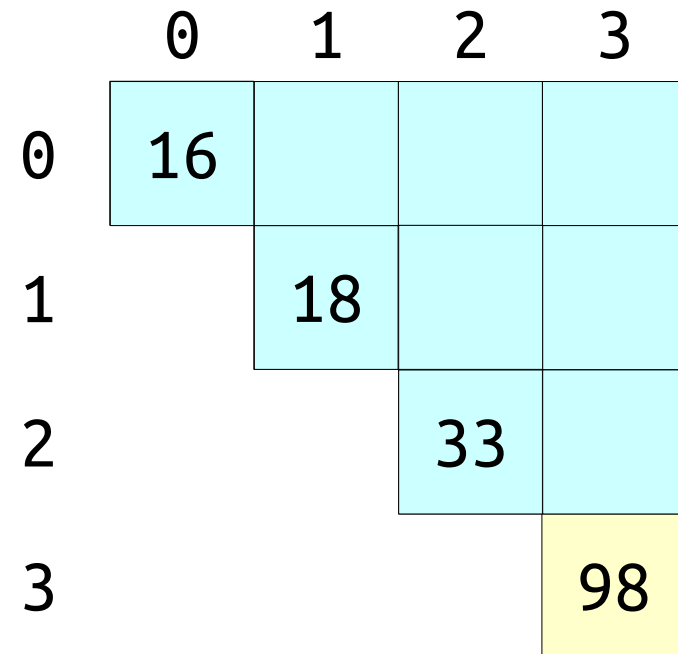
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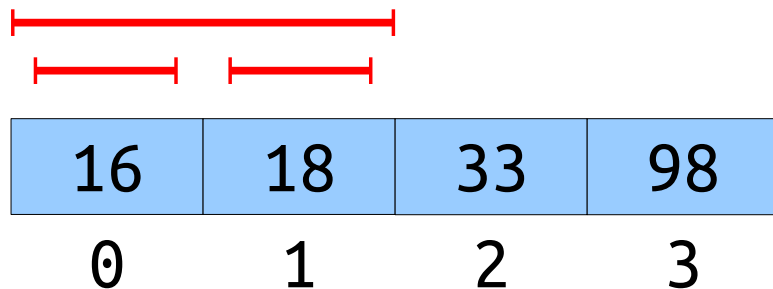


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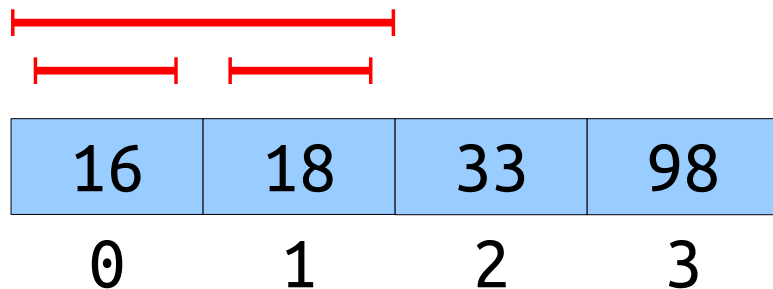
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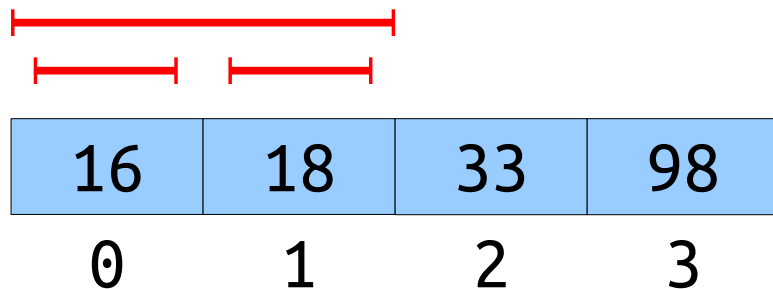
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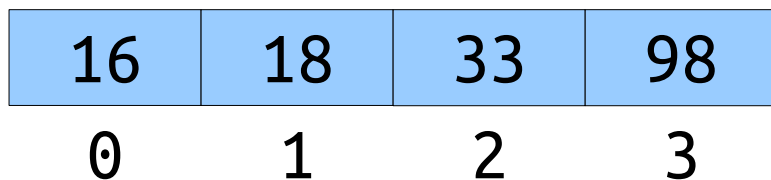
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A diagram above the array shows red horizontal lines with vertical end caps. A long line spans from index 0 to index 3. Below it, two shorter lines span from index 0 to 1 and from index 1 to 2, illustrating the decomposition of the subarray [0, 3] into [0, 1] and [1, 2].

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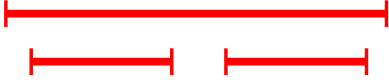


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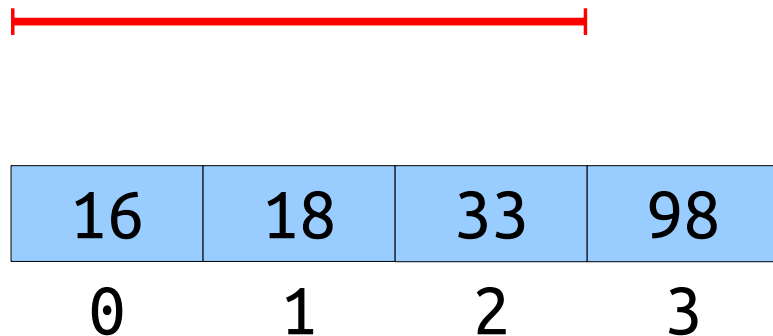
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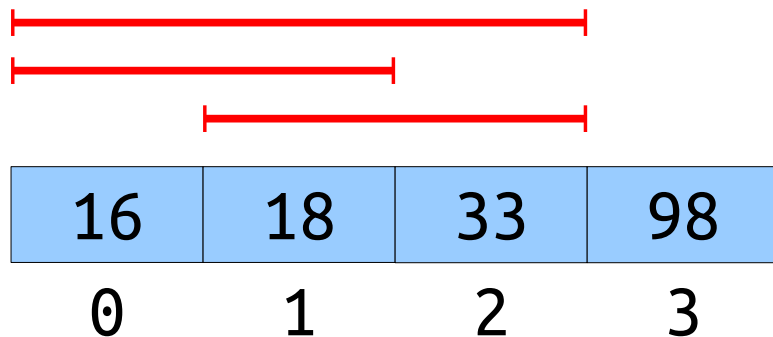
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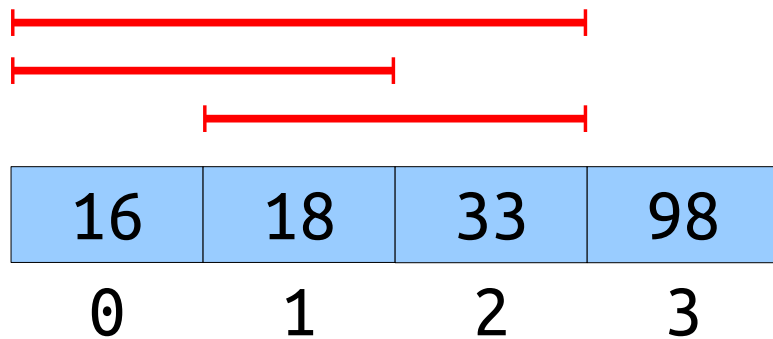
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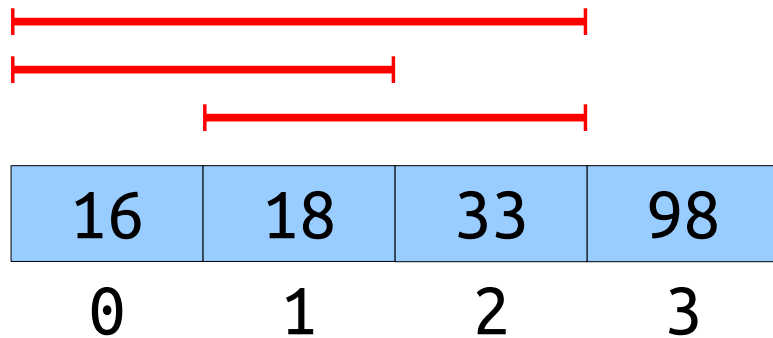
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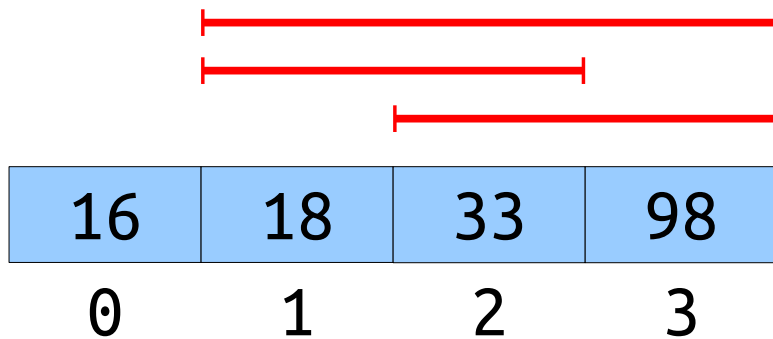


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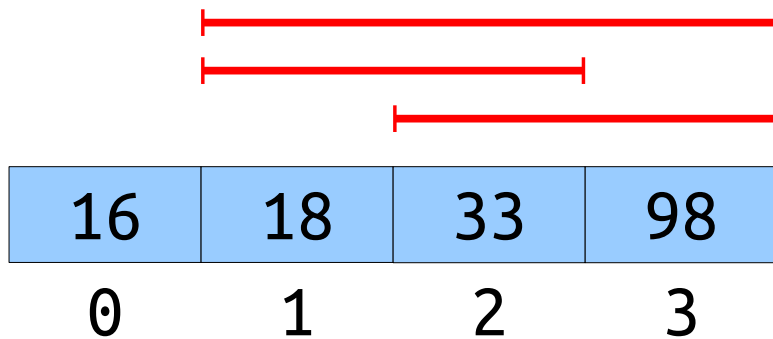
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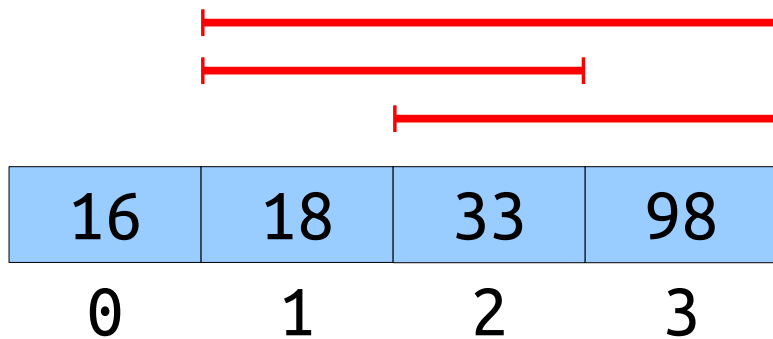
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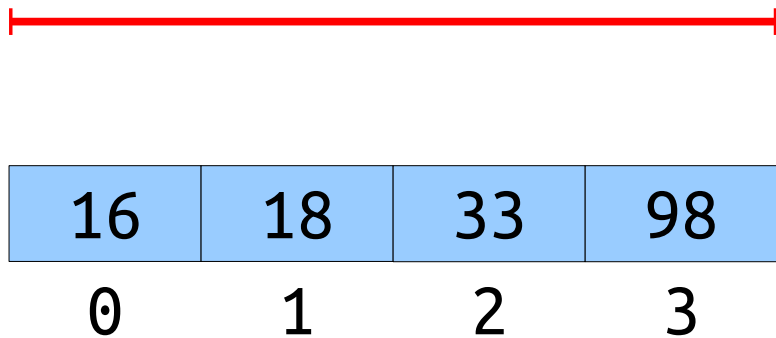
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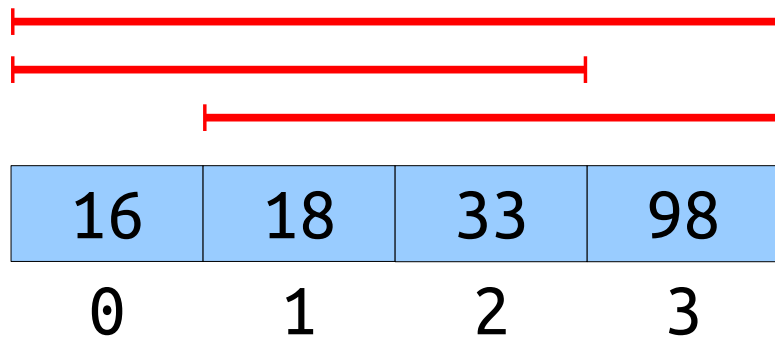
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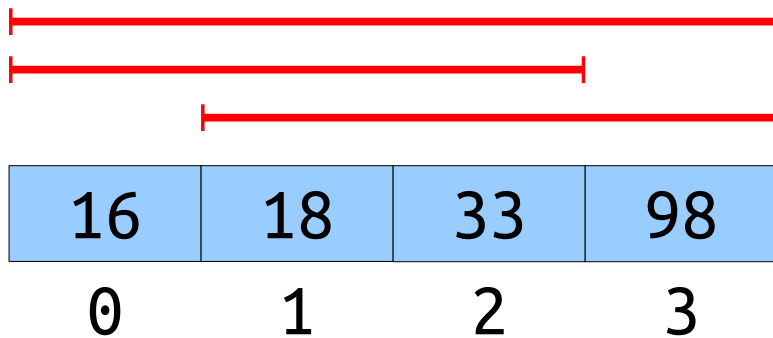
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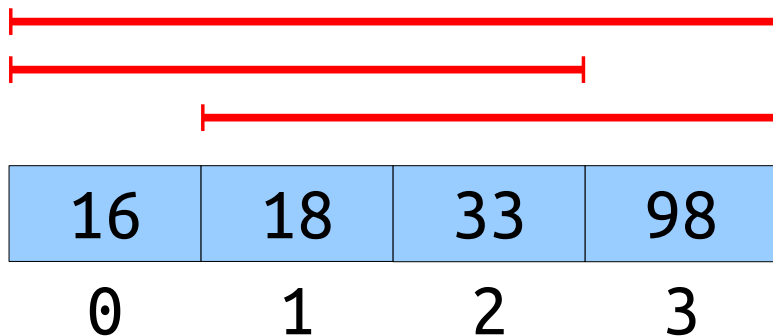
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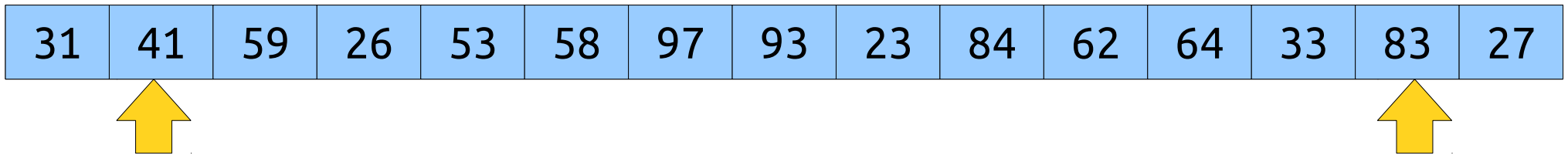
# Some Notation

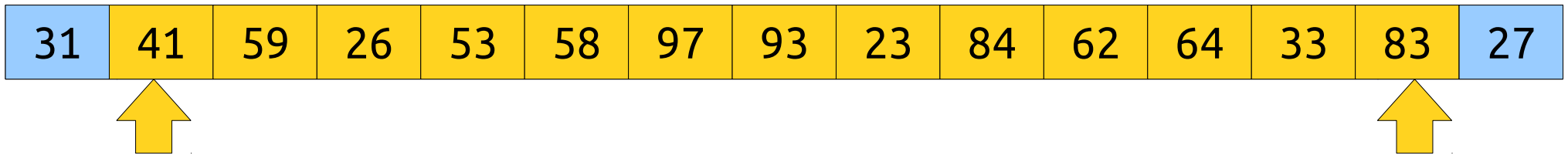
- We'll say that an RMQ data structure has time complexity  $\langle p(n), q(n) \rangle$  if
  - preprocessing takes time at most  $p(n)$  and
  - queries take time at most  $q(n)$ .
- We now have two RMQ data structures:
  - $\langle O(1), O(n) \rangle$  with no preprocessing.
  - $\langle O(n^2), O(1) \rangle$  with full preprocessing.
- These are two extremes on a curve of tradeoffs: no preprocessing versus full preprocessing.
- **Question:** *Is there a “golden mean” between these extremes?*

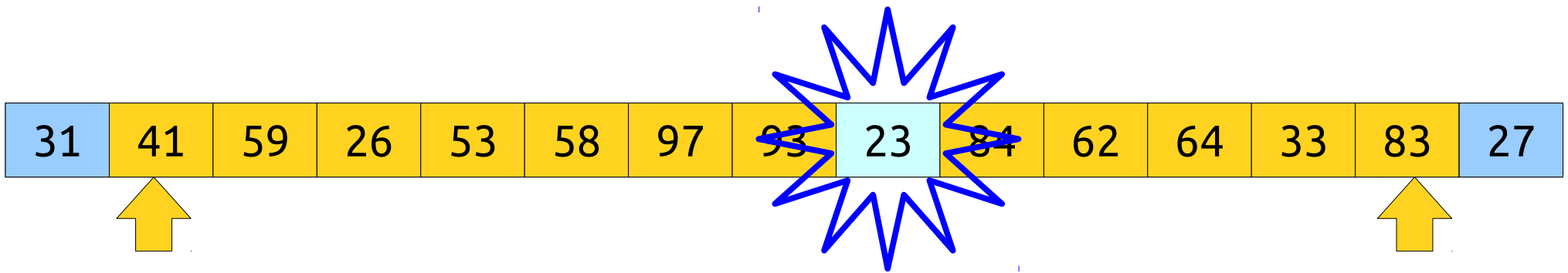
Another Approach: ***Block Decomposition***



31	41	59	26	53	58	97	93	23	84	62	64	33	83	27
----	----	----	----	----	----	----	----	----	----	----	----	----	----	----







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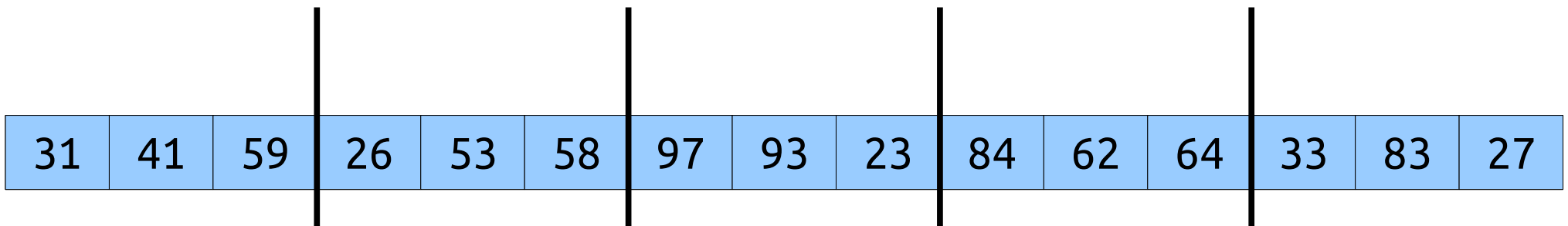
# A Block-Based Approach

- Split the input into  $O(n / b)$  blocks of some “block size”  $b$ .

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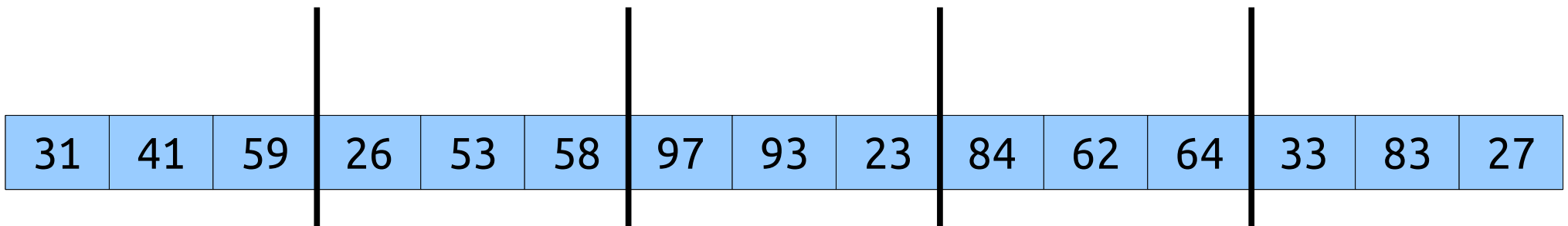
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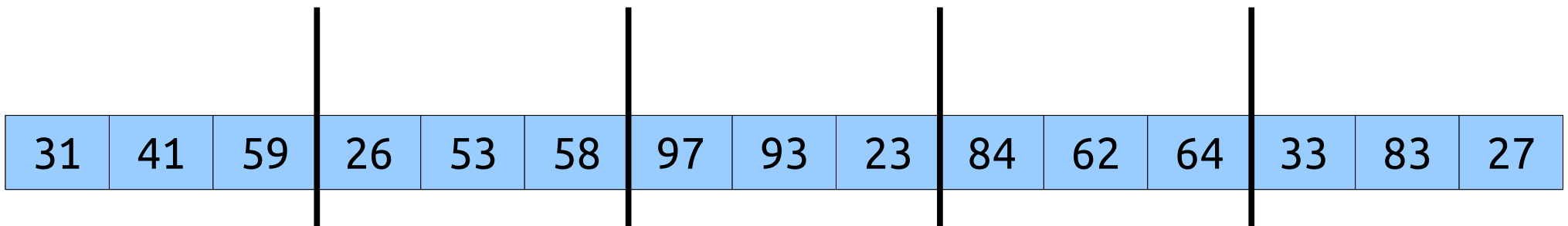
- Split the input into  $O(n / b)$  blocks of some “block size”  $b$ .
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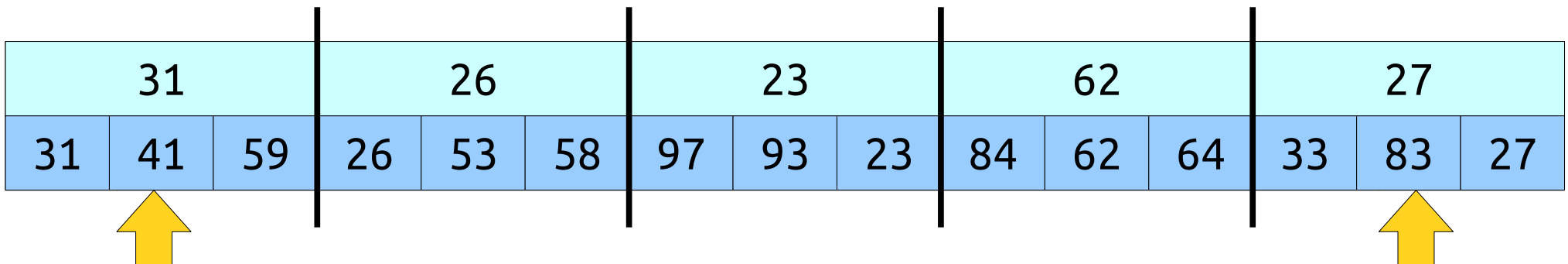
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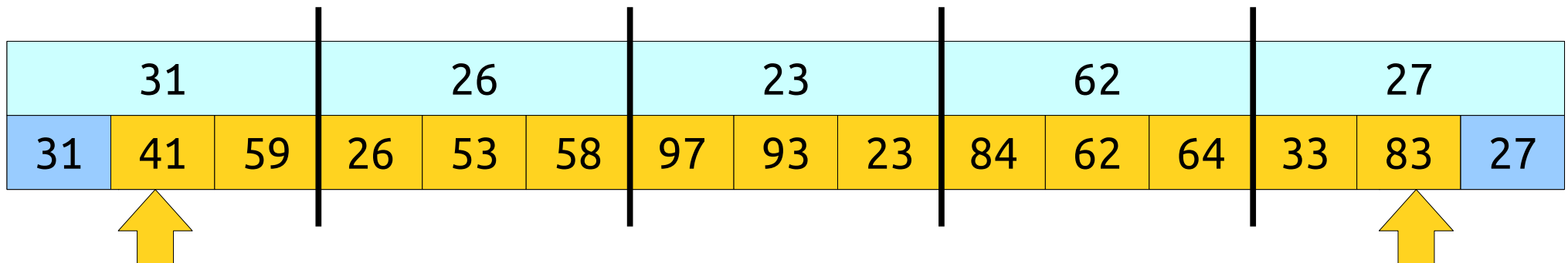
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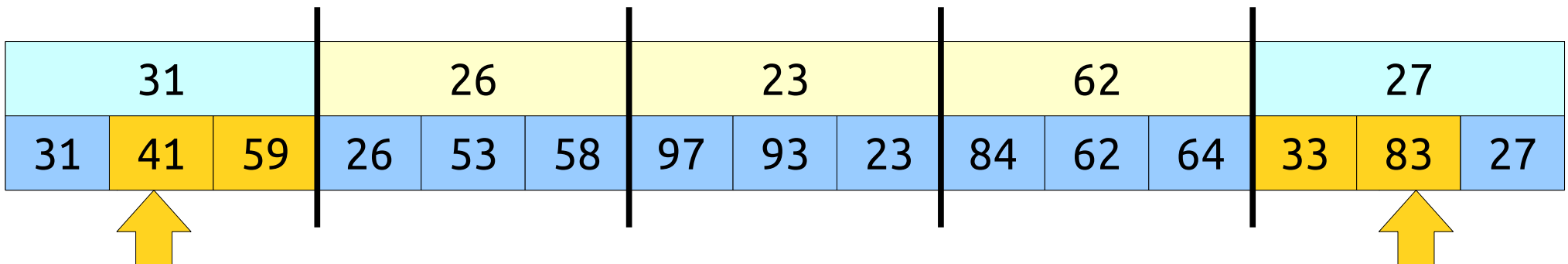
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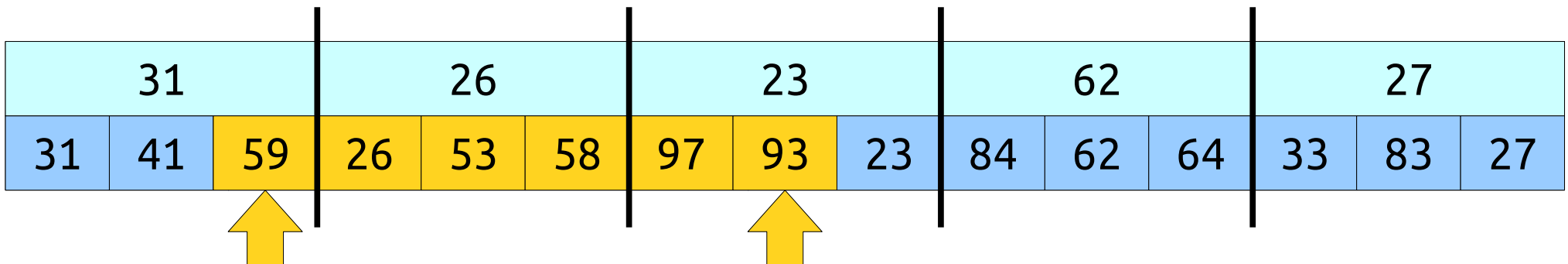
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The diagram illustrates the block-based approach. The input array is split into blocks of size  $b = 3$ . The minimum value of each block is shown above it. Yellow arrows point to the minimum values in the original array.

# A Block-Based Approach

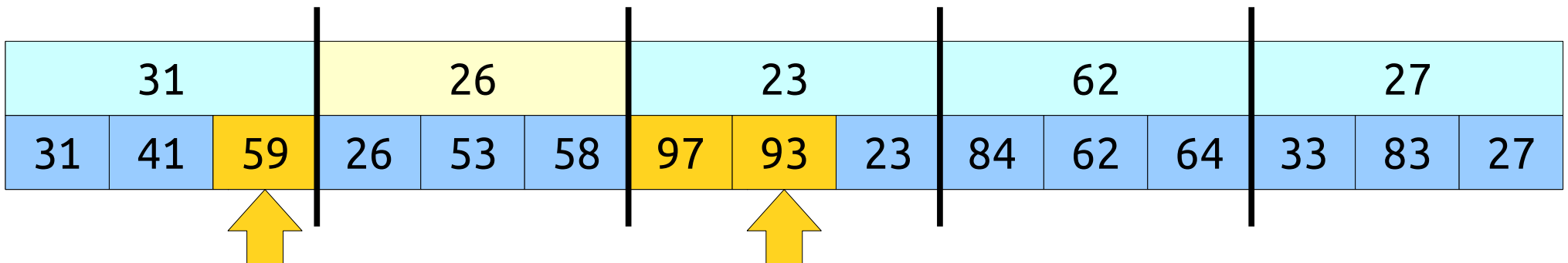
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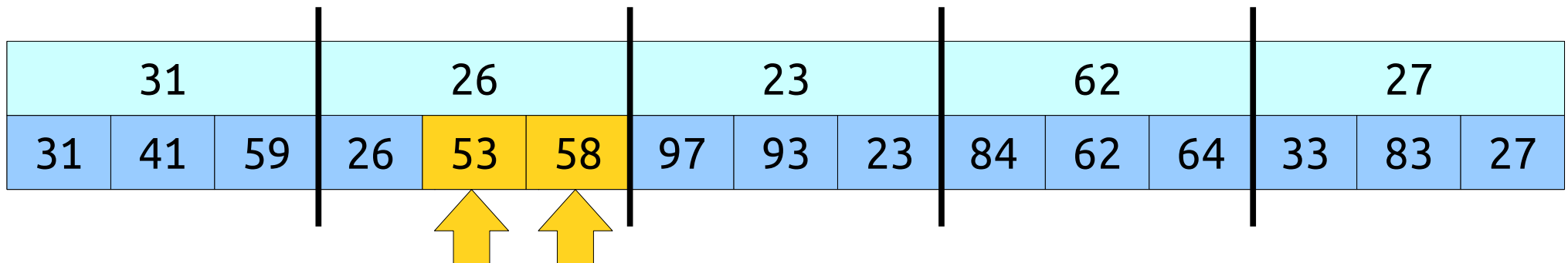
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The diagram illustrates the block-based approach. The input array is divided into 5 blocks of size 3. The minimum value of each block is shown above it. The second block's minimum, 26, is highlighted with two yellow arrows pointing to the elements 53 and 58.

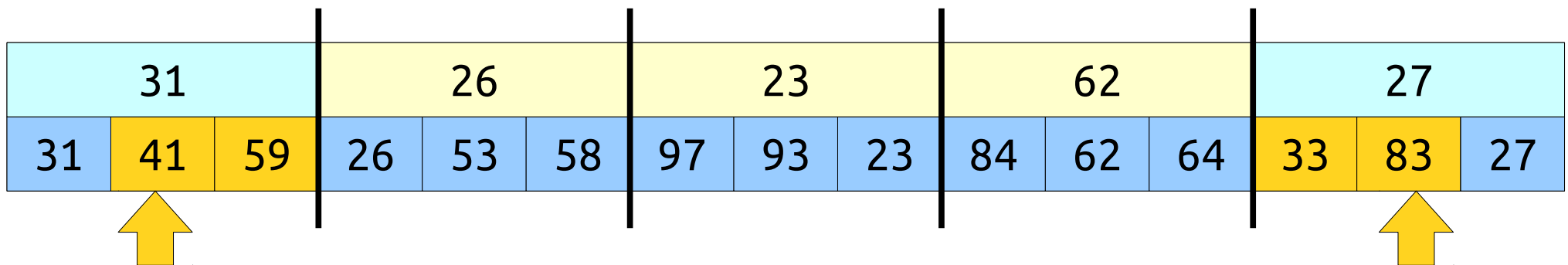
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# Analyzing the Approach

- Let's analyze this approach in terms of  $n$  and  $b$ .
- Preprocessing time:
  - $O(b)$  work on  $O(n / b)$  blocks to find minima.
  - Total work:  **$O(n)$** .
- Time to evaluate  $\text{RMQ}_A(i, j)$ :
  - $O(1)$  work to find block indices (divide by block size).
  - $O(b)$  work to scan inside  $i$  and  $j$ 's blocks.
  - $O(n / b)$  work looking at block minima between  $i$  and  $j$ .
  - Total work:  **$O(b + n / b)$** .



# Intuiting $O(b + n / b)$

- As  $b$  increases:
  - The  $b$  term rises (more elements to scan within each block).
  - The  $n / b$  term drops (fewer blocks to look at).
- As  $b$  decreases:
  - The  $b$  term drops (fewer elements to scan within a block).
  - The  $n / b$  term rises (more blocks to look at).
- Is there an optimal choice of  $b$  given these constraints?

# Optimizing $b$

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$$\begin{aligned} 1 - n/b^2 &= 0 \\ 1 &= n/b^2 \end{aligned}$$

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$$1 = n/b^2$$

$$b^2 = n$$

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- Asymptotically optimal runtime is when  $b = n^{1/2}$ .
- In that case, the runtime is

$$O(b + n / b) = O(n^{1/2} + n / n^{1/2}) = O(n^{1/2} + n^{1/2}) = \mathbf{O(n^{1/2})}$$

# Summary of Approaches

- Three solutions so far:
  - Full preprocessing:  $\langle O(n^2), O(1) \rangle$ .
  - Block partition:  $\langle O(n), O(n^{1/2}) \rangle$ .
  - No preprocessing:  $\langle O(1), O(n) \rangle$ .
- Modest preprocessing yields modest performance increases.
- **Question:** Can we do better?

A Second Approach: ***Sparse Tables***

# An Intuition

- The  $\langle O(n^2), O(1) \rangle$  solution gives fast queries because every range we might look up has already been precomputed.
- This solution is slow overall because we have to compute the minimum of every possible range.
- **Question:** Can we still get constant-time queries without preprocessing all possible ranges?

# An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26	26	26	26	26
1		41	41	26	26	26	26	26
2			59	26	26	26	26	26
3				26	26	26	26	26
4					53	53	53	53
5						58	58	58
6							97	93
7								93

# An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26	26	26	26	26
1		41	41	26	26	26	26	26
2			59	26	26	26	26	26
3				26	26	26	26	26
4					53	53	53	53
5						58	58	58
6							97	93
7								93



# An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93


# An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

# An Observation


31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7



	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

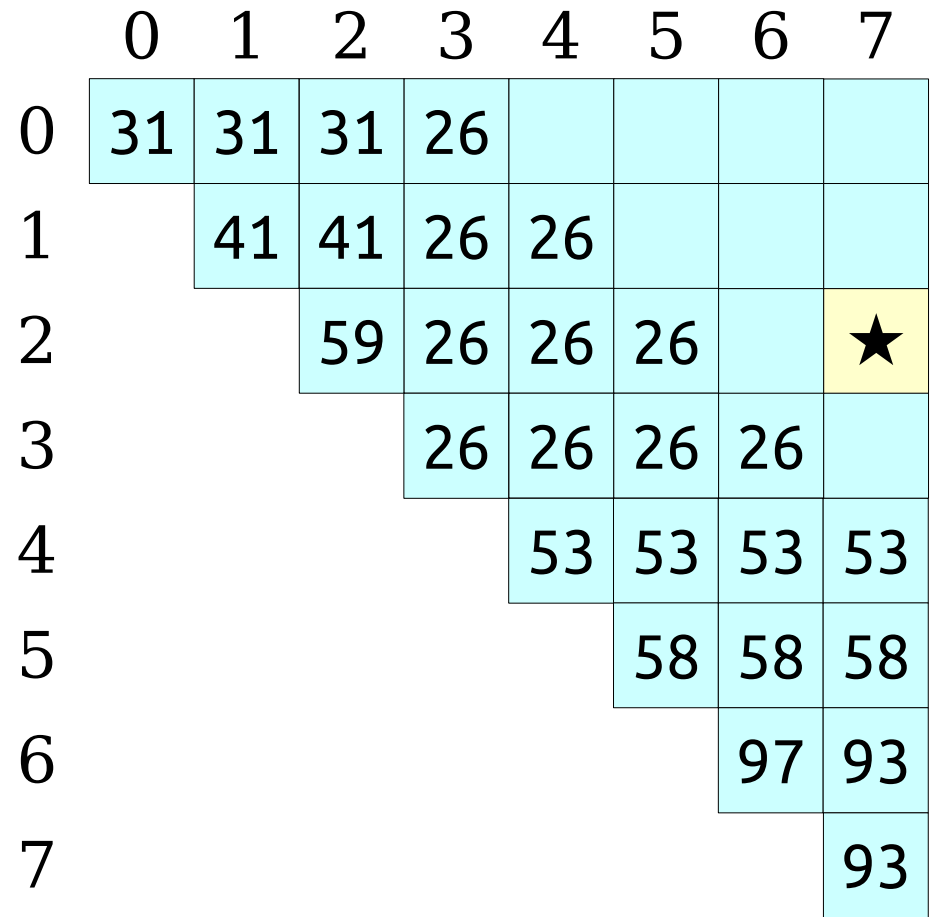
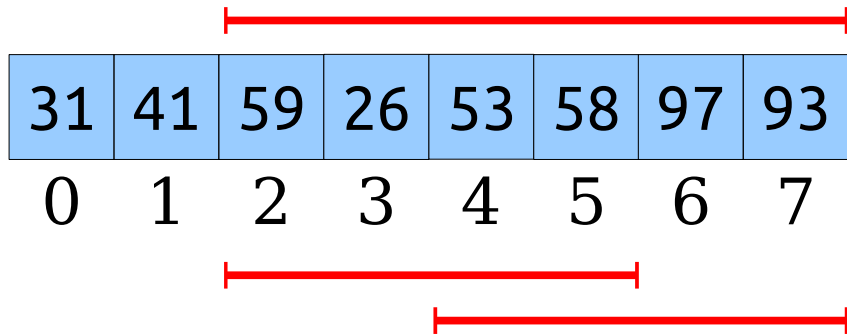
# An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

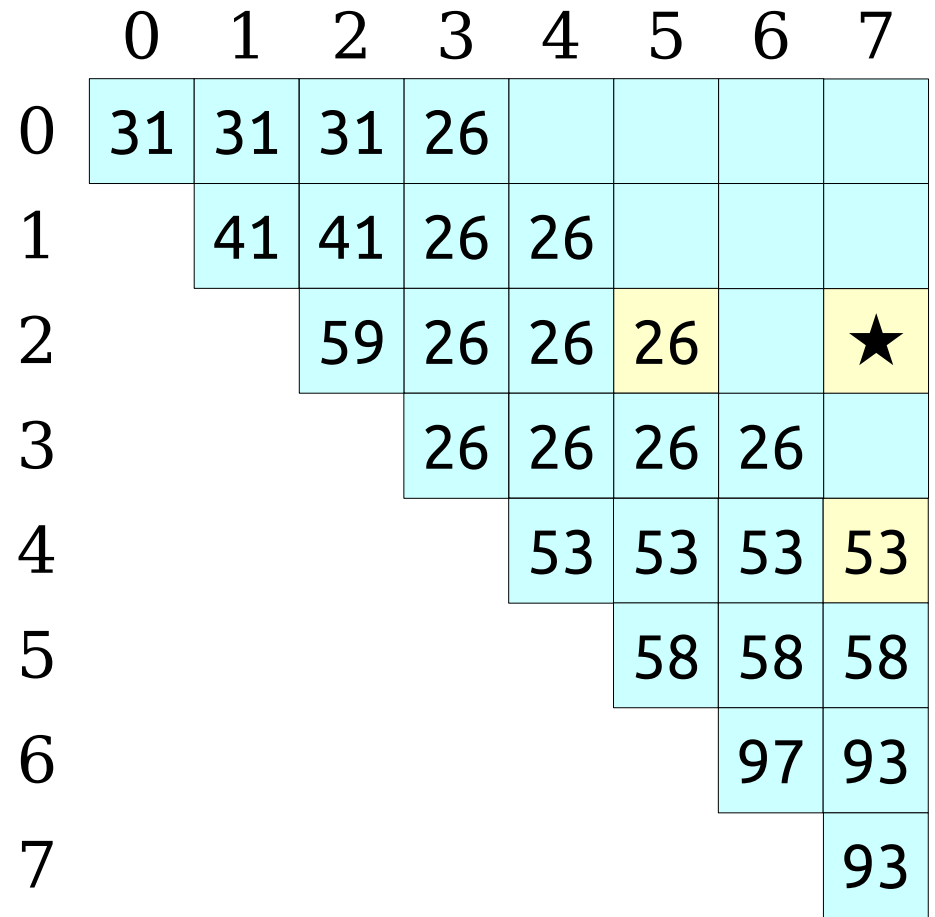
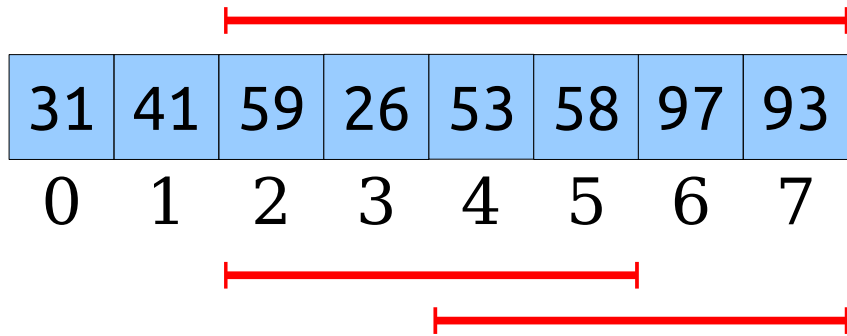


	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		★
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

# An Observation



# An Observation

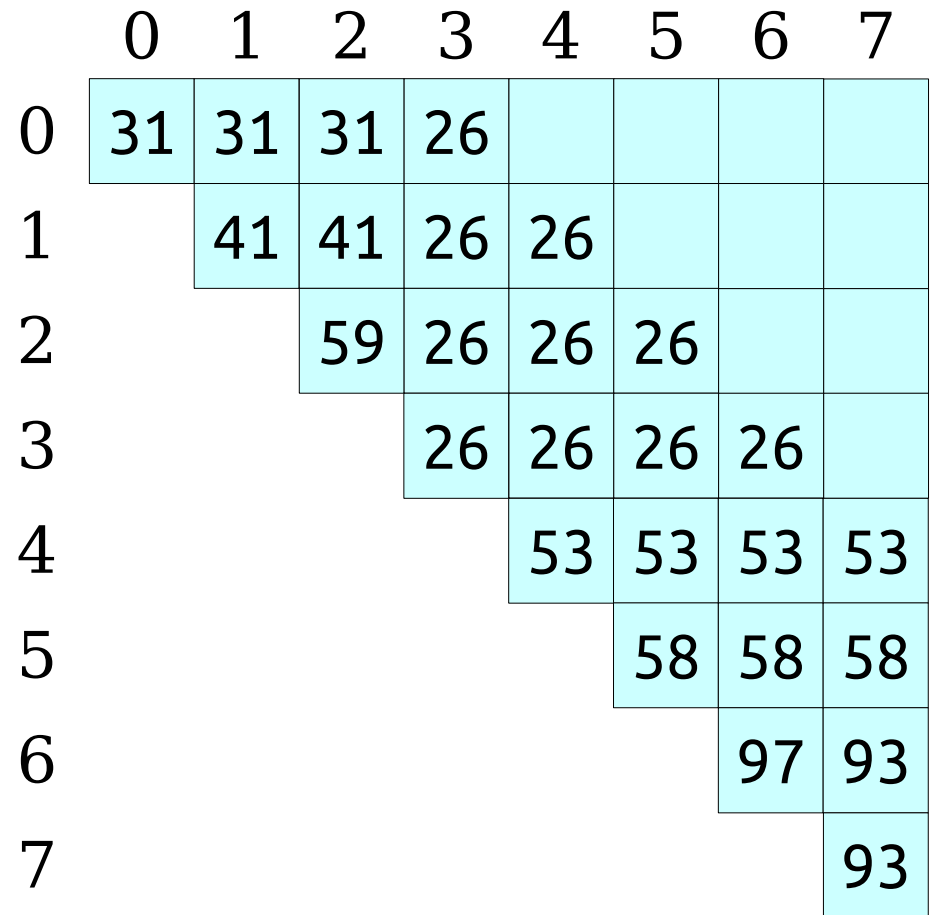
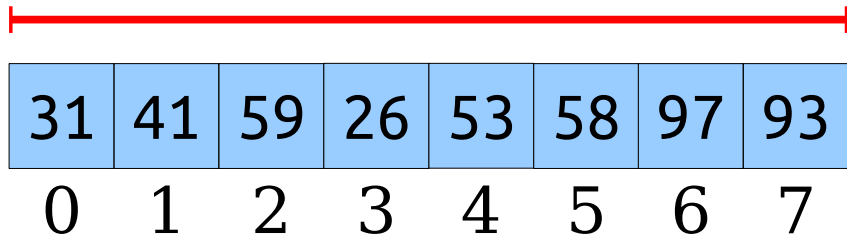


# An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

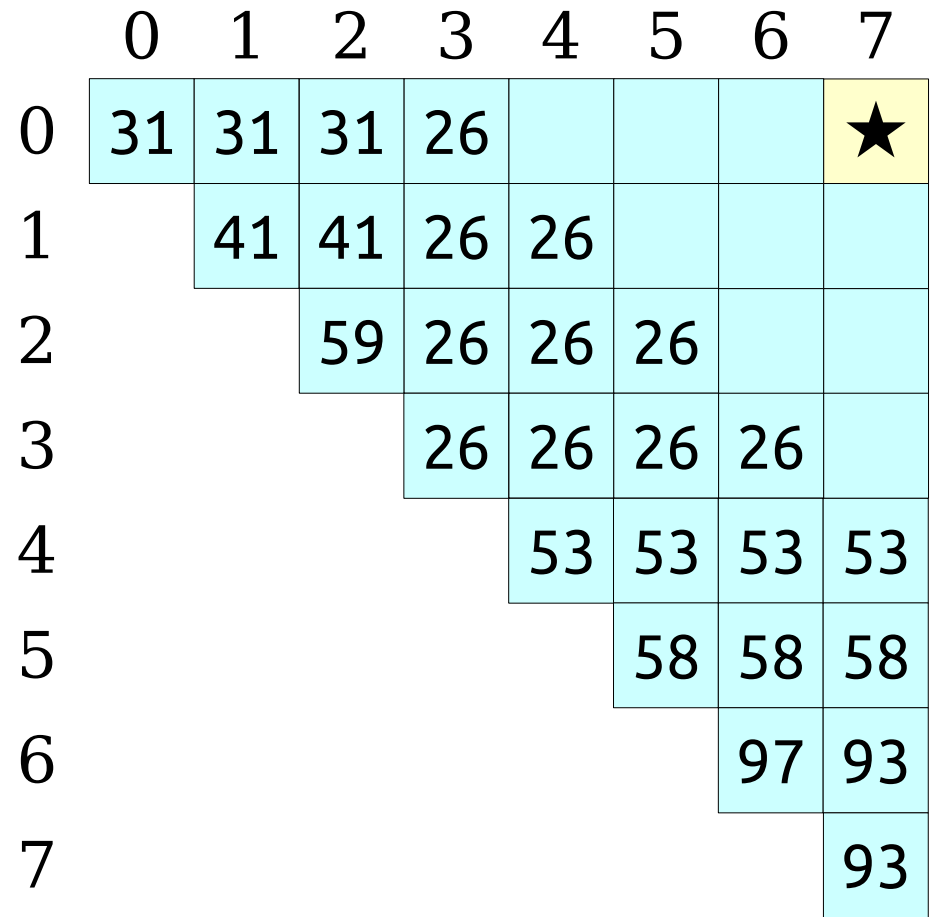
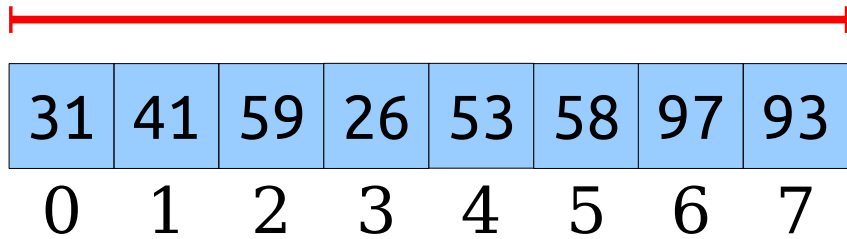
	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

# An Observation

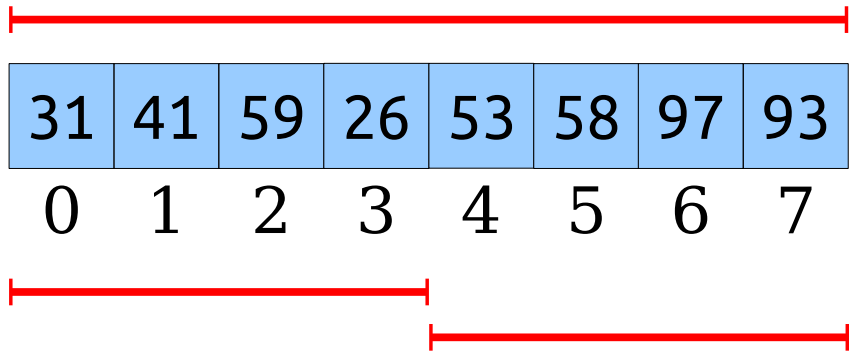




# An Observation

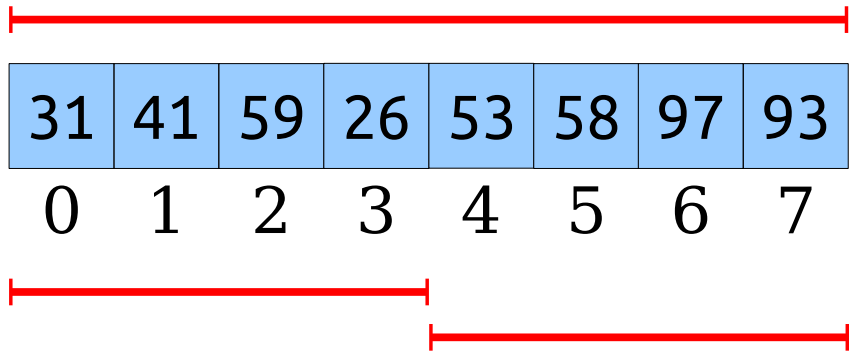


# An Observation



	0	1	2	3	4	5	6	7
0	31	31	31	26				★
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

# An Observation



	0	1	2	3	4	5	6	7
0	31	31	31	26				★
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

# An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

# An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

# An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

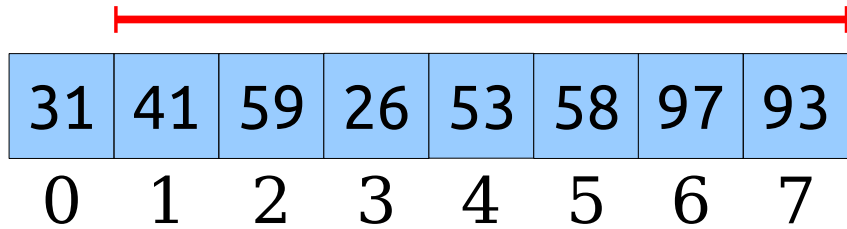
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0	31	31	31					
1		41	41	26				
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

# An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

# An Observation

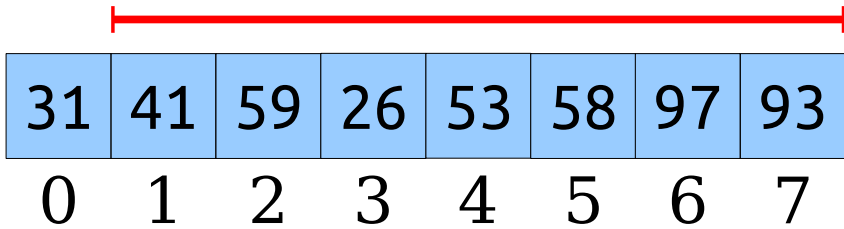


31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93



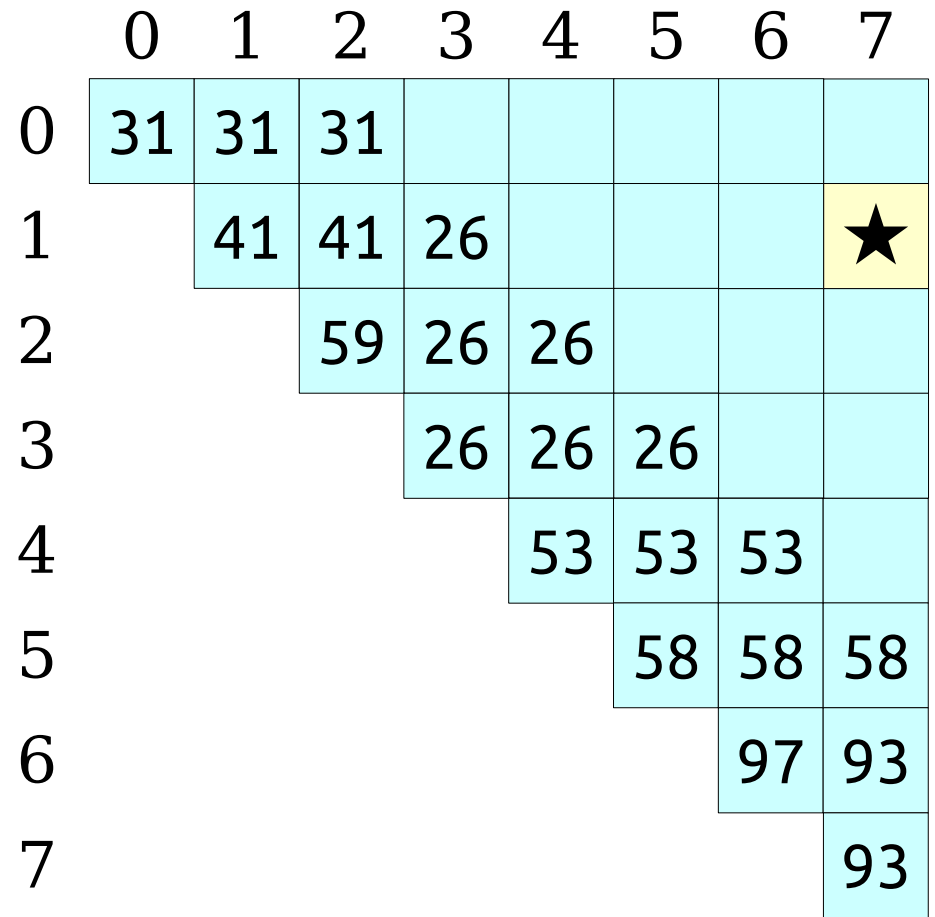
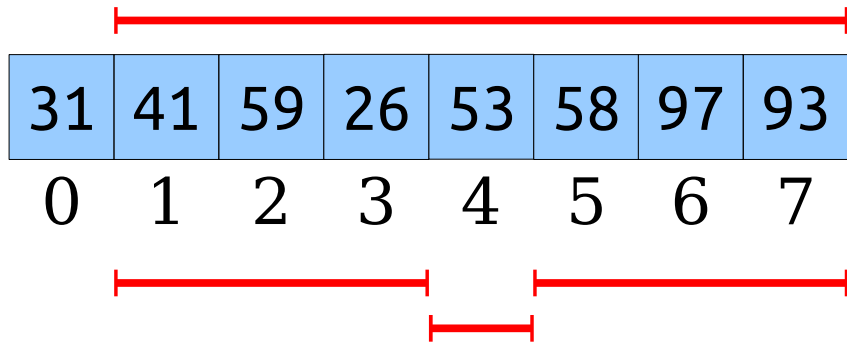
# An Observation



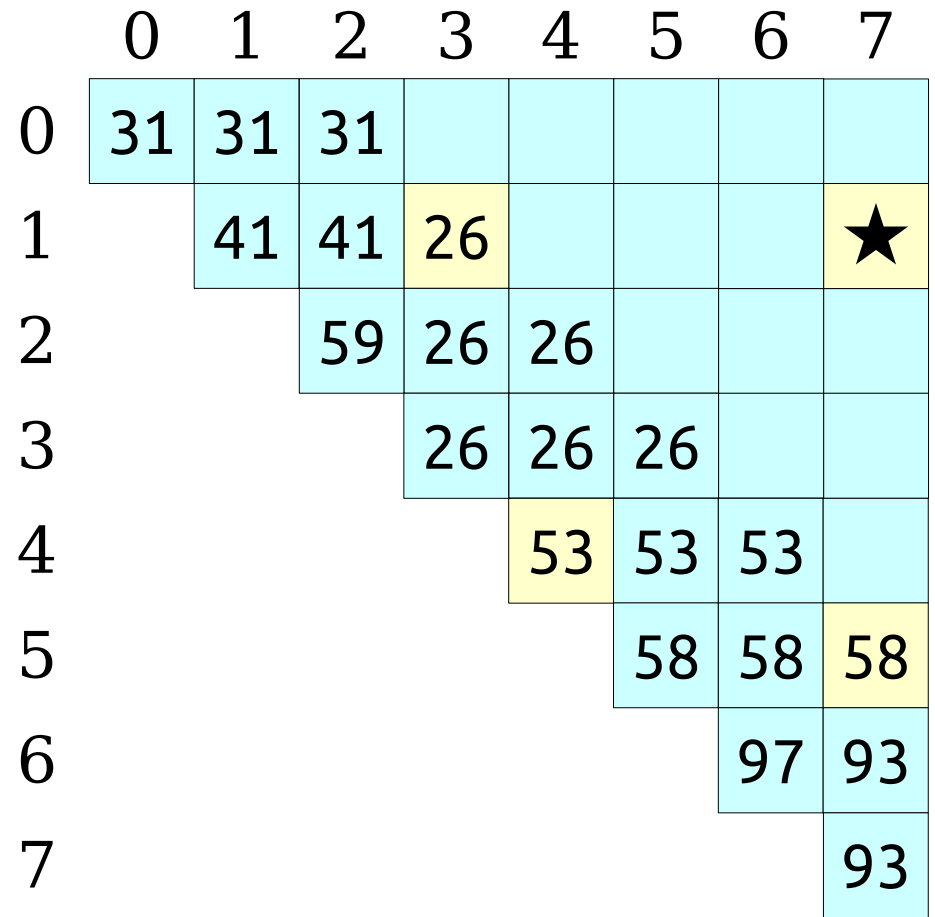
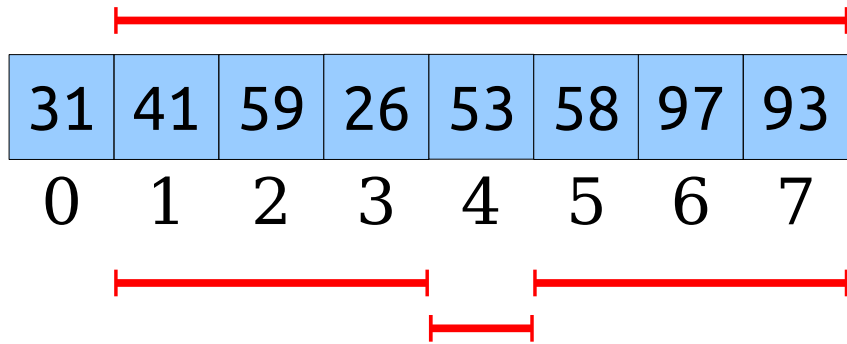
31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				★
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

# An Observation



# An Observation



# An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

# An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

# An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

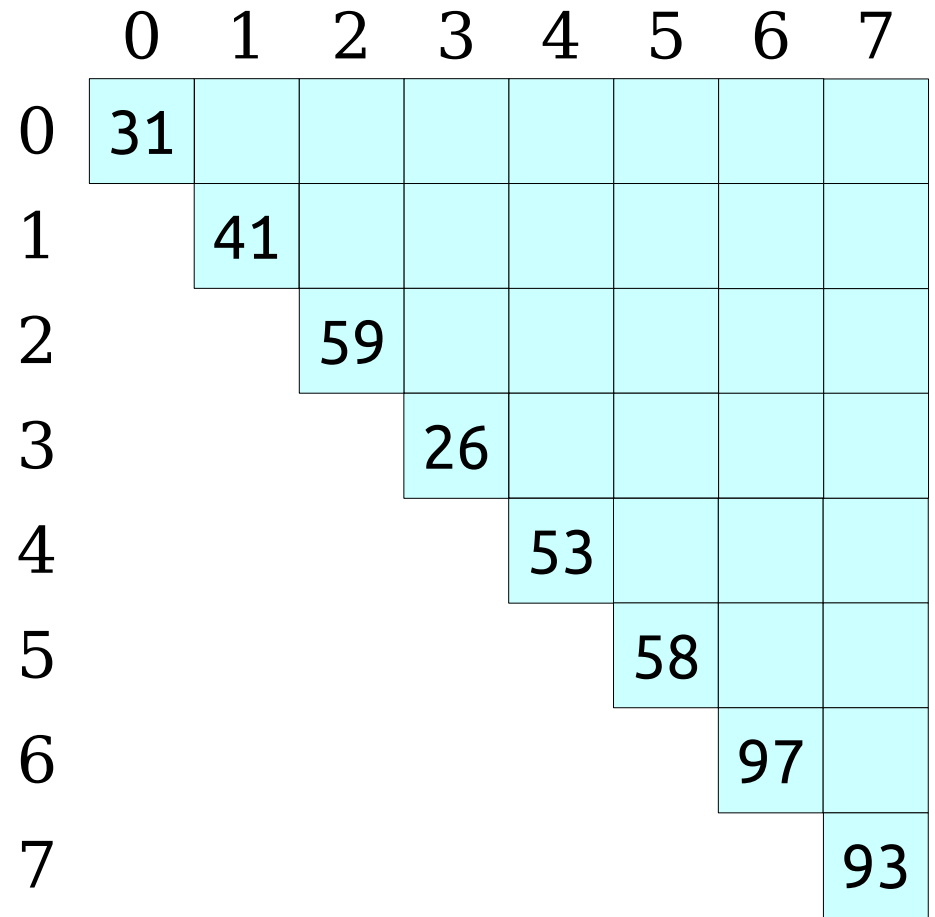
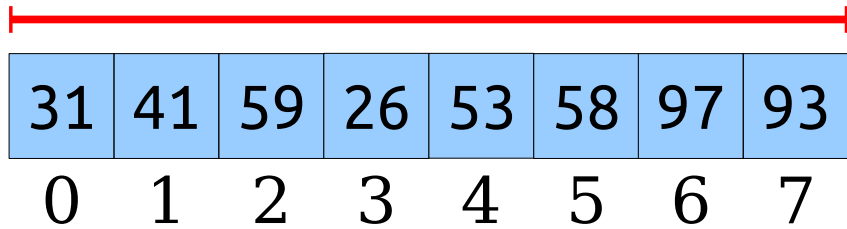
	0	1	2	3	4	5	6	7
0	31							
1		41						
2			59					
3				26				
4					53			
5						58		
6							97	
7								93

# An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

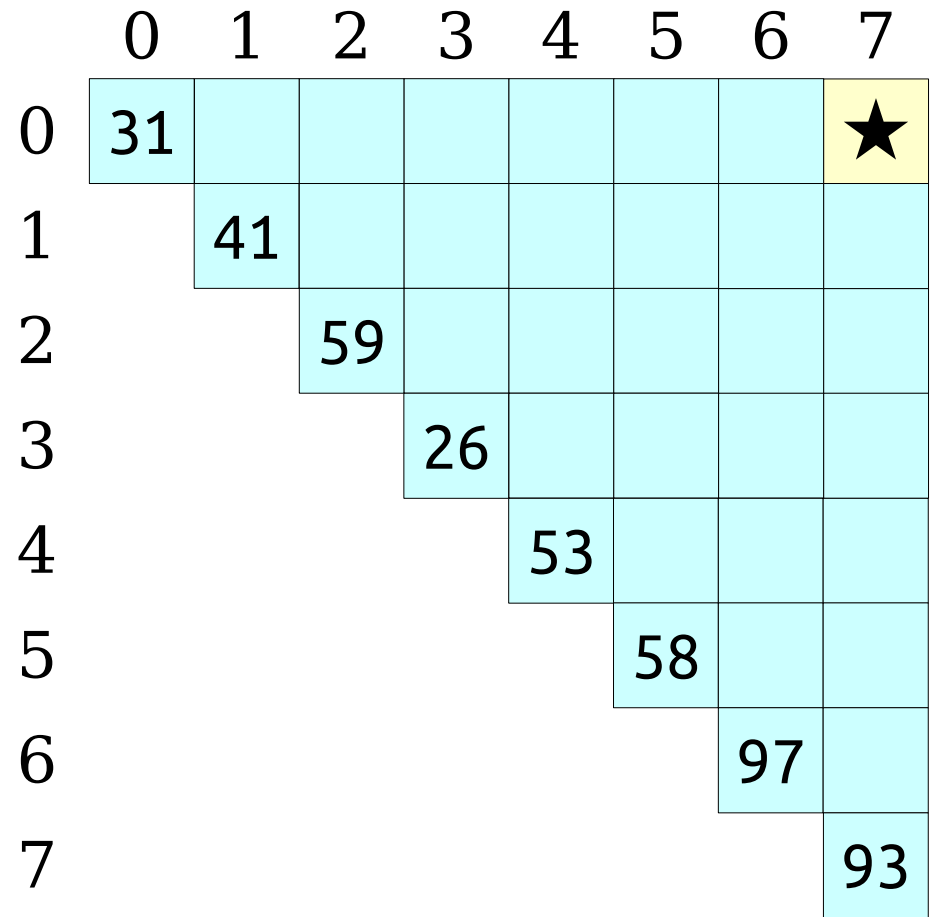
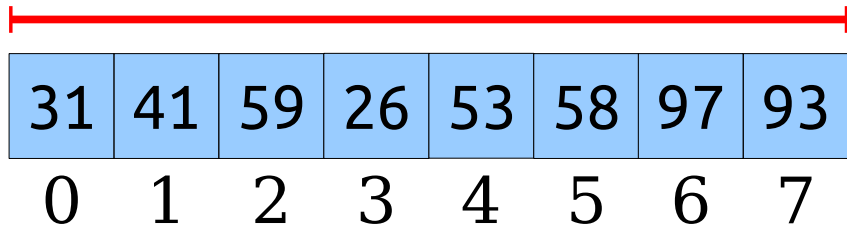
	0	1	2	3	4	5	6	7
0	31							
1		41						
2			59					
3				26				
4					53			
5						58		
6							97	
7								93

# An Observation

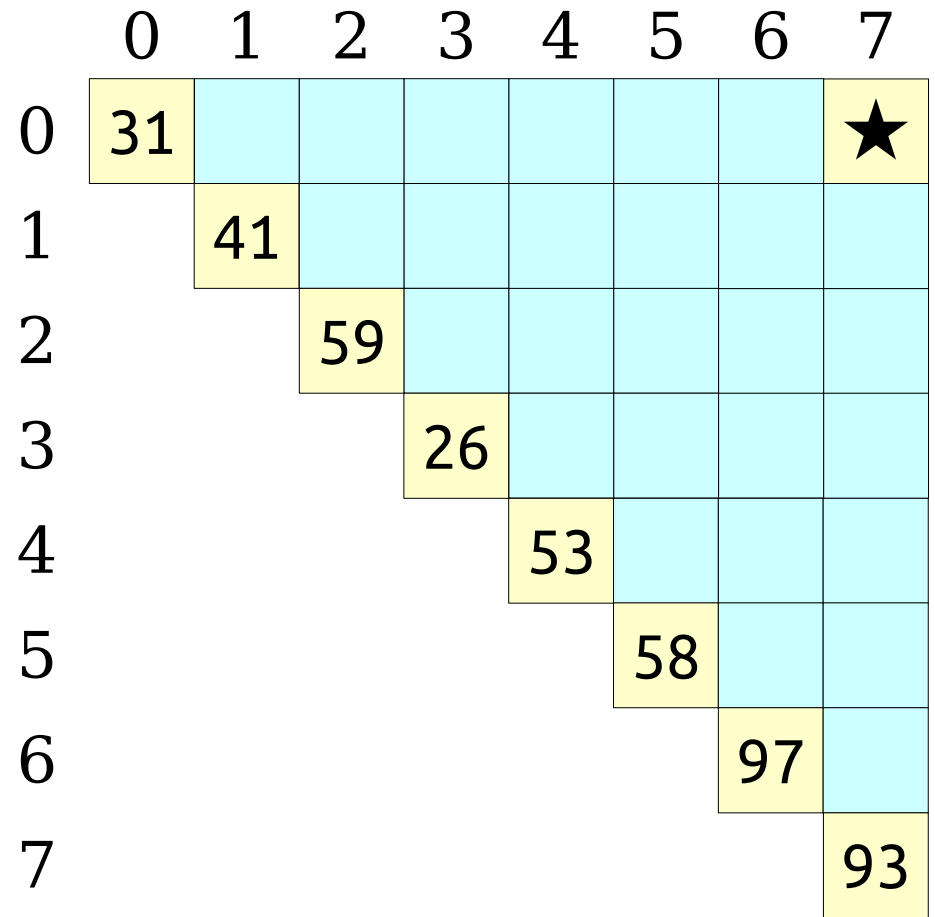
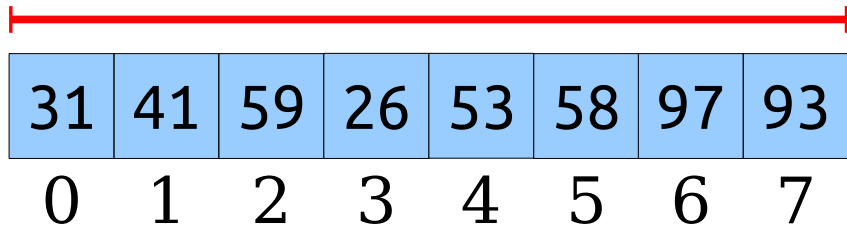




# An Observation



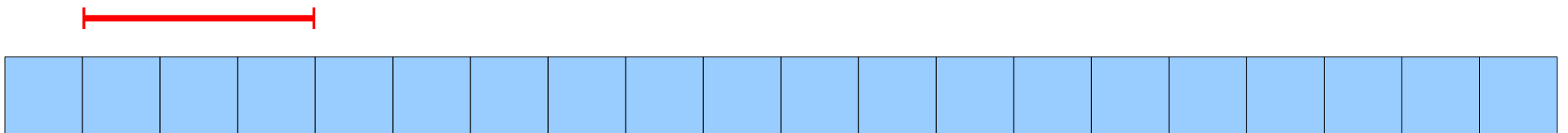
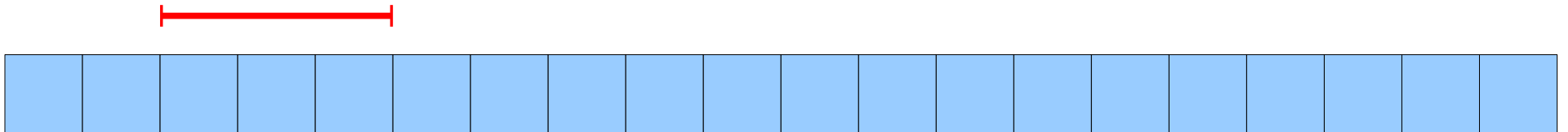
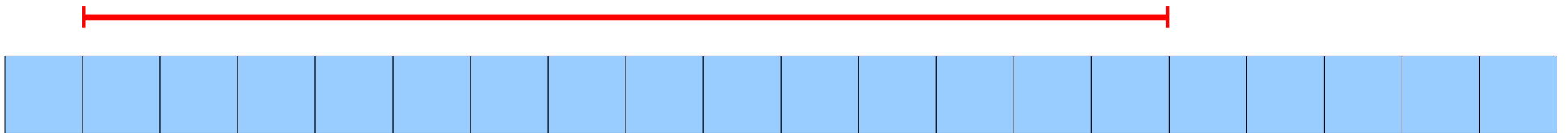
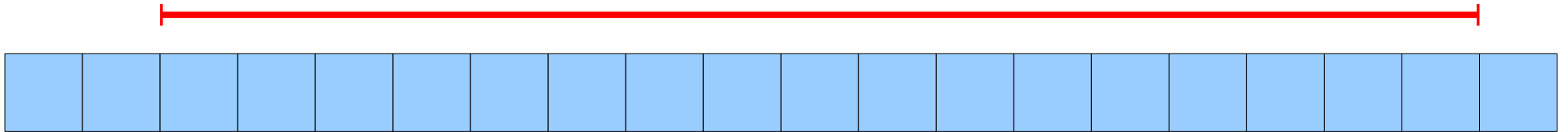
# An Observation



# The Intuition

- It's still possible to answer any query in time  $O(1)$  without precomputing RMQ over all ranges.
- If we precompute the answers over too many ranges, the preprocessing time will be too large.
- If we precompute the answers over too few ranges, the query time won't be  $O(1)$ .
- **Goal:** Precompute RMQ over a set of ranges such that
  - There are  $o(n^2)$  total ranges, but
  - there are enough ranges to support  $O(1)$  query times.

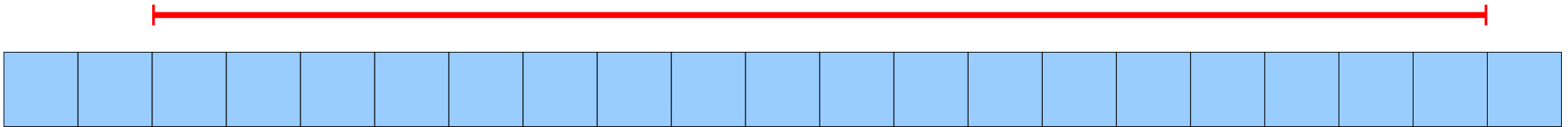
# Some Observations



# The Approach

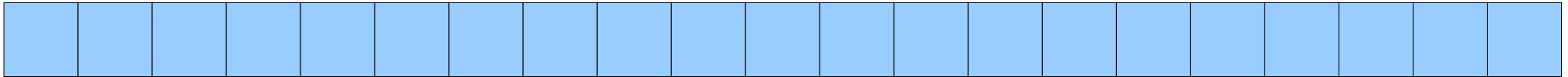
- For each index  $i$ , compute RMQ for ranges starting at  $i$  of size 1, 2, 4, 8, 16, ...,  $2^k$  as long as they fit in the array.
  - Gives both large and small ranges starting at any point in the array.
  - Only  $O(\log n)$  ranges computed for each array element.
  - Total number of ranges:  $O(n \log n)$ .
- **Claim:** Any range in the array can be formed as the union of two of these ranges.

# Creating Ranges

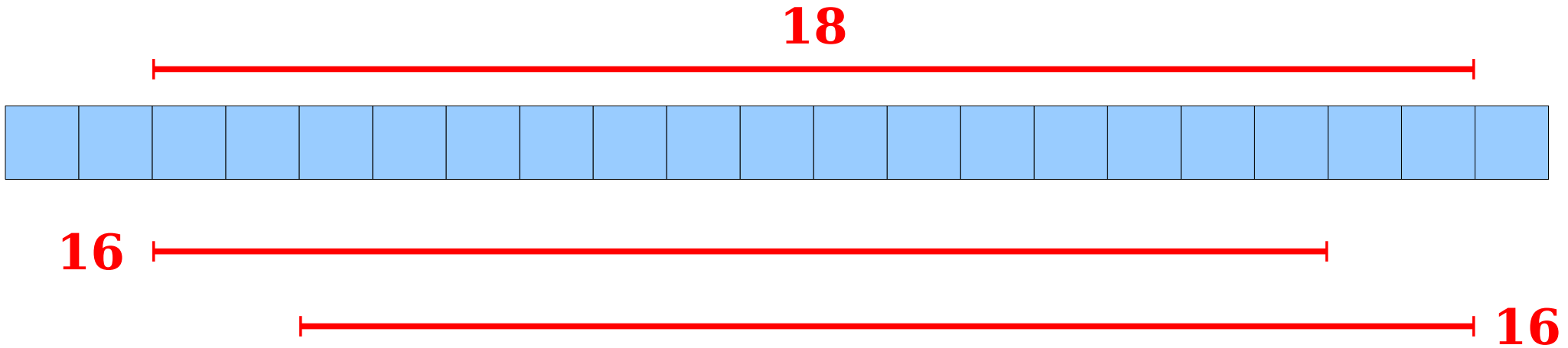


# Creating Ranges

**18**

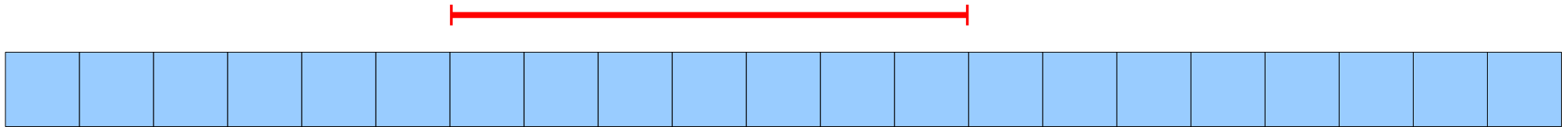


# Creating Ranges

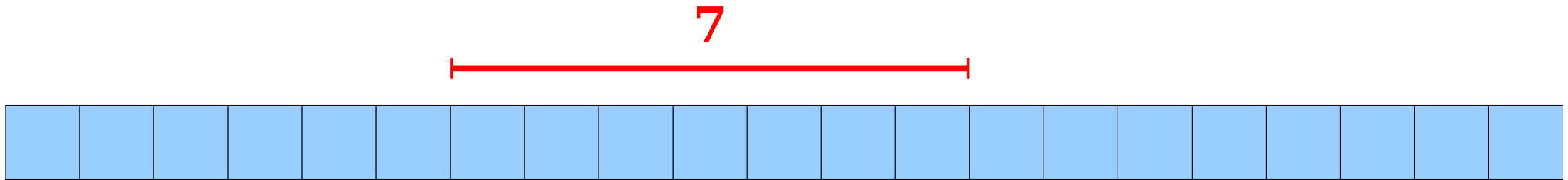




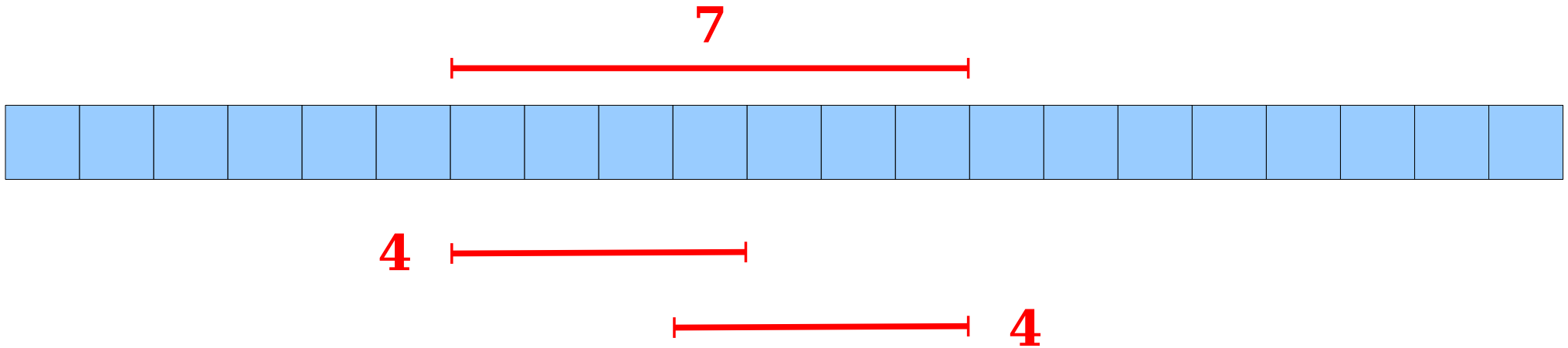
# Creating Ranges



# Creating Ranges



# Creating Ranges



# Doing a Query

- To answer  $\text{RMQ}_A(i, j)$ :
  - Find the largest  $k$  such that  $2^k \leq j - i + 1$ .
    - With the right preprocessing, this can be done in time  $O(1)$ ; you'll figure out how in Problem Set One.
  - The range  $[i, j]$  can be formed as the overlap of the ranges  $[i, i + 2^k - 1]$  and  $[j - 2^k + 1, j]$ .
  - Each range can be looked up in time  $O(1)$ .
  - Total time:  **$O(1)$** .

# Precomputing the Ranges

- There are  $O(n \log n)$  ranges to precompute.
- Using dynamic programming, we can compute all of them in time  $O(n \log n)$ .

31	41	59	26	53	58	97	93
<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>

	$2^0$	$2^1$	$2^2$	$2^3$
<b>0</b>				
<b>1</b>				
<b>2</b>				
<b>3</b>				
<b>4</b>				
<b>5</b>				
<b>6</b>				
<b>7</b>				

# Precomputing the Ranges

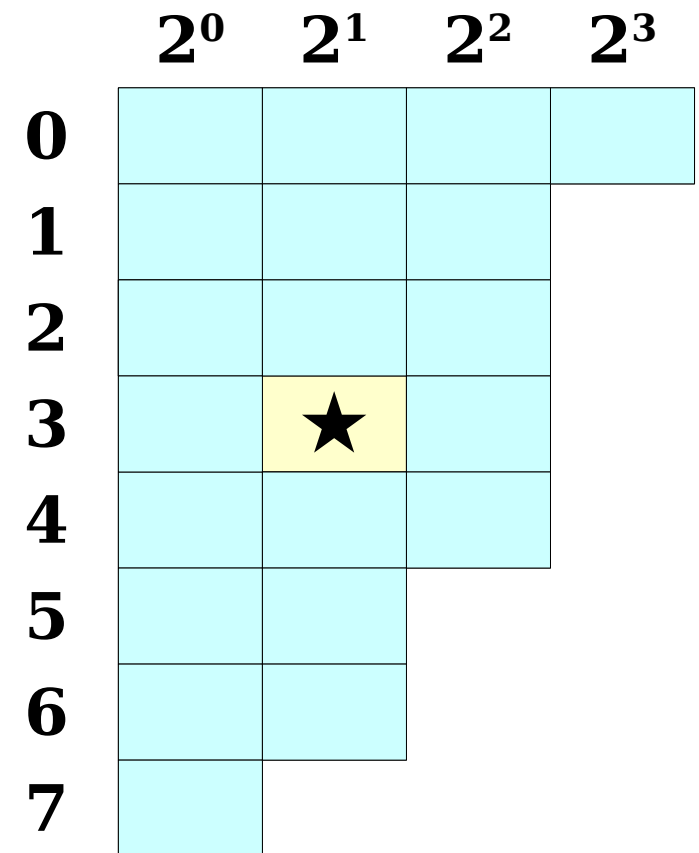
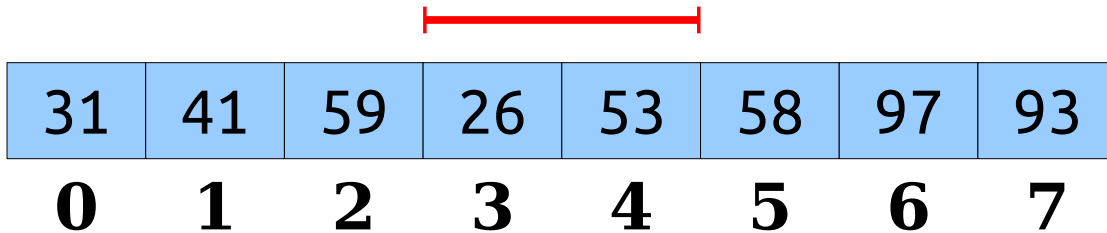
- There are  $O(n \log n)$  ranges to precompute.
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31	41	59	26	53	58	97	93
<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>

	$2^0$	$2^1$	$2^2$	$2^3$
<b>0</b>				
<b>1</b>				
<b>2</b>				
<b>3</b>		★		
<b>4</b>				
<b>5</b>				
<b>6</b>				
<b>7</b>				

# Precomputing the Ranges

- There are  $O(n \log n)$  ranges to precompute.
- Using dynamic programming, we can compute all of them in time  $O(n \log n)$ .



# Precomputing the Ranges

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- Using dynamic programming, we can compute all of them in time  $O(n \log n)$ .

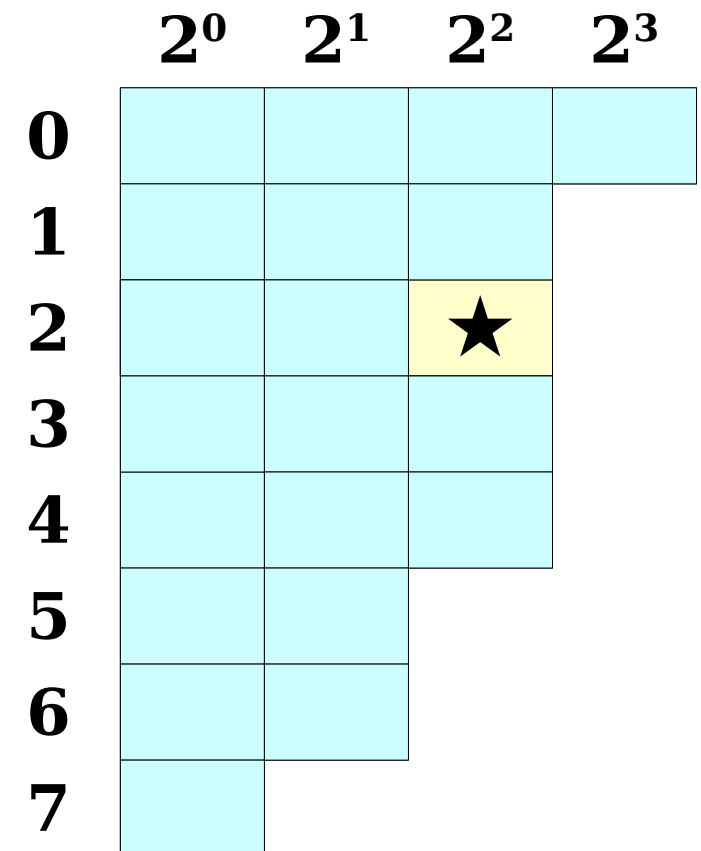
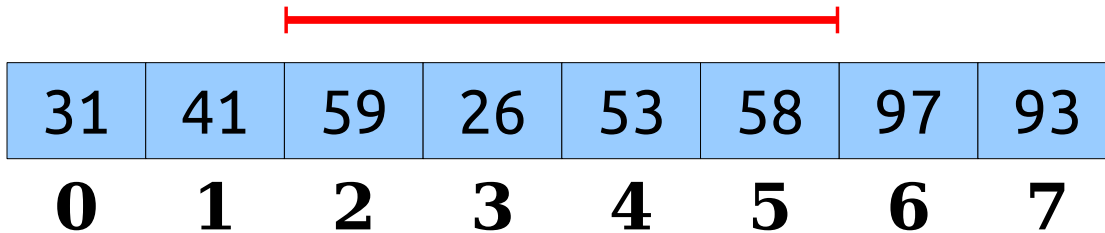
31	41	59	26	53	58	97	93
<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>

	$2^0$	$2^1$	$2^2$	$2^3$
<b>0</b>				
<b>1</b>				
<b>2</b>			★	
<b>3</b>				
<b>4</b>				
<b>5</b>				
<b>6</b>				
<b>7</b>				



# Precomputing the Ranges

- There are  $O(n \log n)$  ranges to precompute.
- Using dynamic programming, we can compute all of them in time  $O(n \log n)$ .



# Precomputing the Ranges

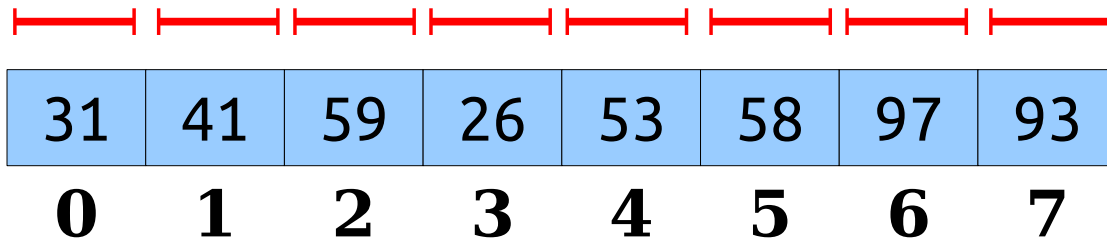
- There are  $O(n \log n)$  ranges to precompute.
- Using dynamic programming, we can compute all of them in time  $O(n \log n)$ .

31	41	59	26	53	58	97	93
<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>

	$2^0$	$2^1$	$2^2$	$2^3$
<b>0</b>				
<b>1</b>				
<b>2</b>				
<b>3</b>				
<b>4</b>				
<b>5</b>				
<b>6</b>				
<b>7</b>				

# Precomputing the Ranges

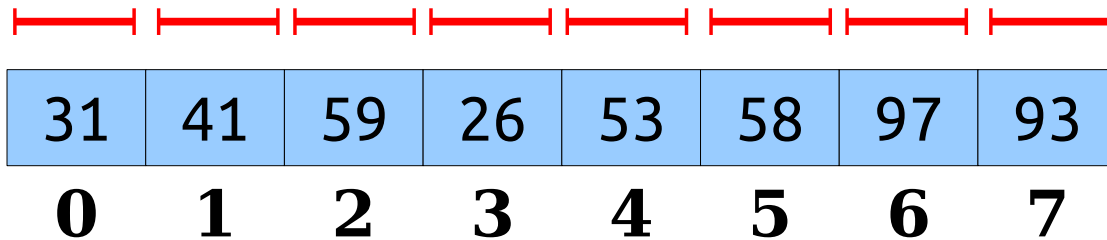
- There are  $O(n \log n)$  ranges to precompute.
- Using dynamic programming, we can compute all of them in time  $O(n \log n)$ .



	$2^0$	$2^1$	$2^2$	$2^3$
<b>0</b>				
<b>1</b>				
<b>2</b>				
<b>3</b>				
<b>4</b>				
<b>5</b>				
<b>6</b>				
<b>7</b>				

# Precomputing the Ranges

- There are  $O(n \log n)$  ranges to precompute.
- Using dynamic programming, we can compute all of them in time  $O(n \log n)$ .



	$2^0$	$2^1$	$2^2$	$2^3$
<b>0</b>	31			
<b>1</b>	41			
<b>2</b>	59			
<b>3</b>	26			
<b>4</b>	53			
<b>5</b>	58			
<b>6</b>	97			
<b>7</b>	93			

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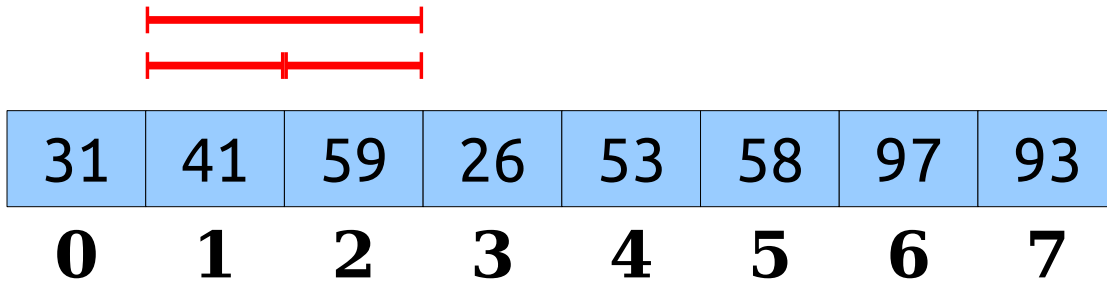
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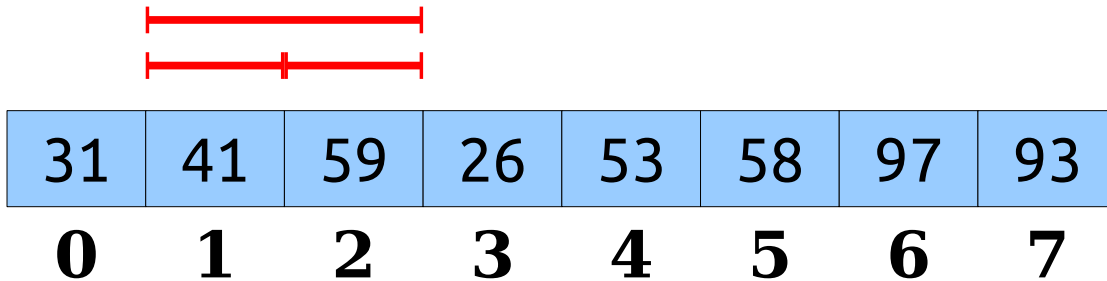
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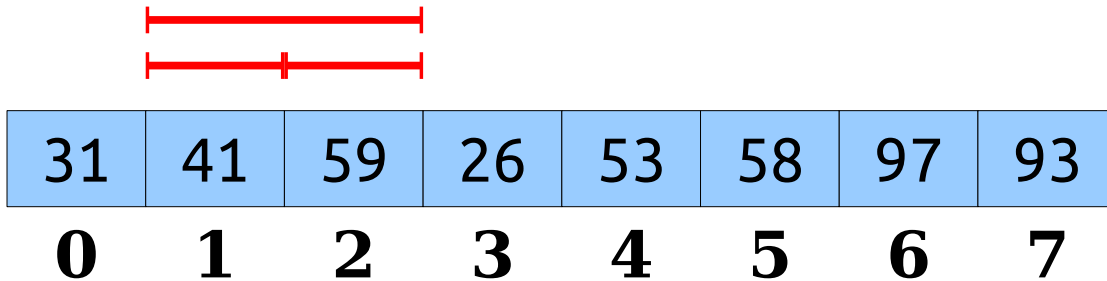
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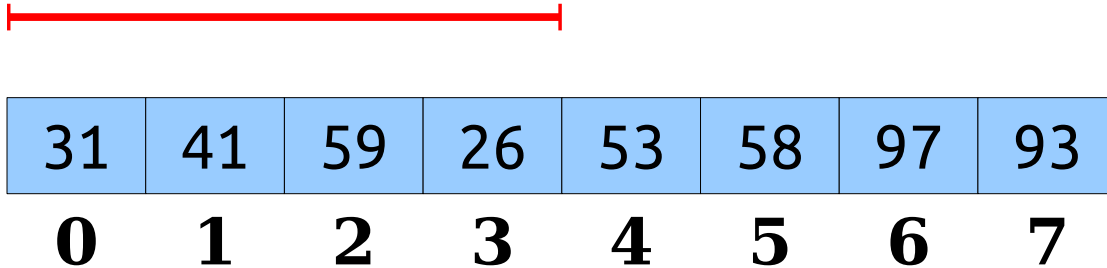
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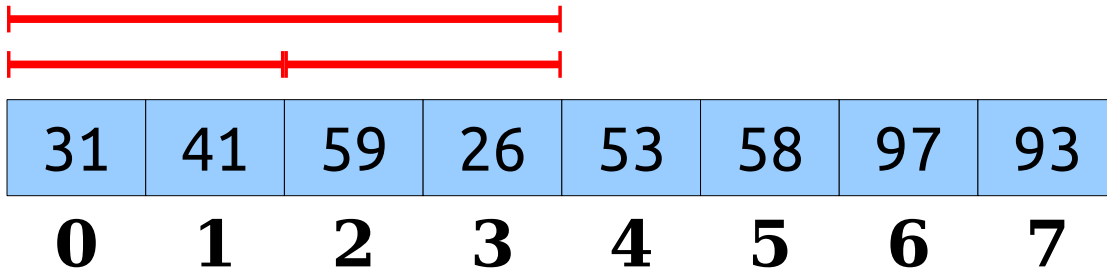
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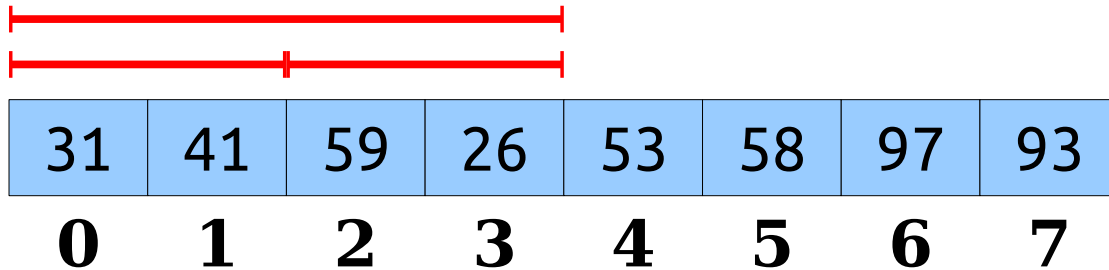
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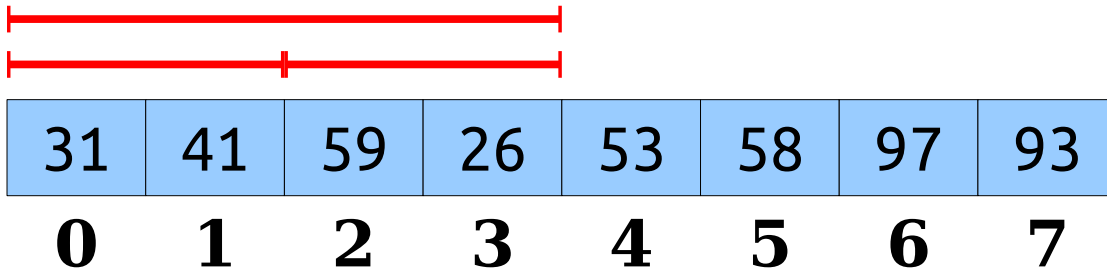
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# Sparse Tables

- This data structure is called a ***sparse table***.
- It gives an  **$\langle O(n \log n), O(1) \rangle$**  solution to RMQ.
- This is asymptotically better than precomputing all possible ranges!

# The Story So Far

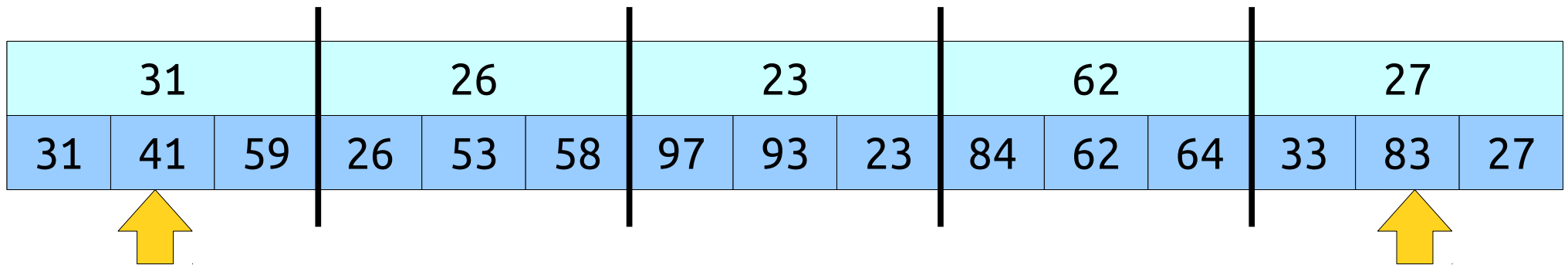
- We now have the following solutions for RMQ:
  - Precompute all:  $\langle O(n^2), O(1) \rangle$ .
  - Sparse table:  $\langle O(n \log n), O(1) \rangle$ .
  - Blocking:  $\langle O(n), O(n^{1/2}) \rangle$ .
  - Precompute none:  $\langle O(1), O(n) \rangle$ .
- ***Can we do better?***

A Third Approach: ***Hybrid Strategies***

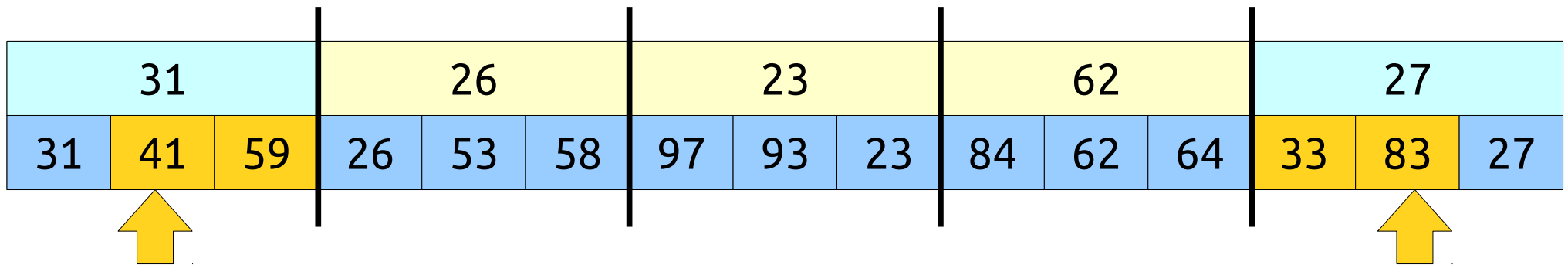
# Blocking Revisited

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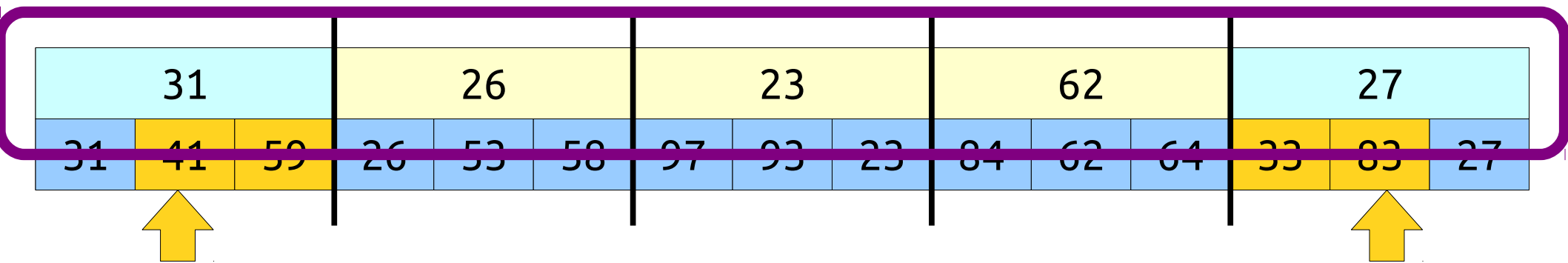
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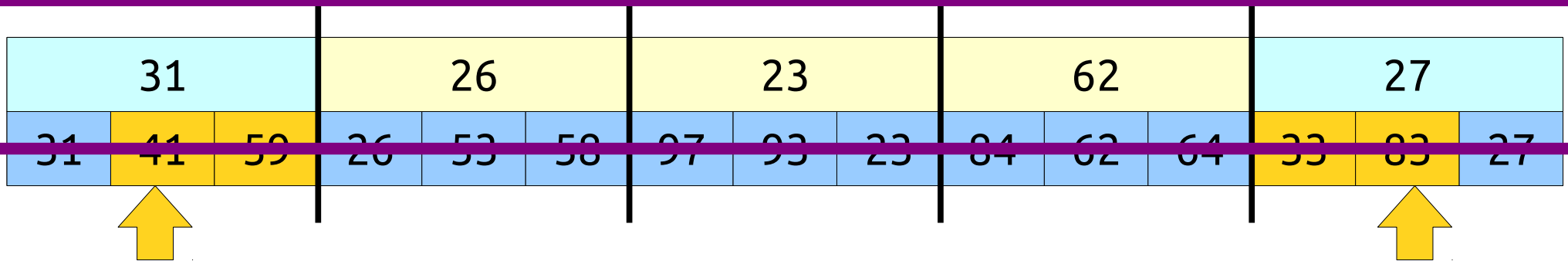
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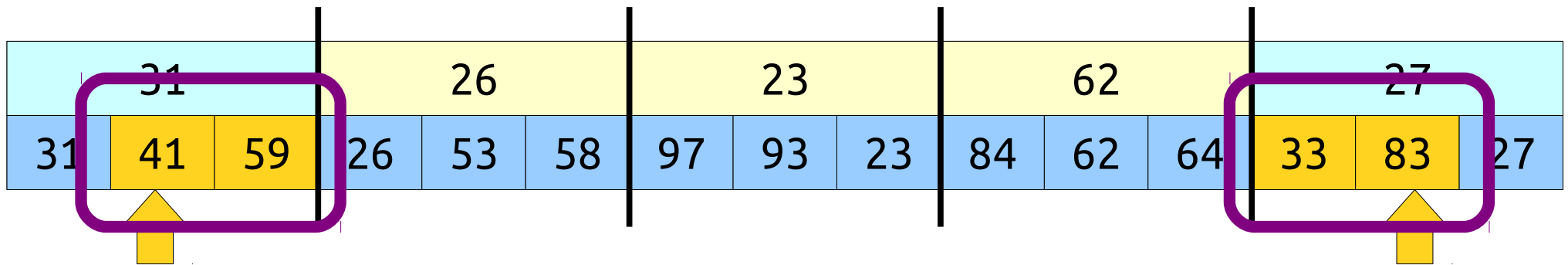


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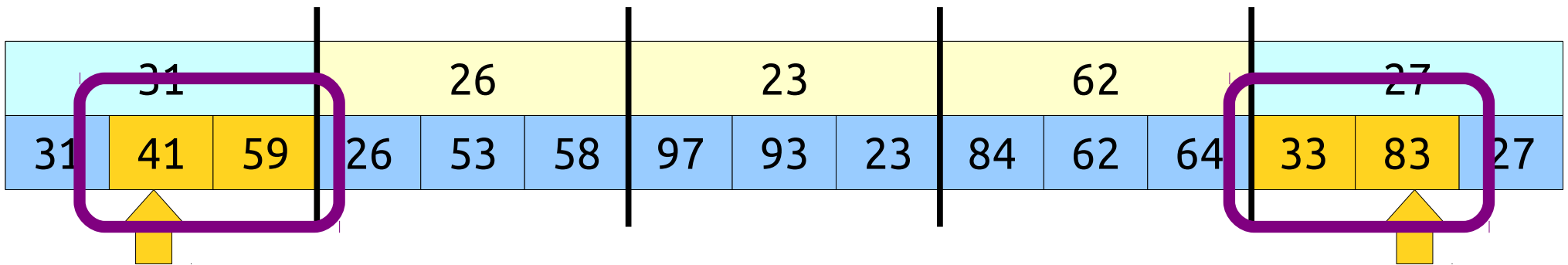
*This is just RMQ on the block minima!*



# Blocking Revisited



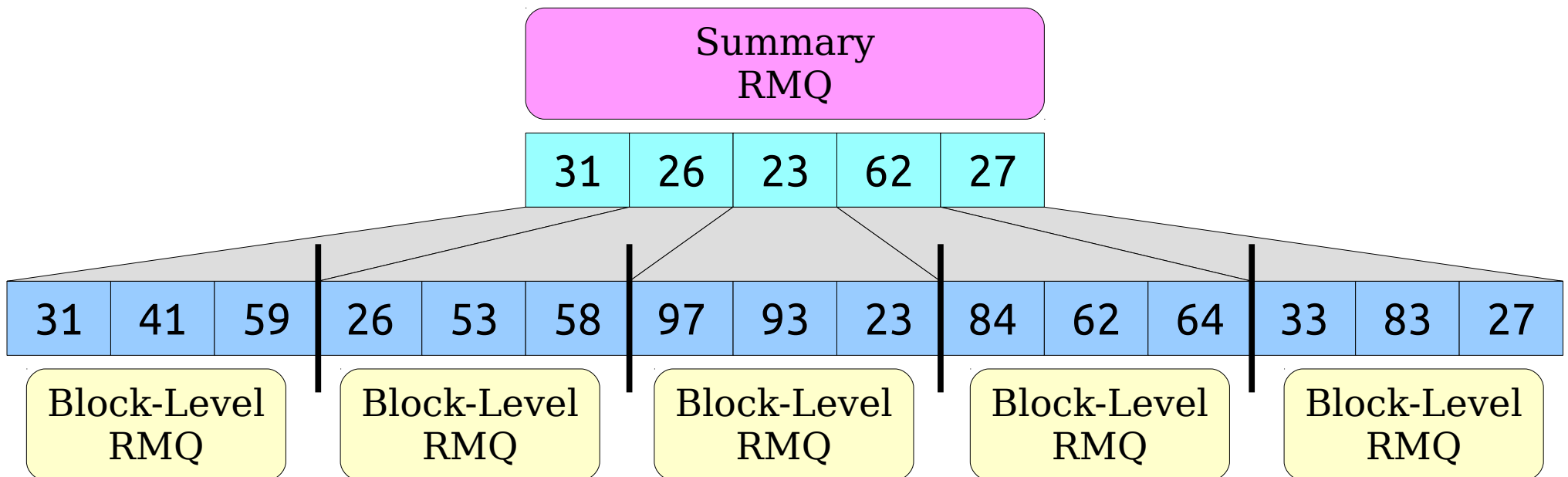
# Blocking Revisited



*This is just RMQ  
inside the blocks!*

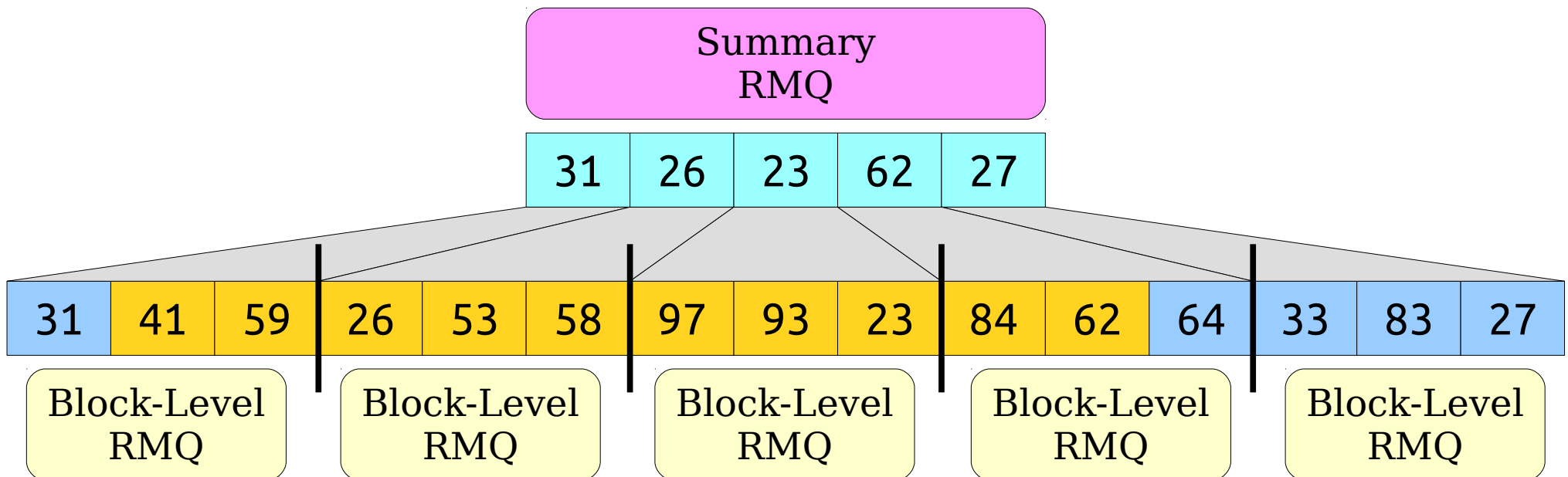
# The Framework

- Split the input into blocks of size  $b$ .
- Form an array of the block minima.
- Construct a “summary” RMQ structure over the block minima.
- Construct “block” RMQ structures for each block.
- Aggregate the results together.



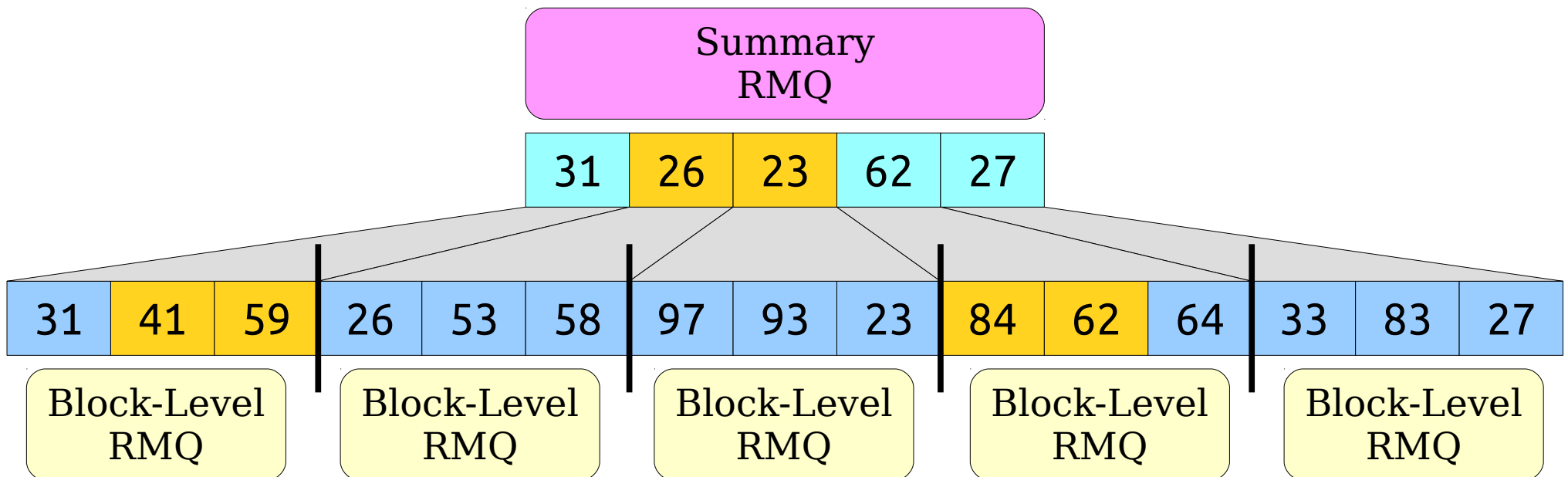
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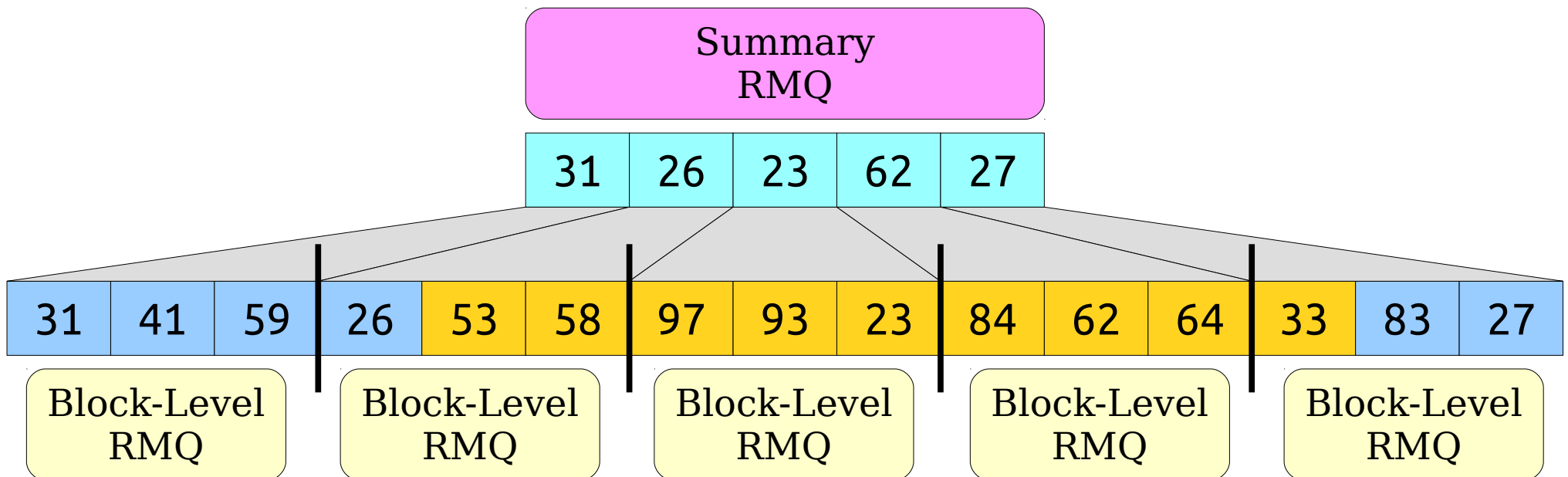
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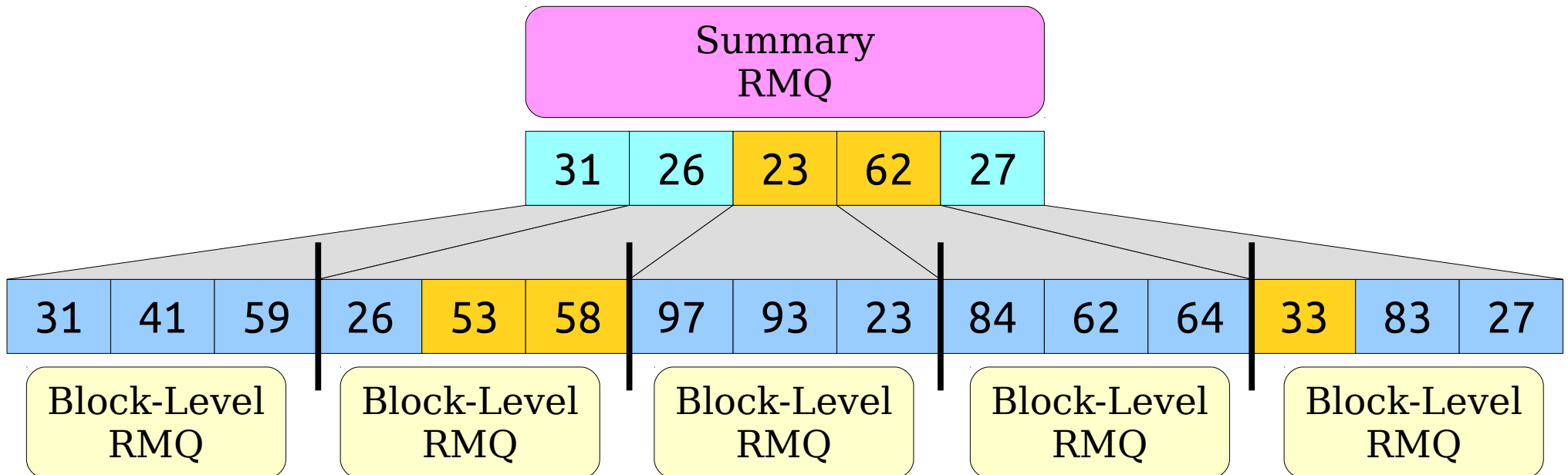
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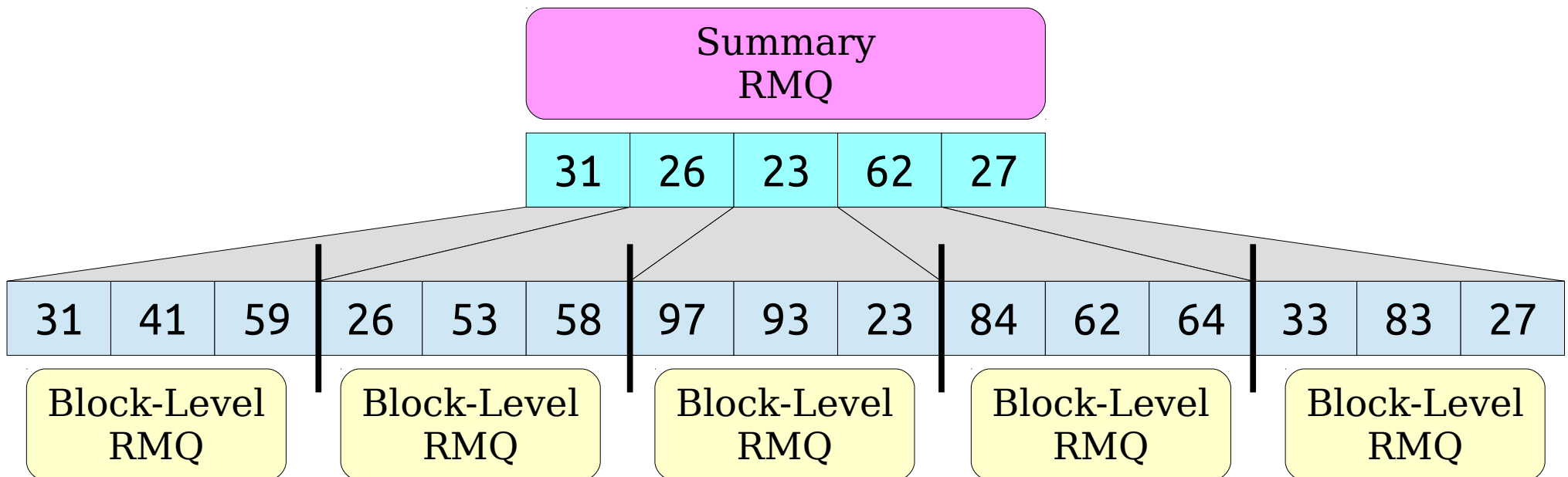
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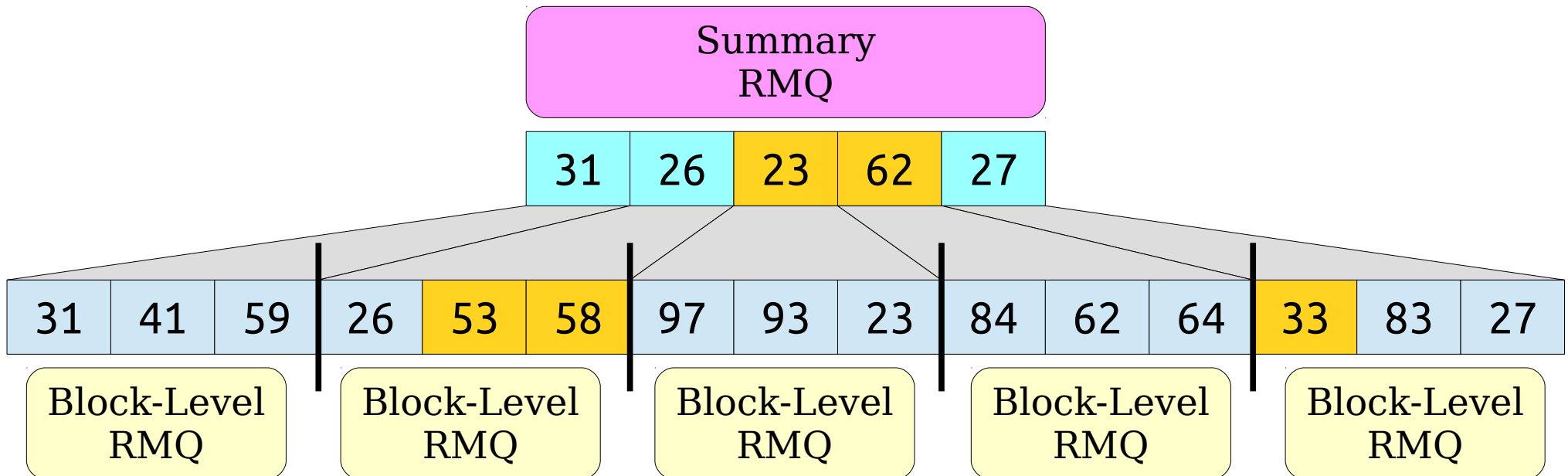
# Analyzing Efficiency

- Suppose we use a  $\langle p_1(n), q_1(n) \rangle$ -time RMQ for the block minima and a  $\langle p_2(n), q_2(n) \rangle$ -time RMQ within each block, with block size  $b$ .
- What is the preprocessing time for this hybrid structure?
  - $O(n)$  time to compute the minima of each block.
  - $O(p_1(n/b))$  time to construct RMQ on the minima.
  - $O((n/b) p_2(b))$  time to construct the block RMQs
- Total construction time is  $O(n + p_1(n/b) + (n/b) p_2(b))$ .



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- What is the query time for this hybrid structure?
  - $O(q_1(n / b))$  time to query the summary RMQ.
  - $O(q_2(b))$  time to query the block RMQs.
- Total query time:  $O(q_1(n / b) + q_2(b))$ .



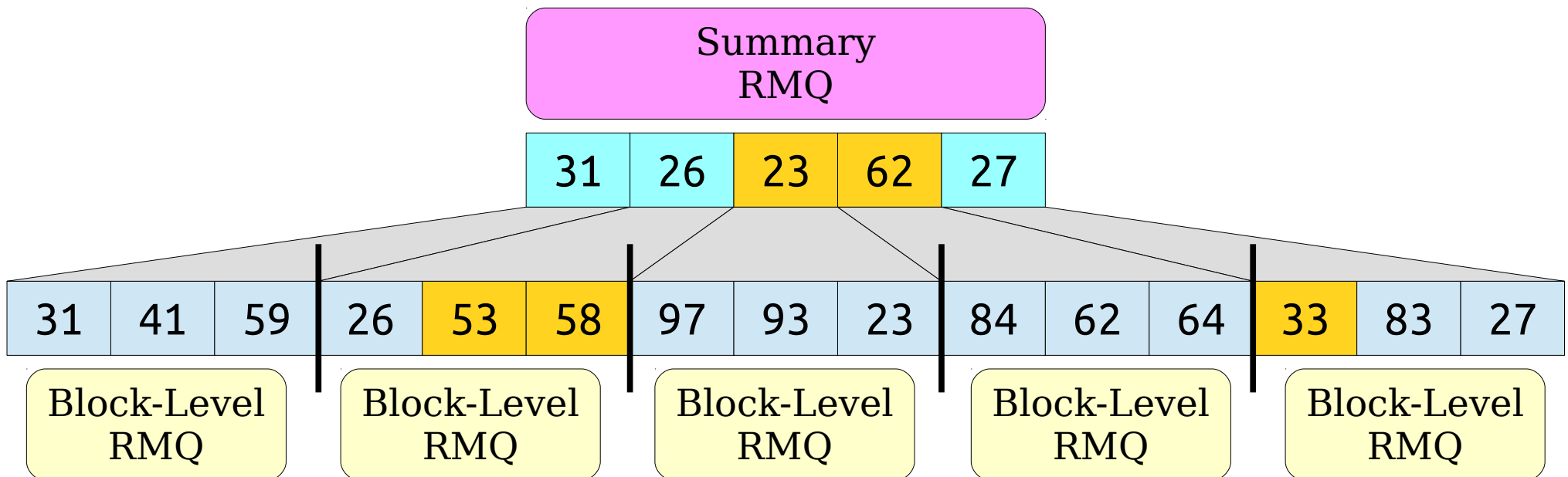
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- Hybrid preprocessing time:

$$O(n + p_1(n/b) + (n/b)p_2(b))$$

- Hybrid query time:

$$O(q_1(n/b) + q_2(b))$$



# A Sanity Check

- The  $\langle O(n), O(n^{1/2}) \rangle$  block-based structure from earlier uses this framework with the  $\langle O(1), O(n) \rangle$  no-preprocessing RMQ structure and  $b = n^{1/2}$ .

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- Looks good so far!

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# An Observation

- We can use any data structures we'd like for the summary and block RMQs.
- Suppose we use an  $\langle O(n \log n), O(1) \rangle$  sparse table for the summary RMQ.
- If the block size is  $b$ , the time to construct a sparse table over the  $(n / b)$  blocks is  $O((n / b) \log (n / b))$ .
- **Cute trick:** If  $b = \Theta(\log n)$ , the time to construct a sparse table over the minima is

$$\begin{aligned} & O((n / \log n) \log(n / \log n)) \\ &= O((n / \log n) \log n) && \textit{(O is an upper bound)} \\ &= \mathbf{O(n)}. && \textit{(logs cancel out)} \end{aligned}$$

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- Set the block size to  $\log n$ .
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- We have an  $\langle \mathbf{O(n \log \log n)}, \mathbf{O(1)} \rangle$  solution to RMQ!

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# Where We Stand

- We've seen a bunch of RMQ structures today:
  - No preprocessing:  $\langle O(1), O(n) \rangle$
  - Full preprocessing:  $\langle O(n^2), O(1) \rangle$
  - Block partition:  $\langle O(n), O(n^{1/2}) \rangle$
  - Sparse table:  $\langle O(n \log n), O(1) \rangle$
  - Hybrid 1:  $\langle O(n), O(\log n) \rangle$
  - Hybrid 2:  $\langle O(n \log \log n), O(1) \rangle$
  - Hybrid 3:  $\langle O(n), O(\log \log n) \rangle$

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Full preprocessing:  $\langle O(n^2), O(1) \rangle$

- **Block partition:**  $\langle O(n), O(n^{1/2}) \rangle$

Sparse table:  $\langle O(n \log n), O(1) \rangle$

- **Hybrid 1:**  $\langle O(n), O(\log n) \rangle$

Hybrid 2:  $\langle O(n \log \log n), O(1) \rangle$

- **Hybrid 3:**  $\langle O(n), O(\log \log n) \rangle$

Is there an  $\langle O(n), O(1) \rangle$  solution to RMQ?

***Yes!***

# Next Time

- ***Cartesian Trees***
  - A data structure closely related to RMQ.
- ***The Method of Four Russians***
  - A technique for shaving off log factors.
- ***The Fischer-Heun Structure***
  - A deceptively simple, asymptotically optimal RMQ structure.