

Binomial Heaps

Where We're Going

- ***Binomial Heaps (Today)***
 - A simple, flexible, and versatile priority queue.
- ***Lazy Binomial Heaps (Today)***
 - A powerful building block for designing advanced data structures.
- ***Fibonacci Heaps (Tuesday)***
 - A heavyweight and theoretically excellent priority queue.

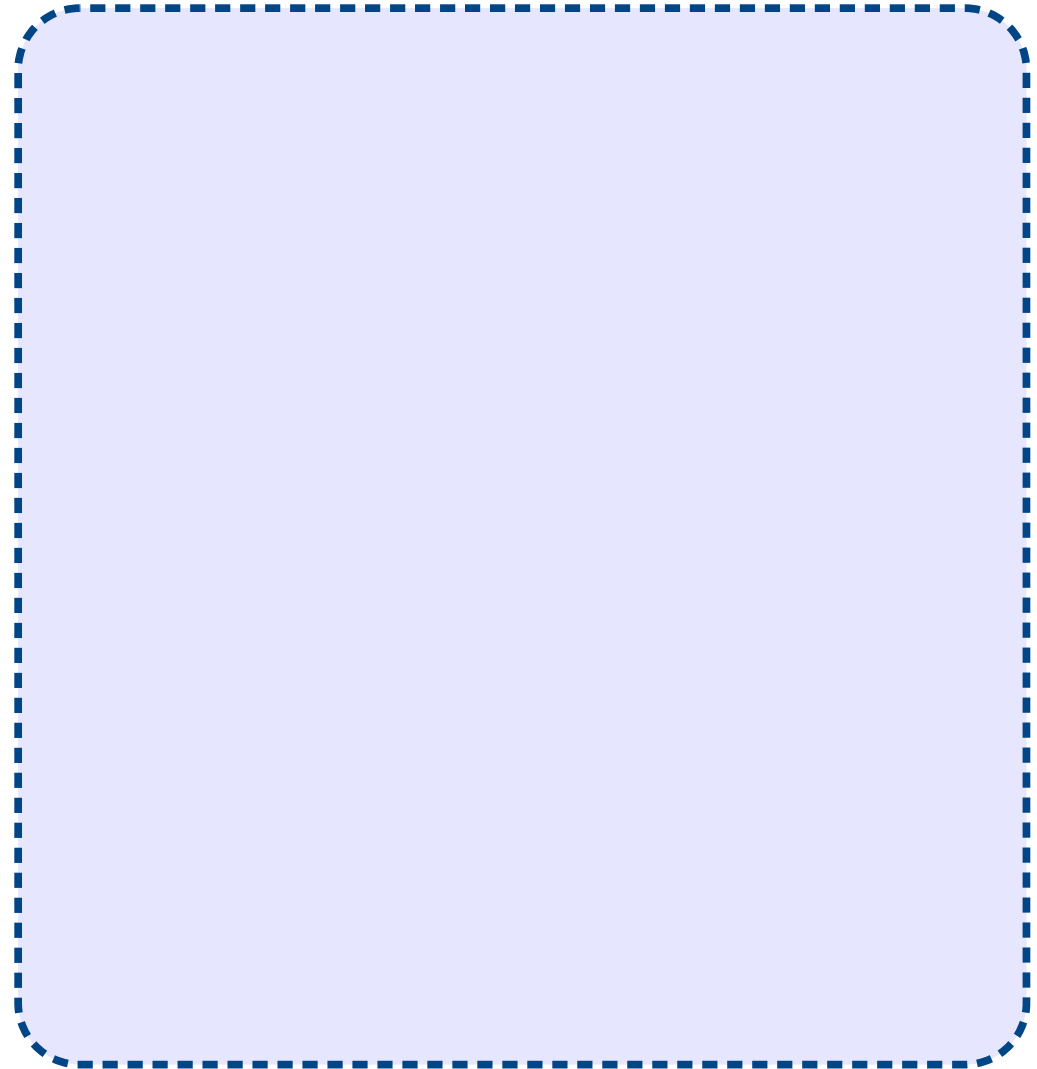
Review: Priority Queues

Priority Queues

- A **priority queue** is a data structure that supports these operations:
 - $pq.enqueue(v, k)$, which enqueues element v with key k ;
 - $pq.find-min()$, which returns the element with the least key; and
 - $pq.extract-min()$, which removes and returns the element with the least key.
- They're useful as building blocks in a *bunch* of algorithms.

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Mt. Giluwe

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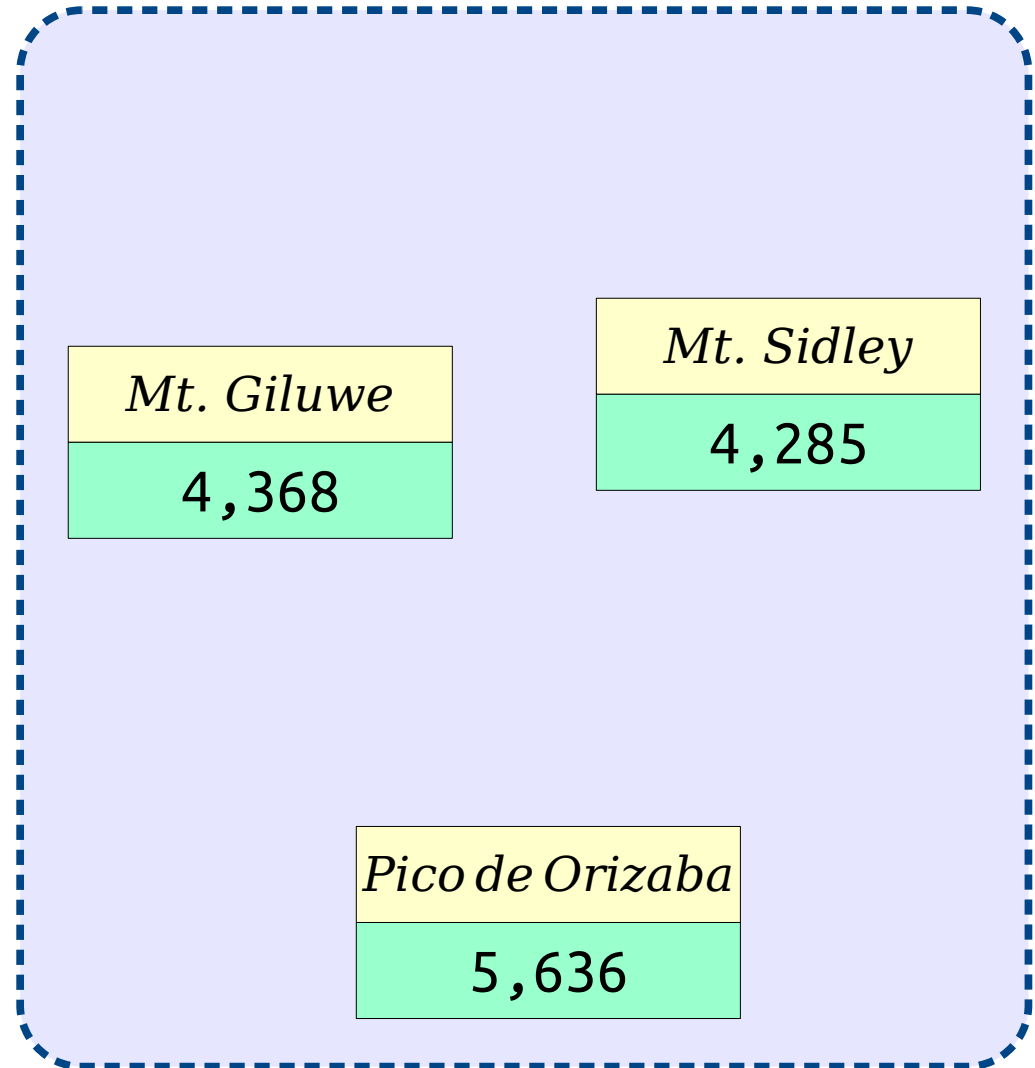
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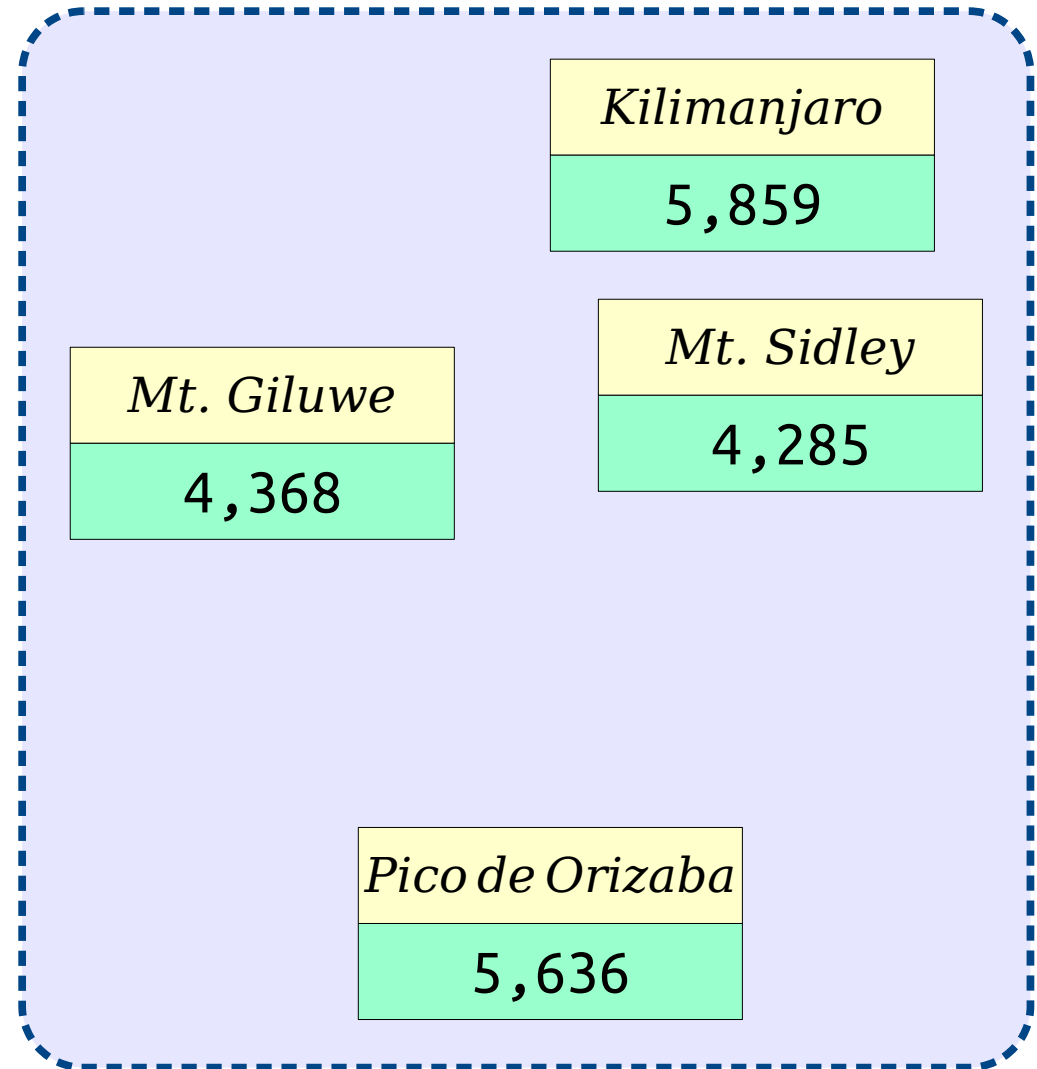
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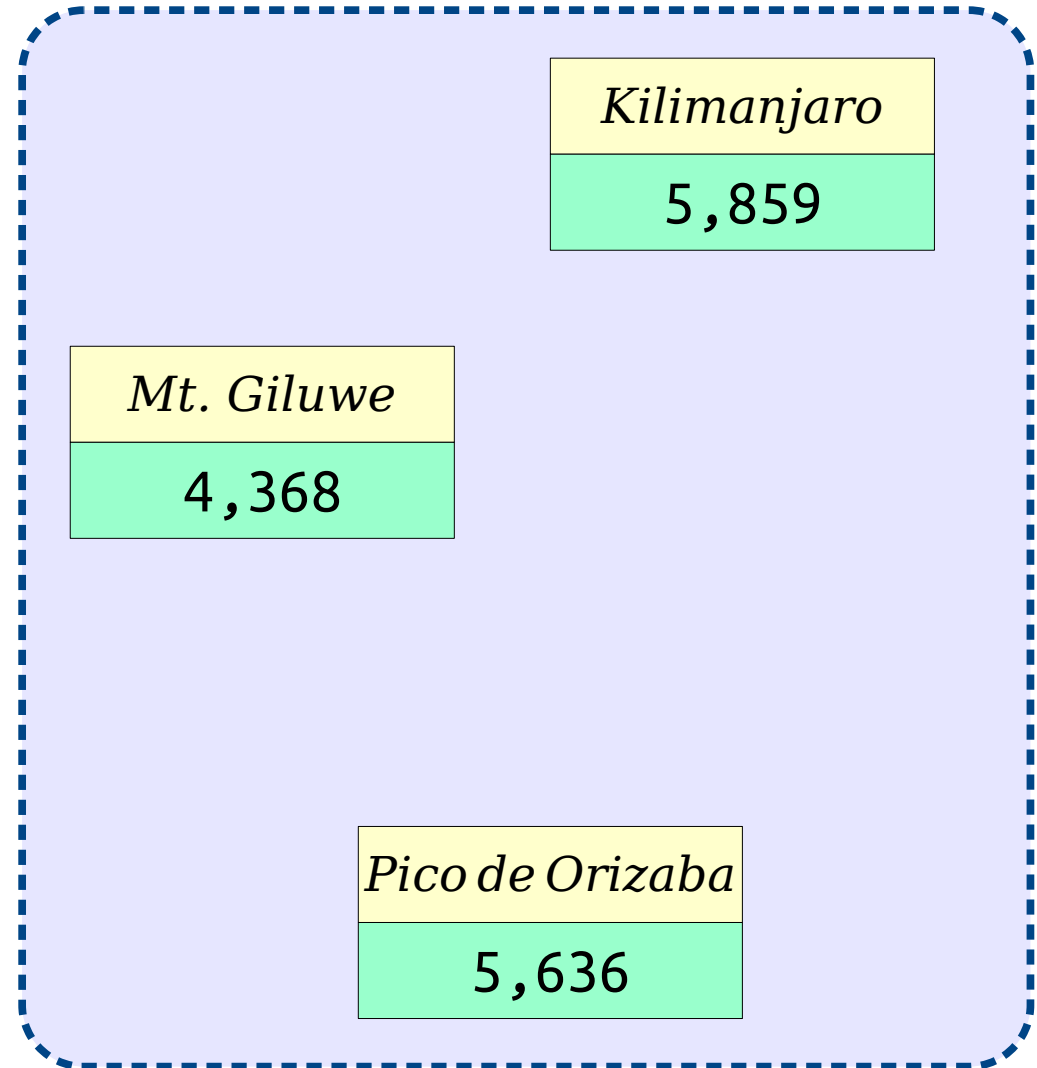
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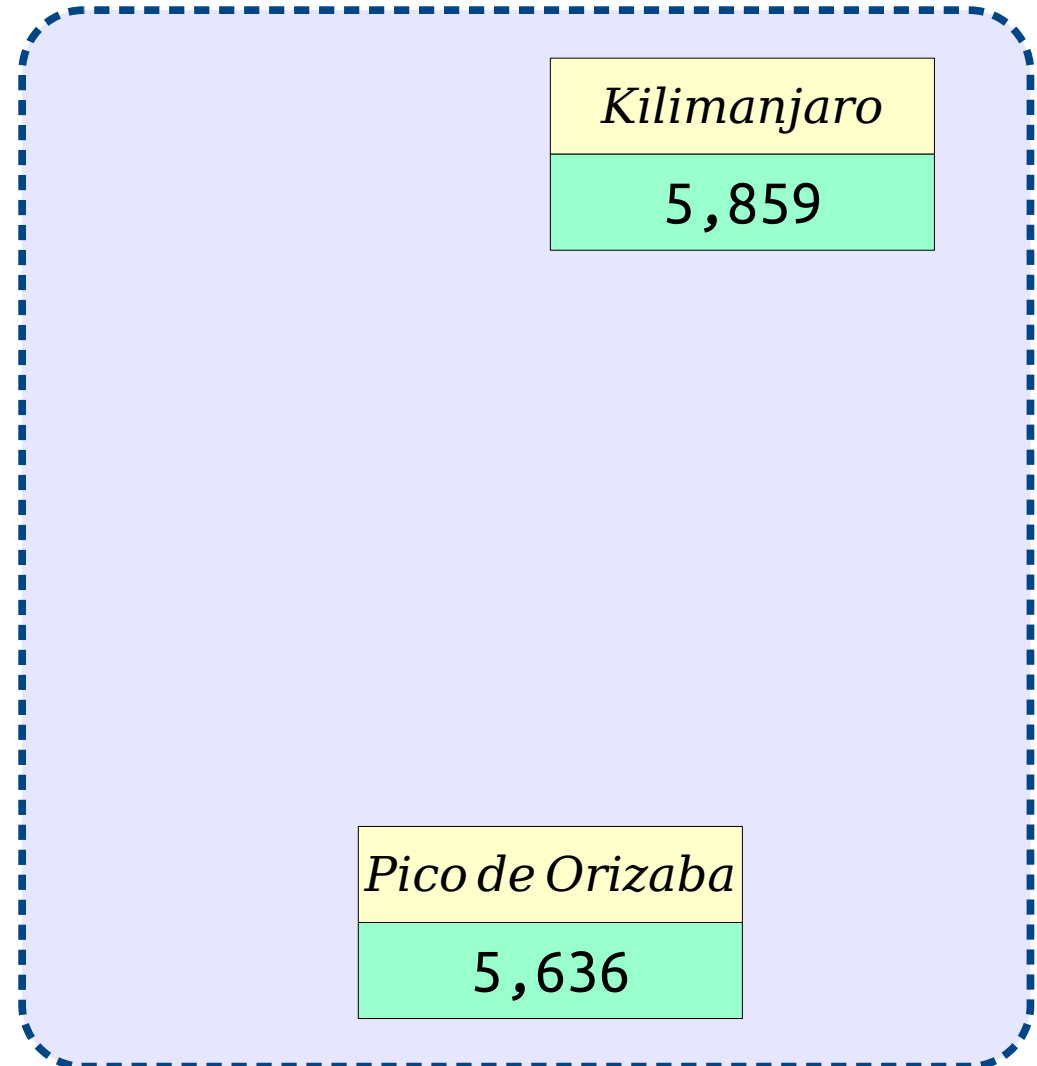
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Binary Heaps

- Priority queues are frequently implemented as **binary heaps**.
 - **enqueue** and **extract-min** run in time $O(\log n)$; **find-min** runs in time $O(1)$.
- These heaps are surprisingly fast in practice. It's tough to beat their performance!
 - d -ary heaps can outperform binary heaps for a well-tuned value of d , and otherwise only the **sequence heap** is known to specifically outperform this family.
 - (Is this information incorrect as of 2021? Let me know and I'll update it.)
- In that case, why do we need other heaps?

Priority Queues in Practice

- Many graph algorithms directly rely on priority queues supporting extra operations:
 - ***meld***(pq_1, pq_2): Destroy pq_1 and pq_2 and combine their elements into a single priority queue. (*MSTs via Cheriton-Tarjan*)
 - pq .***decrease-key***(v, k'): Given a pointer to element v already in the queue, lower its key to have new value k' . (*Shortest paths via Dijkstra, global min-cut via Stoer-Wagner*)
 - pq .***add-to-all***(Δk): Add Δk to the keys of each element in the priority queue, typically used with ***meld***. (*Optimum branchings via Chu-Edmonds-Liu*)
- In lecture, we'll cover binomial heaps to efficiently support ***meld*** and Fibonacci heaps to efficiently support ***meld*** and ***decrease-key***.
- You'll design a priority queue supporting ***meld*** and ***add-to-all*** on the next problem set.

Priority Queues in Practice

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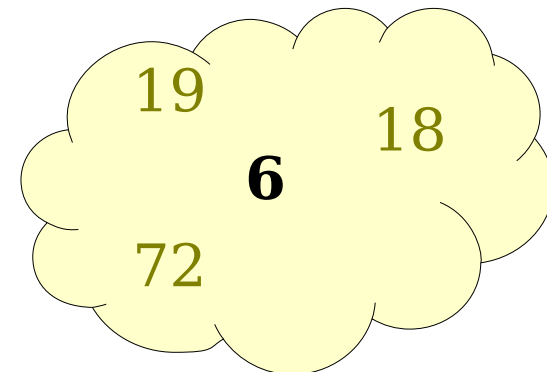
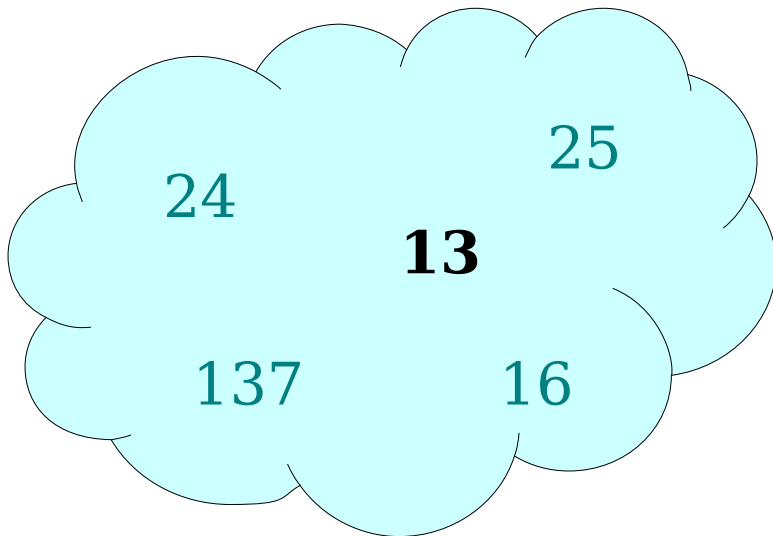
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Meldable Priority Queues

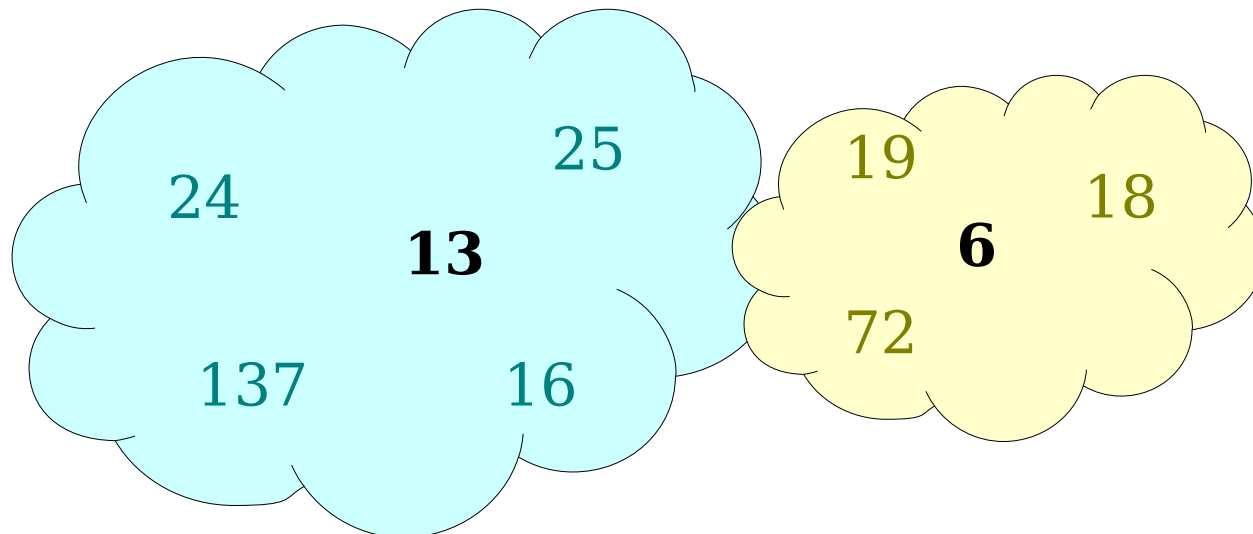
Meldable Priority Queues

- A priority queue supporting the ***meld*** operation is called a ***meldable priority queue***.
- ***meld***(pq_1 , pq_2) destructively modifies pq_1 and pq_2 and produces a new priority queue containing all elements of pq_1 and pq_2 .



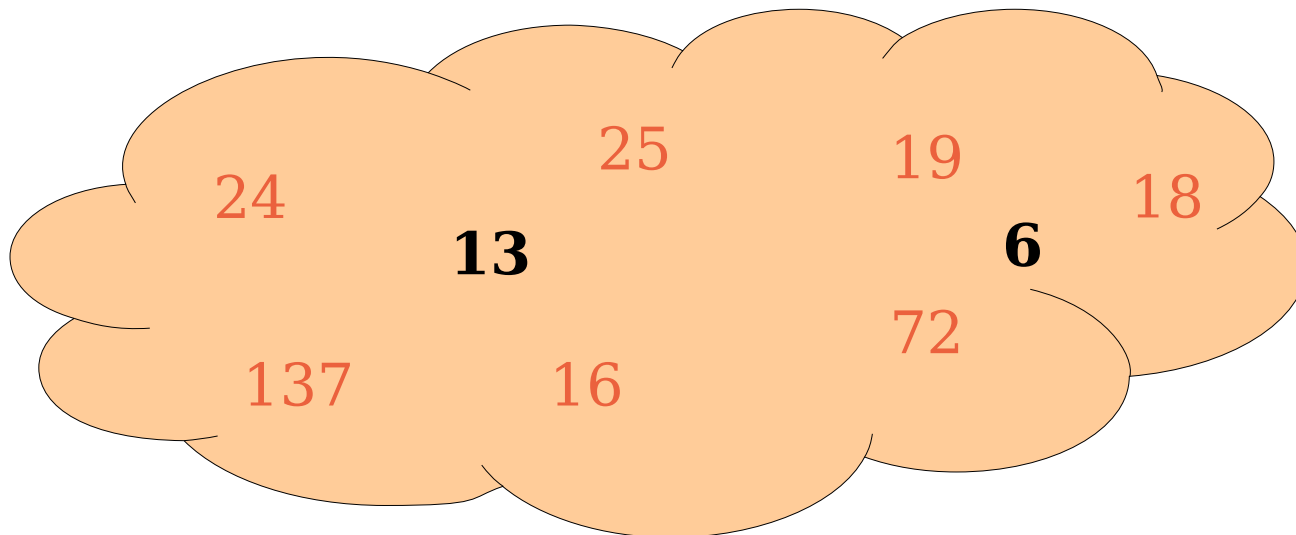
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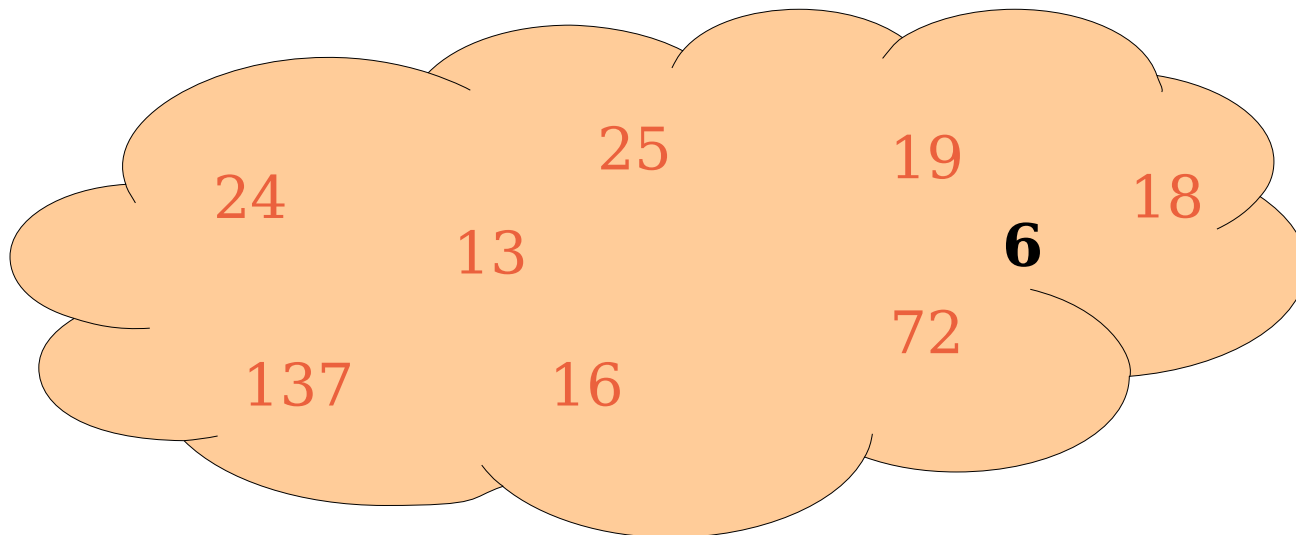
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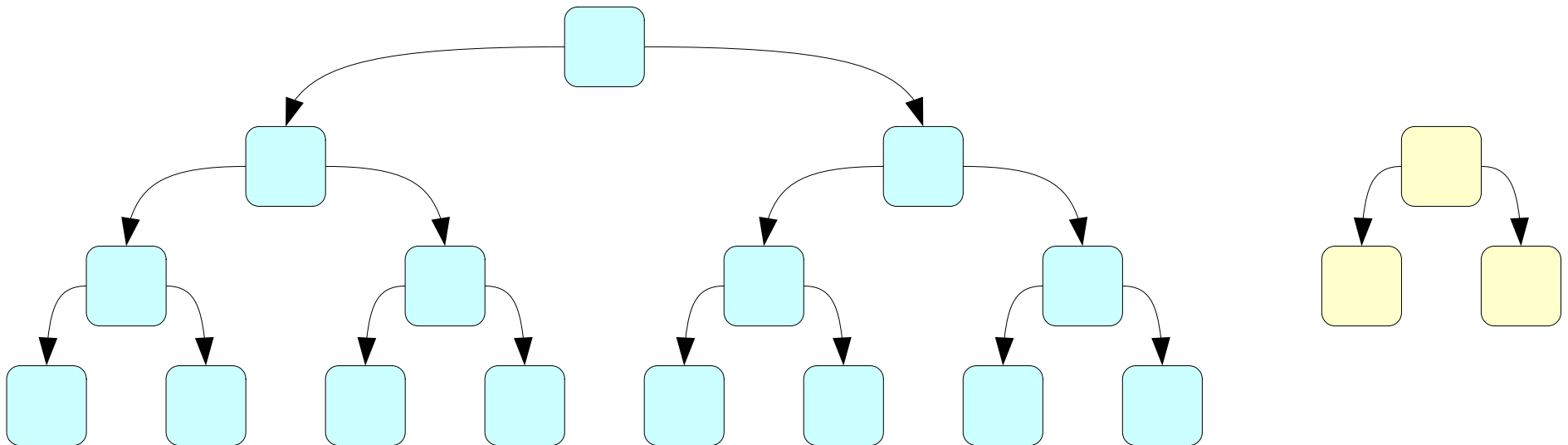
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Efficiently Meldable Queues

- Standard binary heaps do not efficiently support *meld*.
- **Intuition**: Binary heaps are complete binary trees, and two complete binary trees cannot easily be linked to one another.



What things *can* be combined together
efficiently?

Adding Binary Numbers

- Given the binary representations of two numbers n and m , we can add those numbers in time $O(\log m + \log n)$.

Intuition:

Writing out n in any “reasonable” base requires $\Theta(\log n)$ digits.

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A diagram illustrating the addition of two binary numbers. The numbers are aligned by their least significant bits. The first number is 10110 and the second is 01111. A plus sign is to the left of the second number. A horizontal line is drawn under the numbers. The result 0101 is shown below the line. A carry of 1 is shown above the third column from the right. The fourth column from the right (the third column from the left) is highlighted in yellow, showing a 1 in both the addends and the result, with a carry of 1 to the next column.

			1		
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<hr/>					
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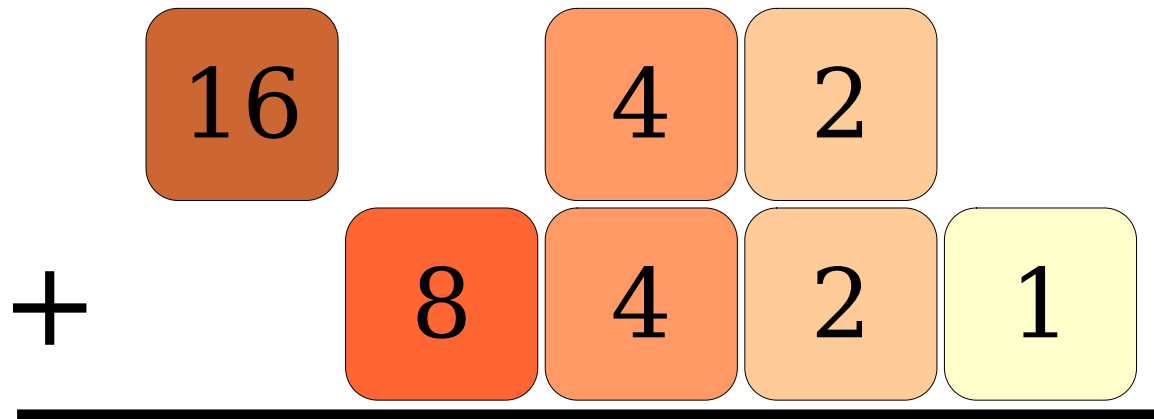
A Different Intuition

- Represent n and m as a collection of “packets” whose sizes are powers of two.
- Adding together n and m can then be thought of as combining the packets together, eliminating duplicates

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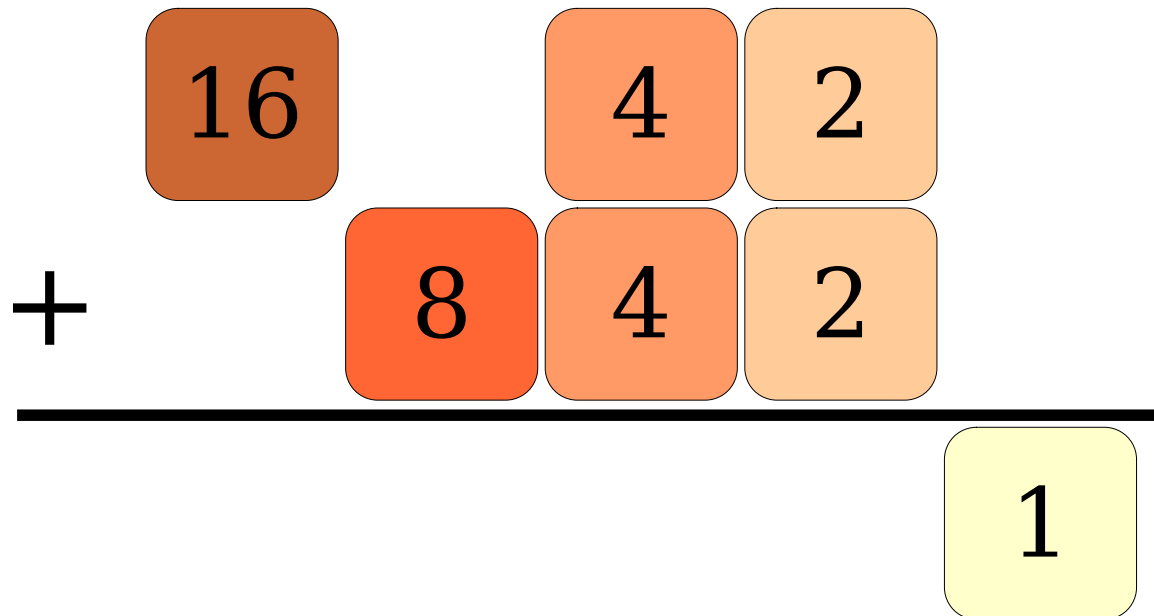
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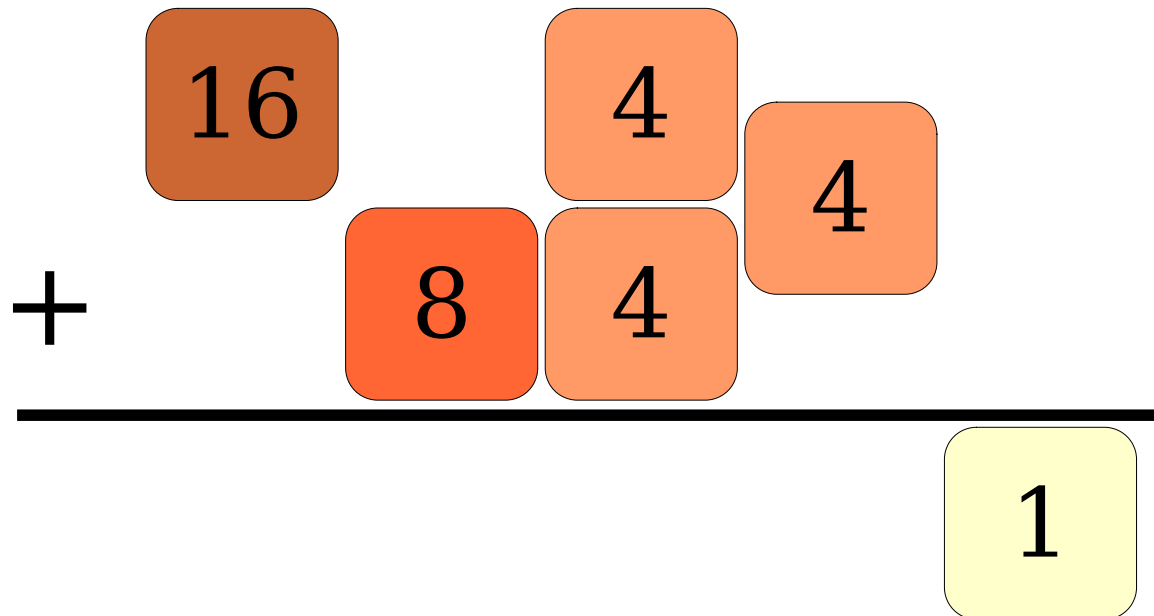
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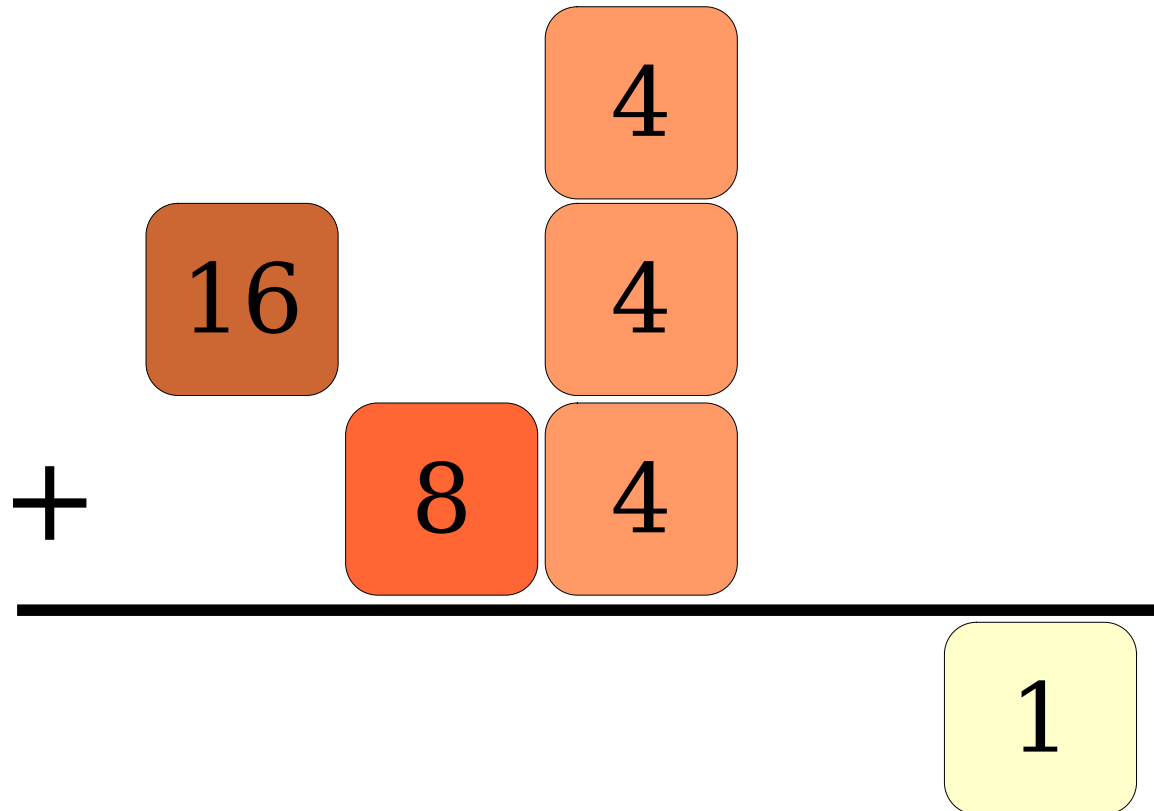
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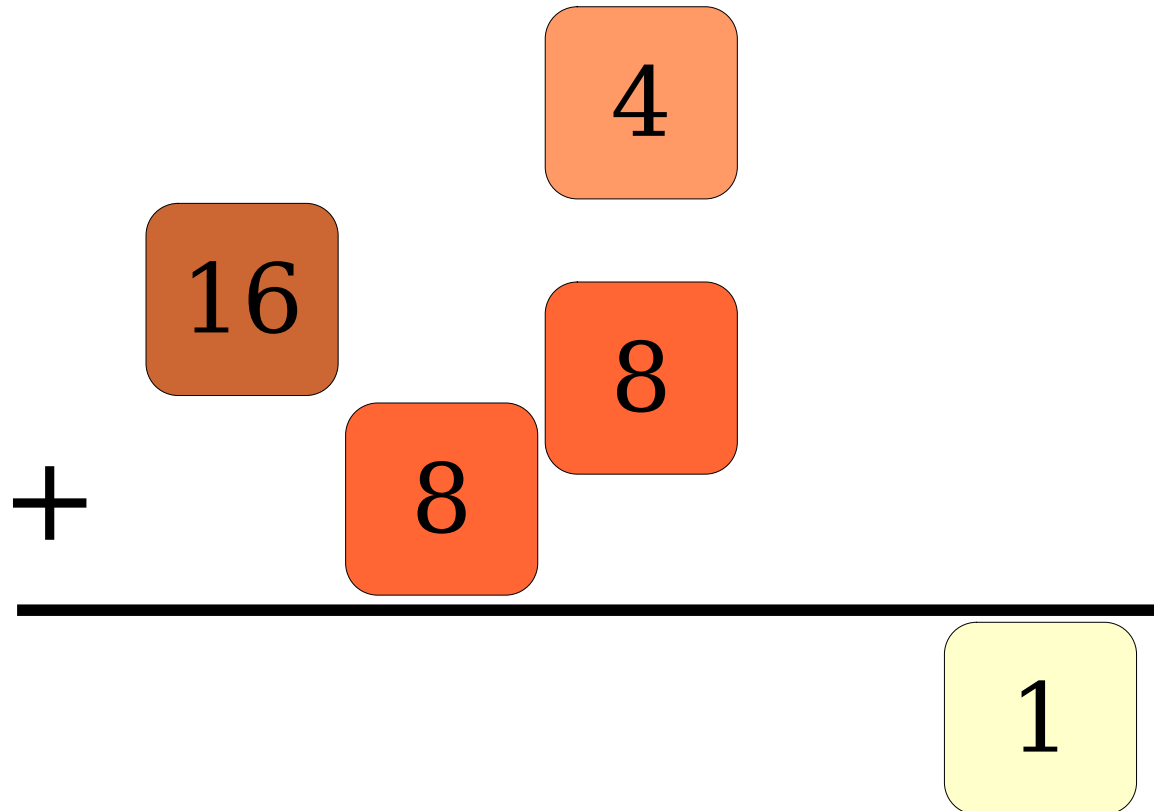
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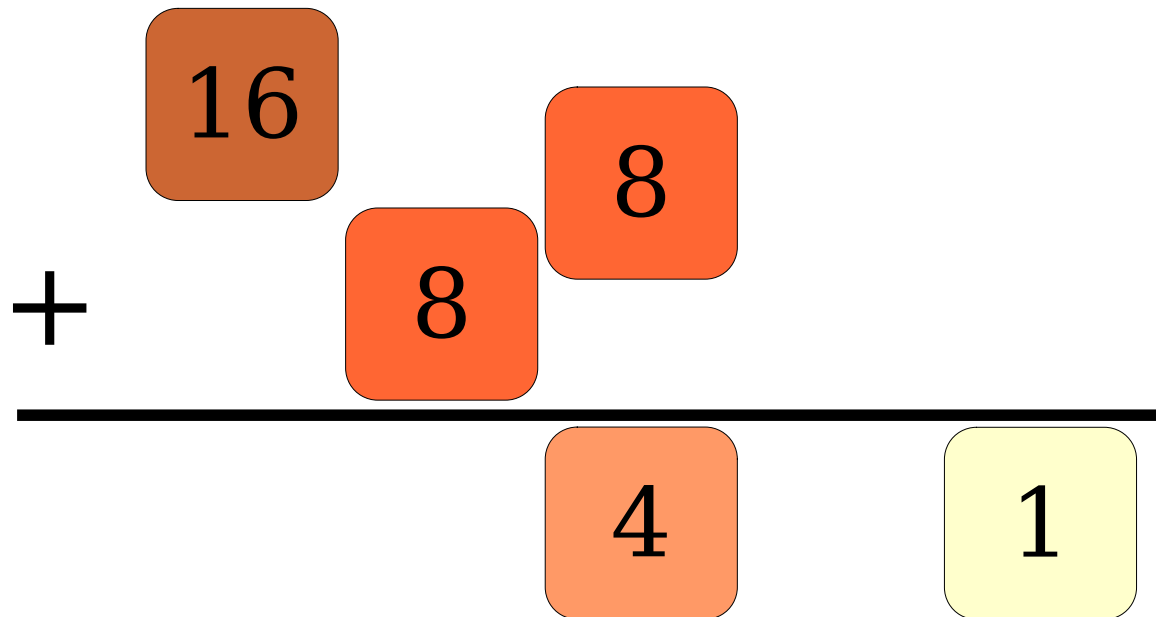
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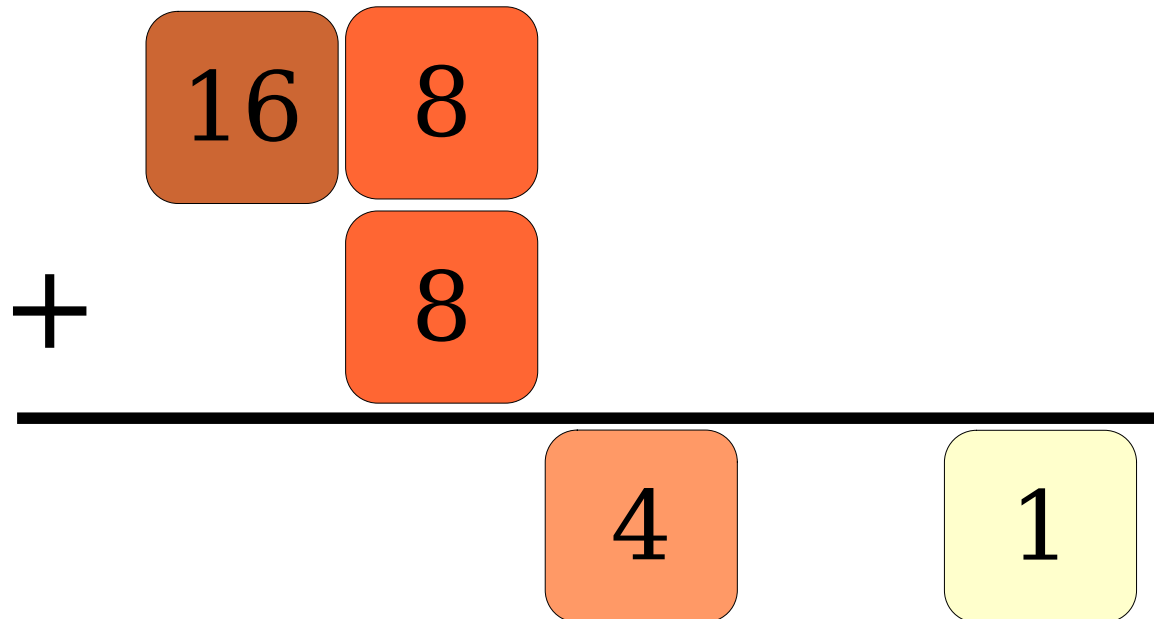
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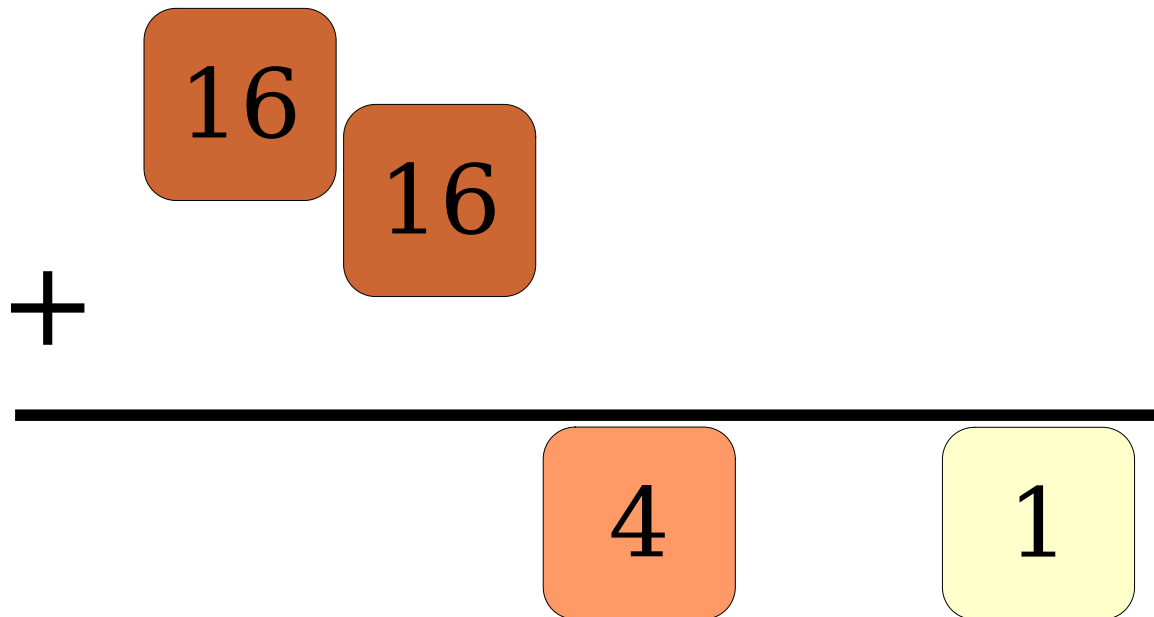
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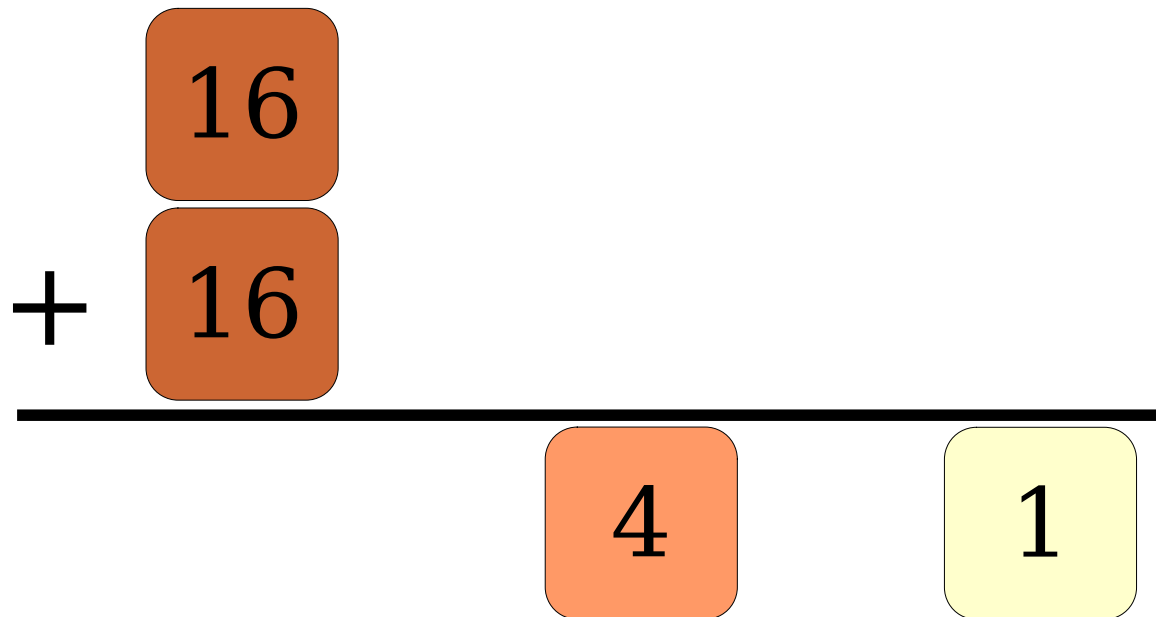
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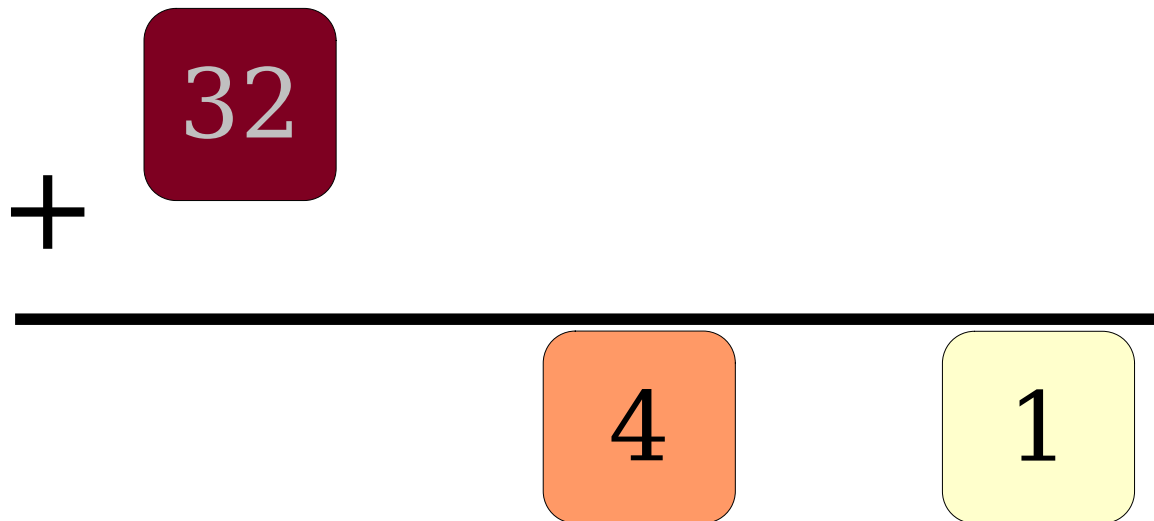
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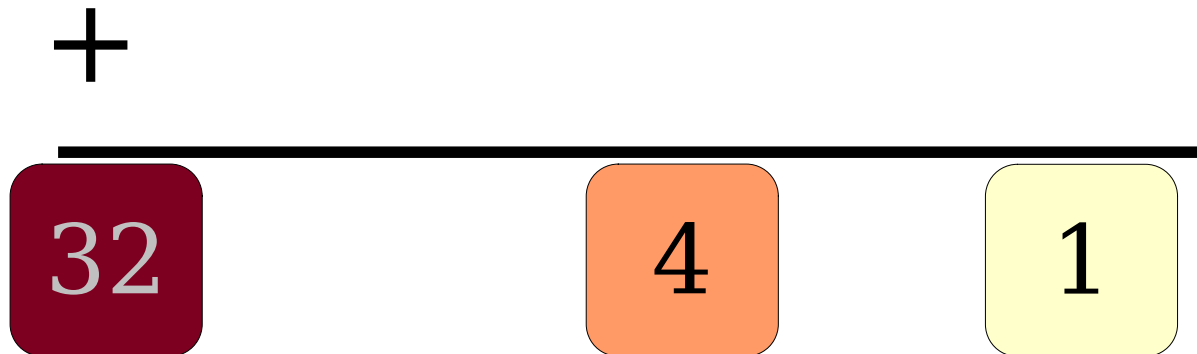
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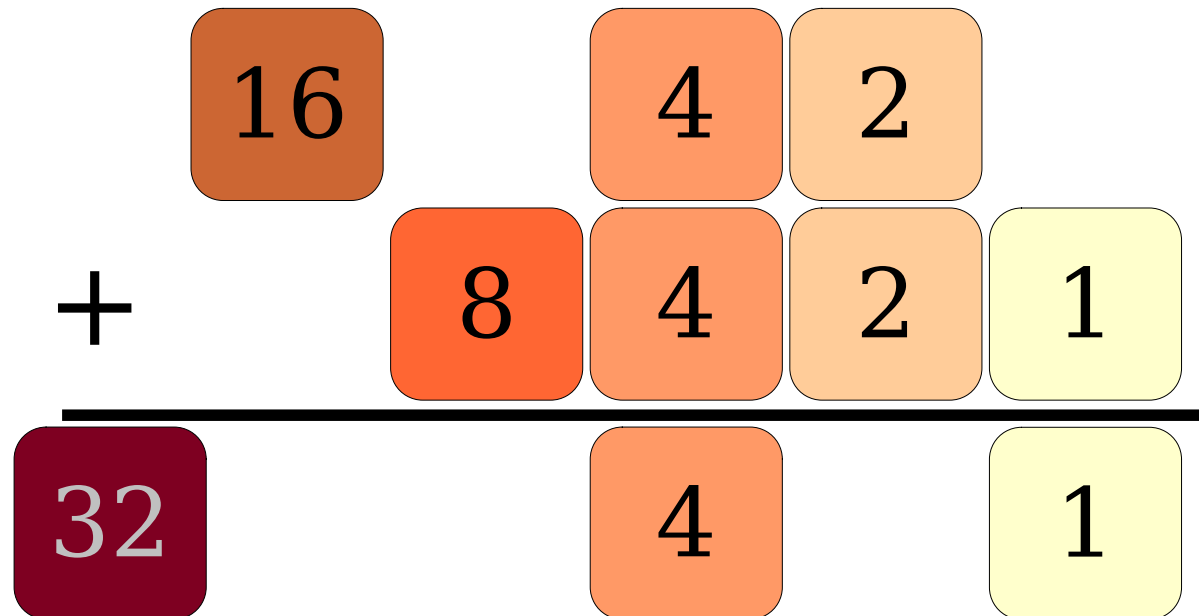
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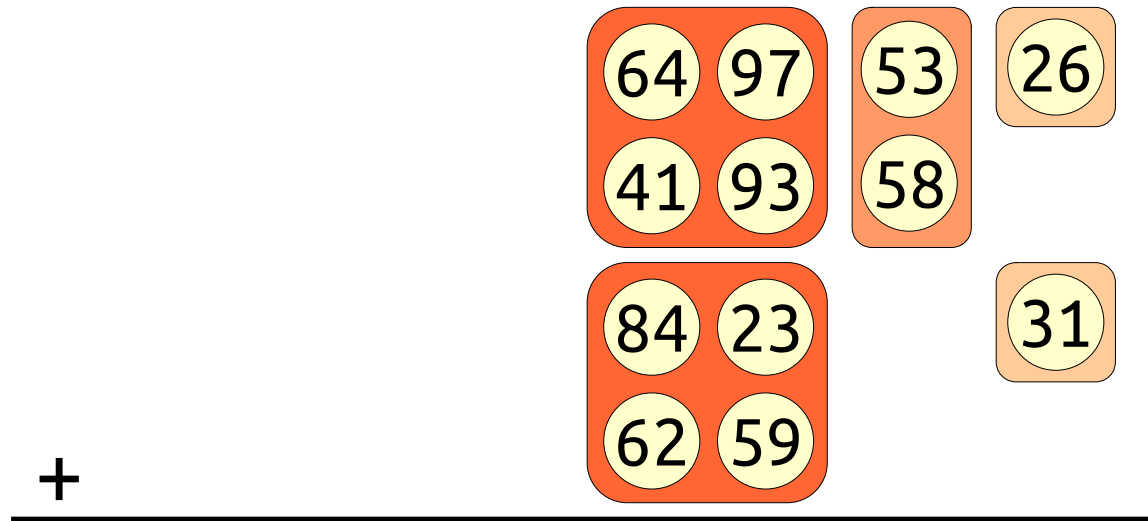
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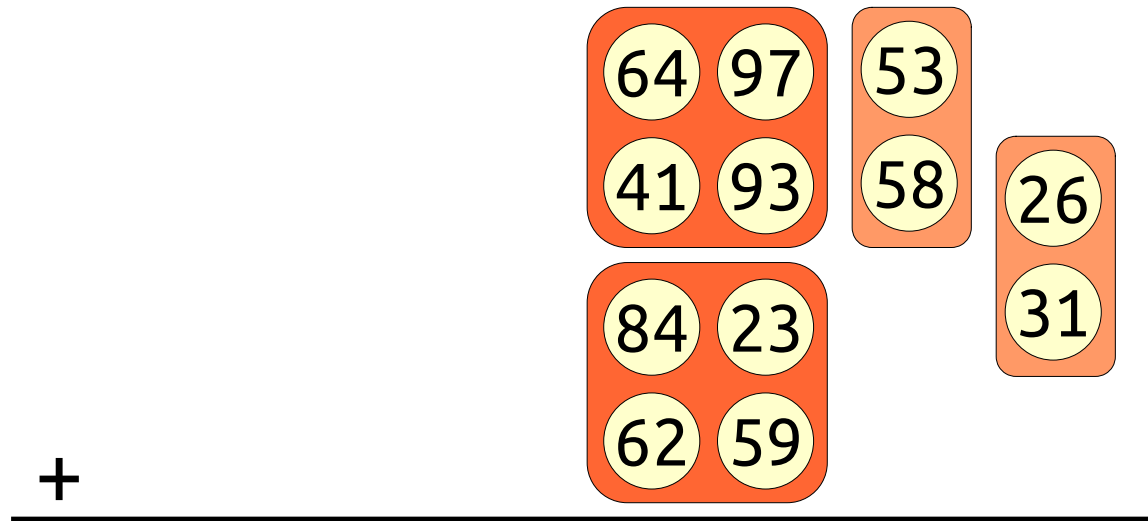
Building a Priority Queue

- **Idea:** Store elements in “packets” whose sizes are powers of two and **meld** by “adding” groups of packets.



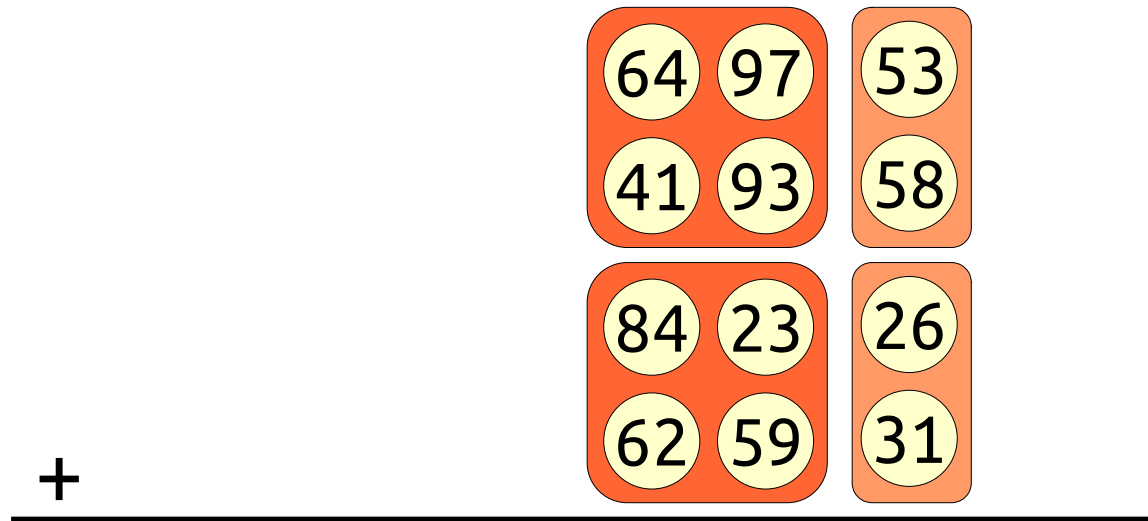
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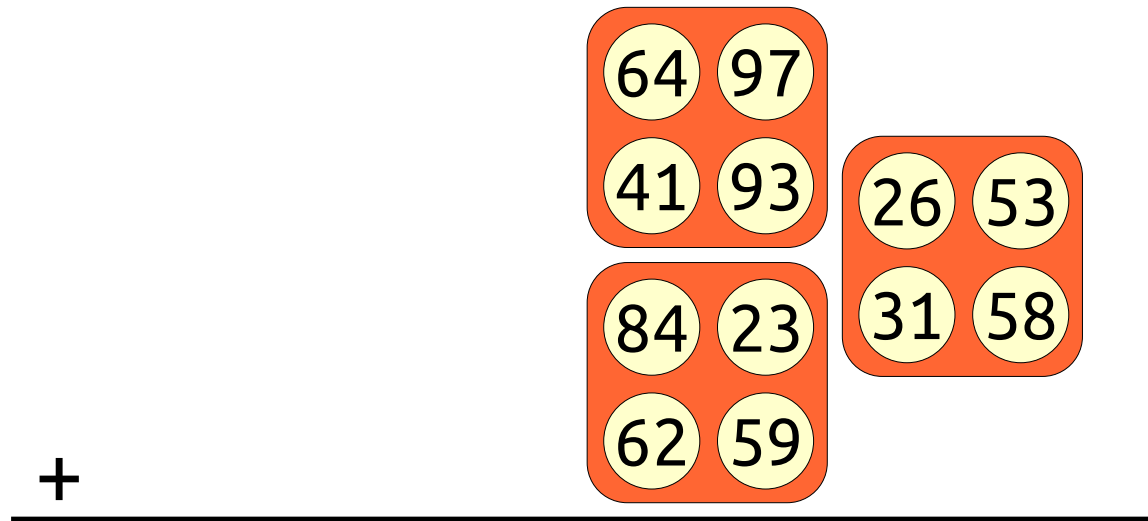
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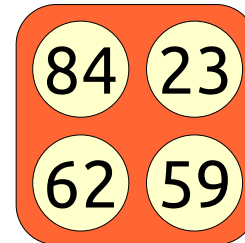
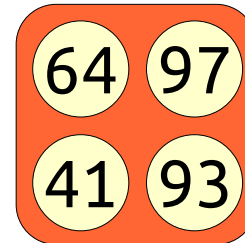
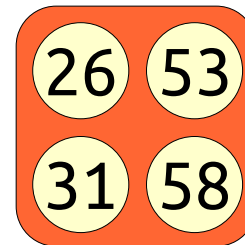
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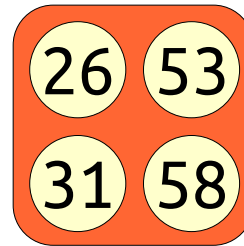
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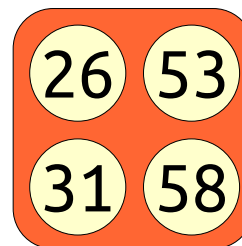
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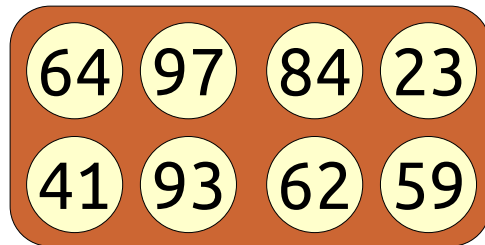


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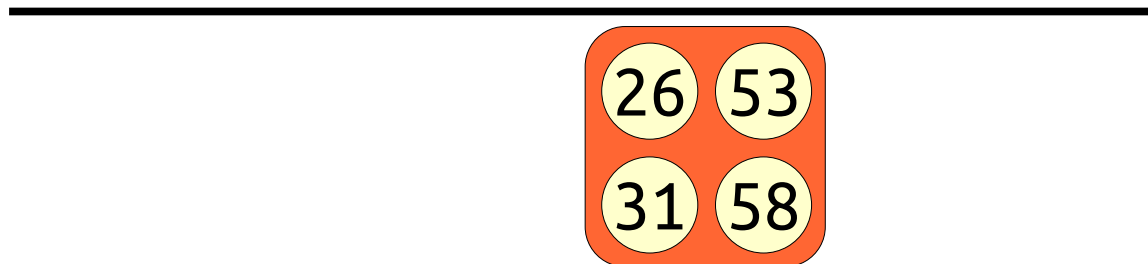


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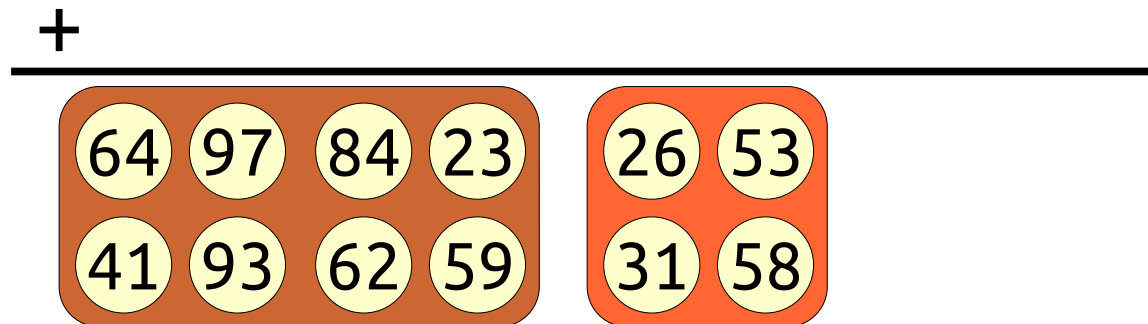


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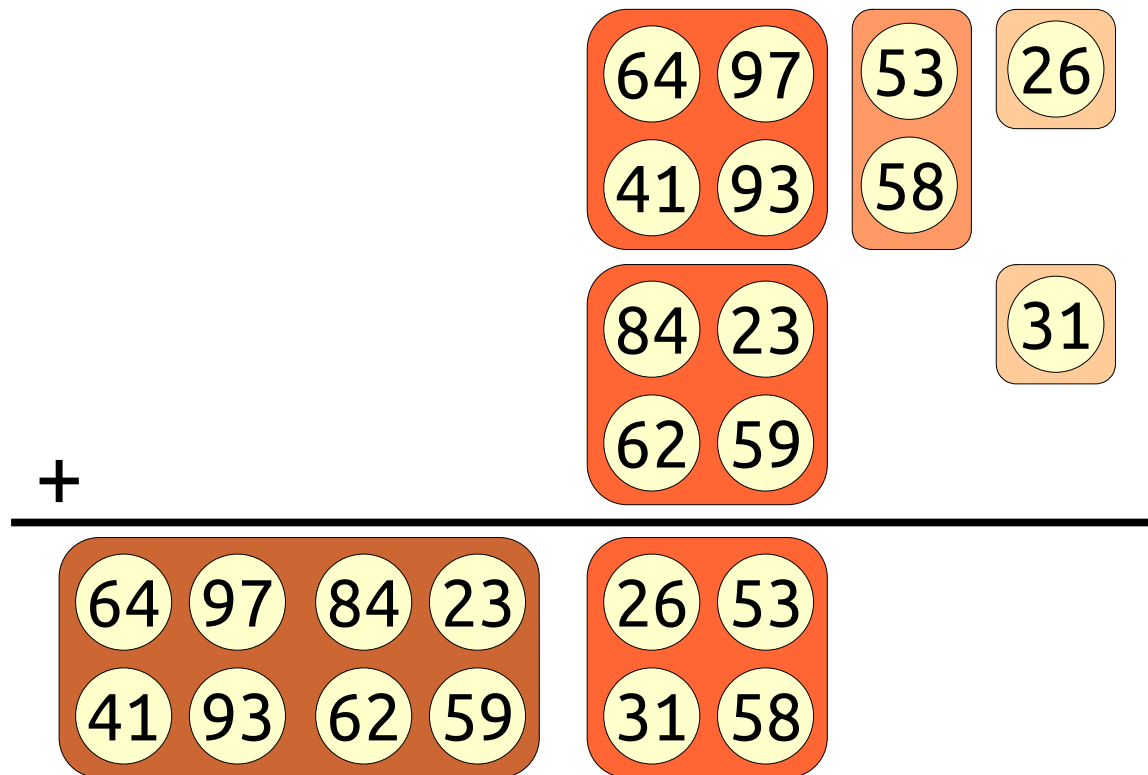
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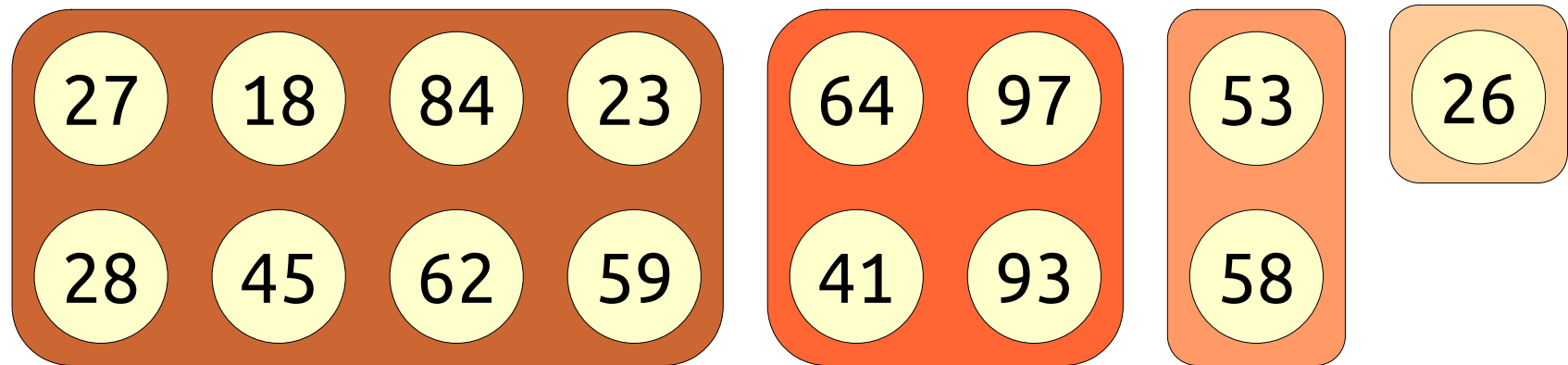


Building a Priority Queue

- What properties must our packets have?

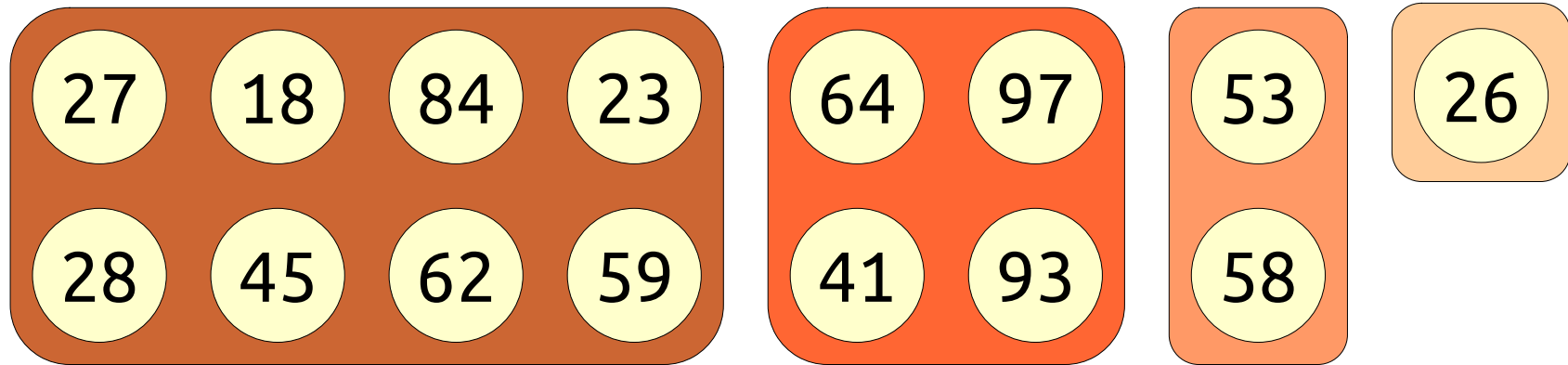
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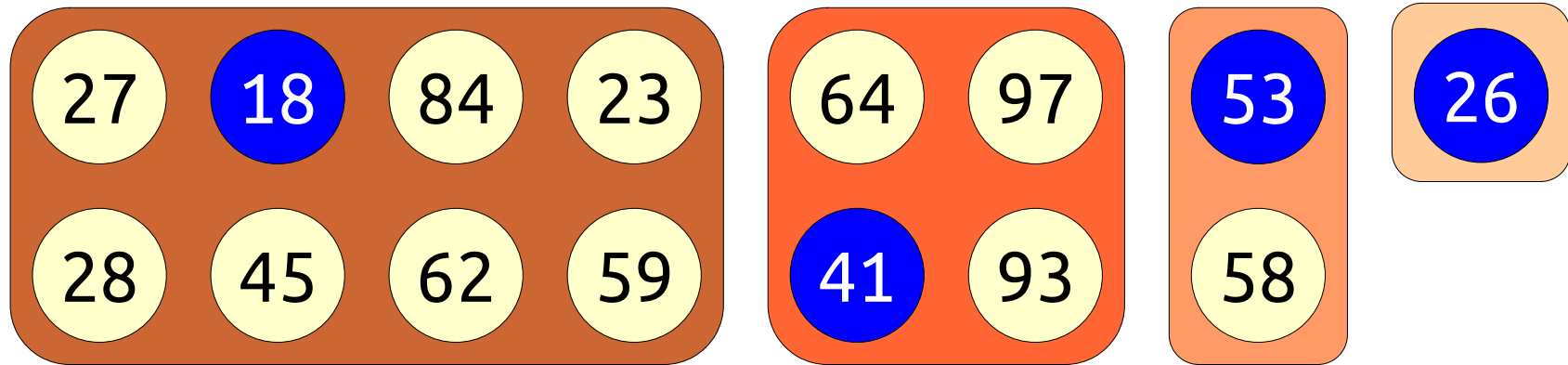
Building a Priority Queue

- What properties must our packets have?
 - Sizes must be powers of two.
 - Can efficiently fuse packets of the same size.



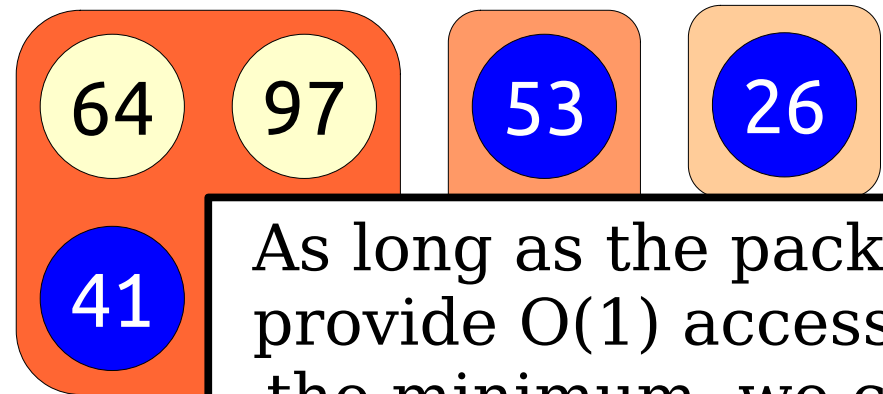
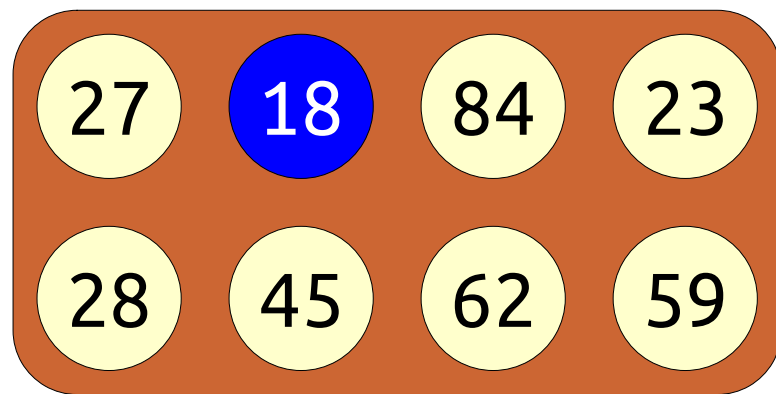
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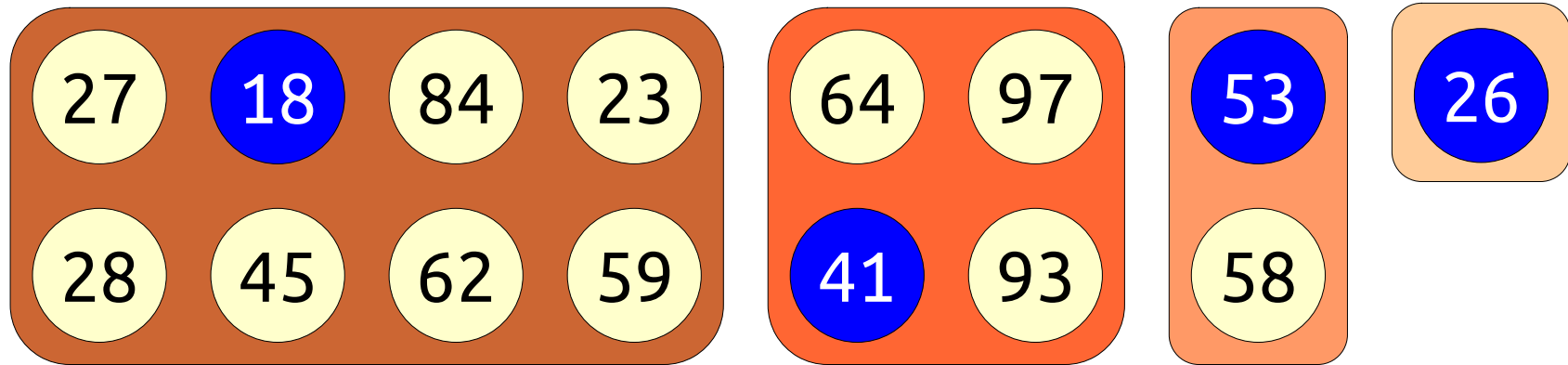
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As long as the packets provide $O(1)$ access to the minimum, we can execute *find-min* in time $O(\log n)$.

Building a Priority Queue

- What properties must our packets have?
 - Sizes must be powers of two.
 - Can efficiently fuse packets of the same size.
 - Can efficiently find the minimum element of each packet.

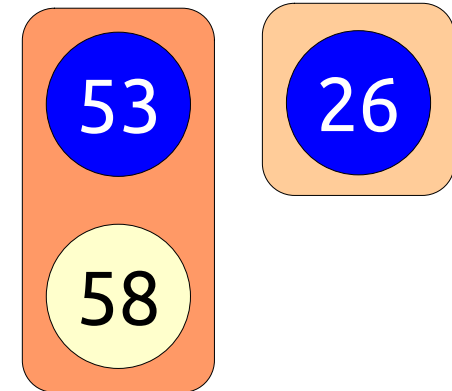
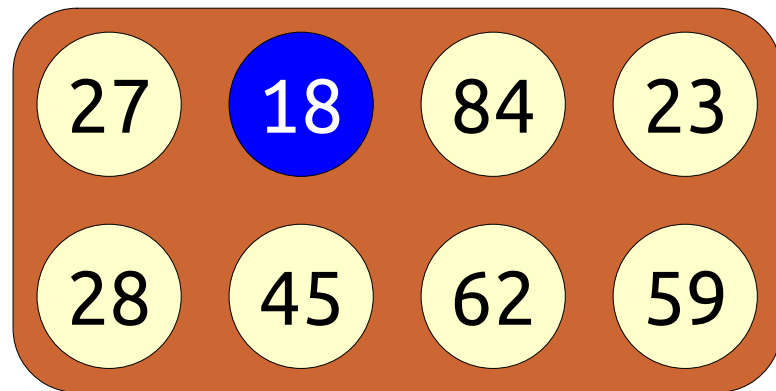


Inserting into the Queue

- If we can efficiently meld two priority queues, we can efficiently enqueue elements to the queue.
- ***Idea:*** Meld together the queue and a new queue with a single packet.

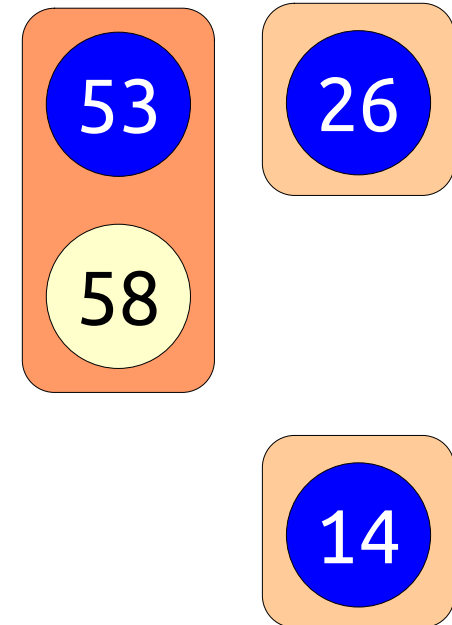
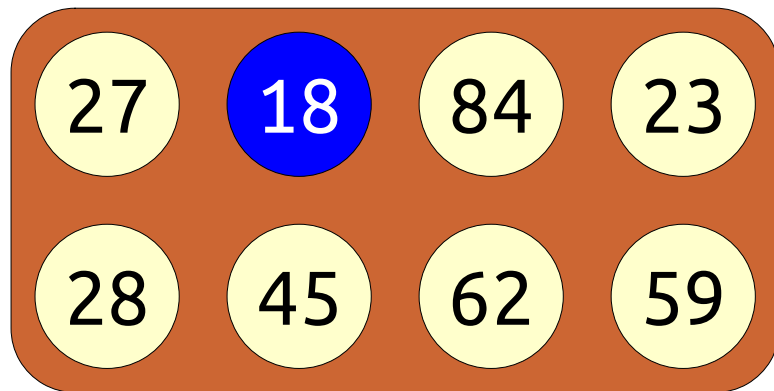
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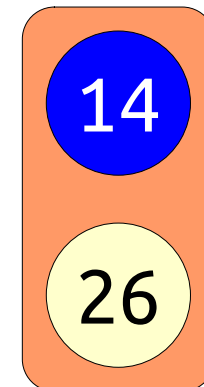
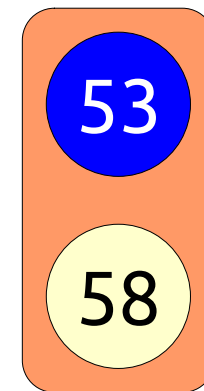
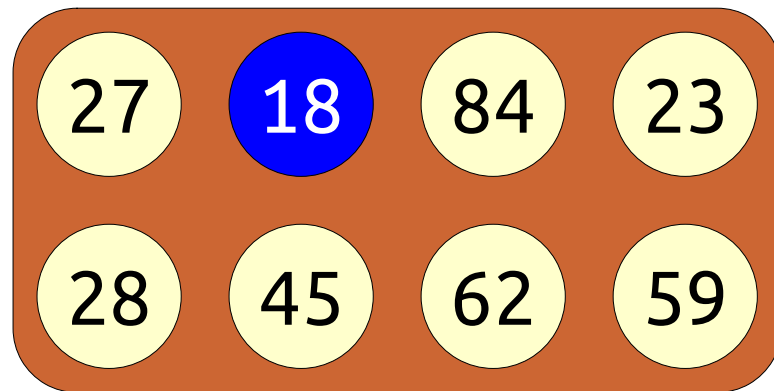
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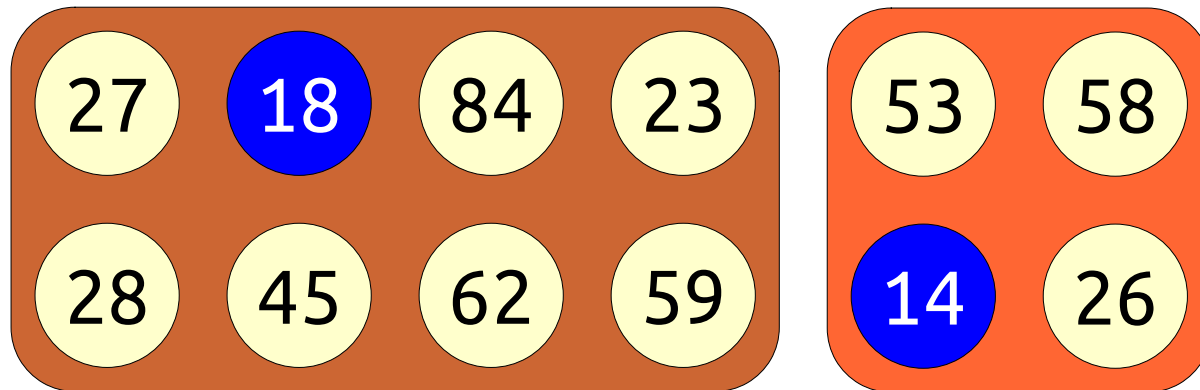
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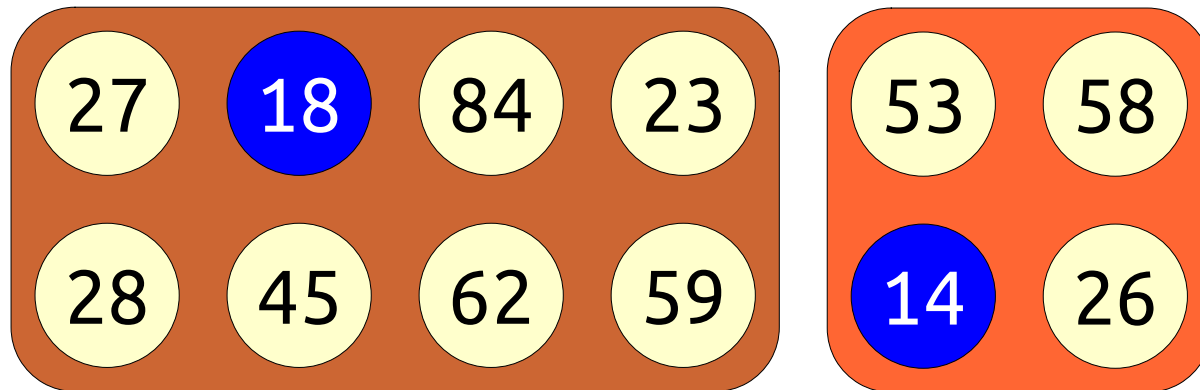
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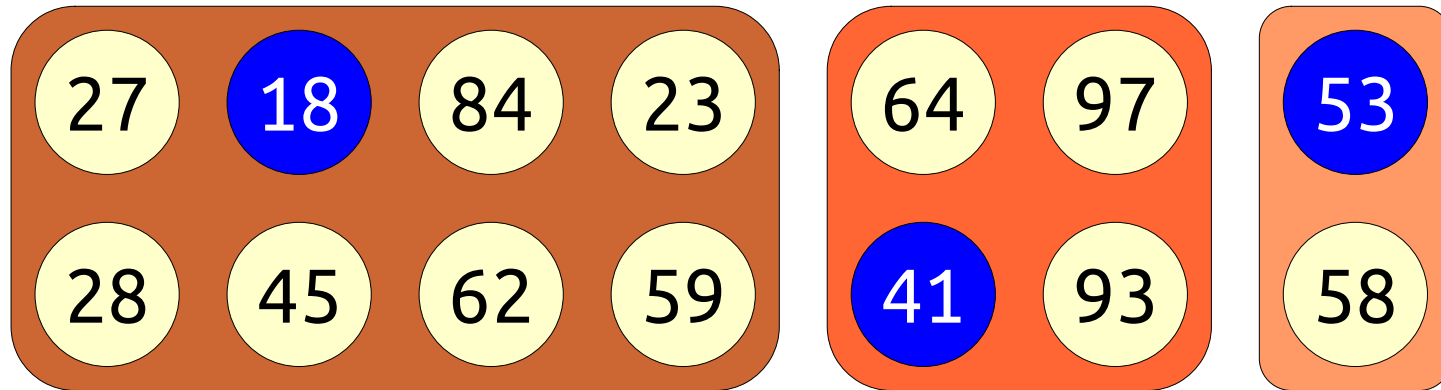
Time required:
 $O(\log n)$ fuses.

Deleting the Minimum

- Our analogy with arithmetic breaks down when we try to remove the minimum element.
- After losing an element, the packet will not necessarily hold a number of elements that is a power of two.

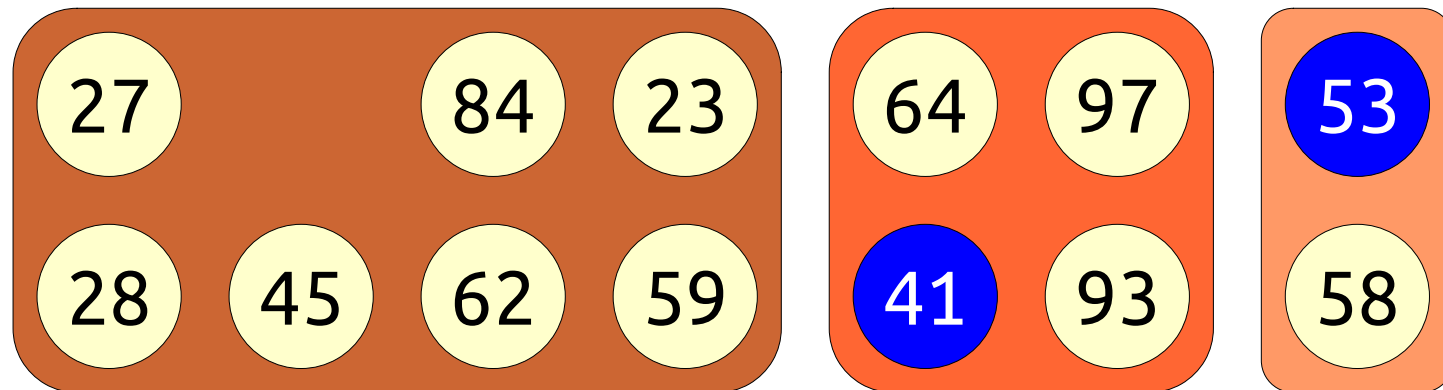
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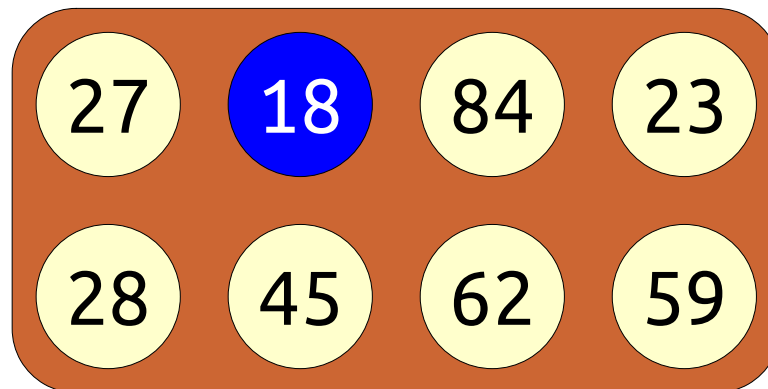
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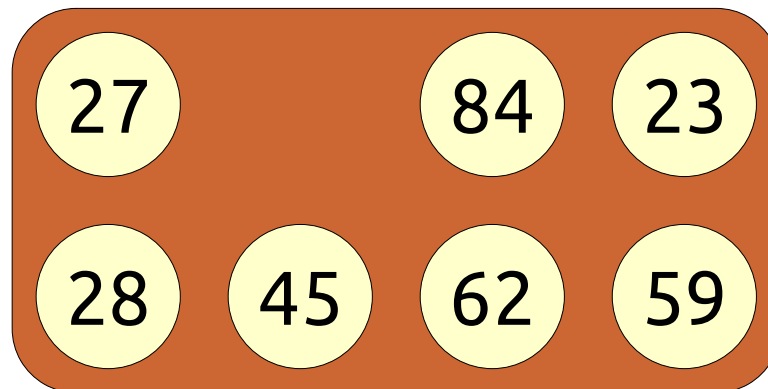
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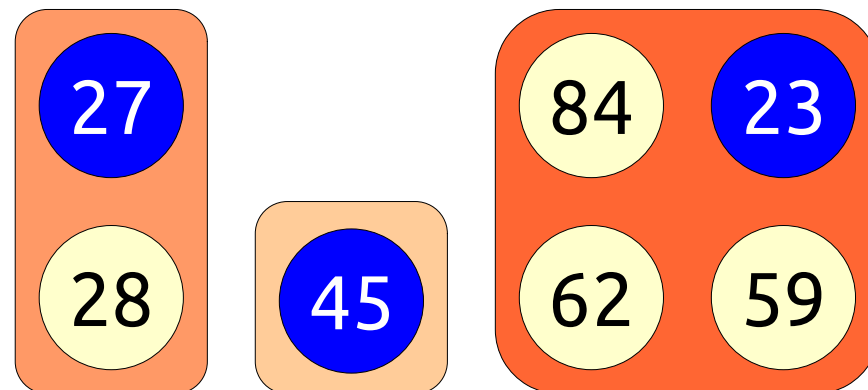
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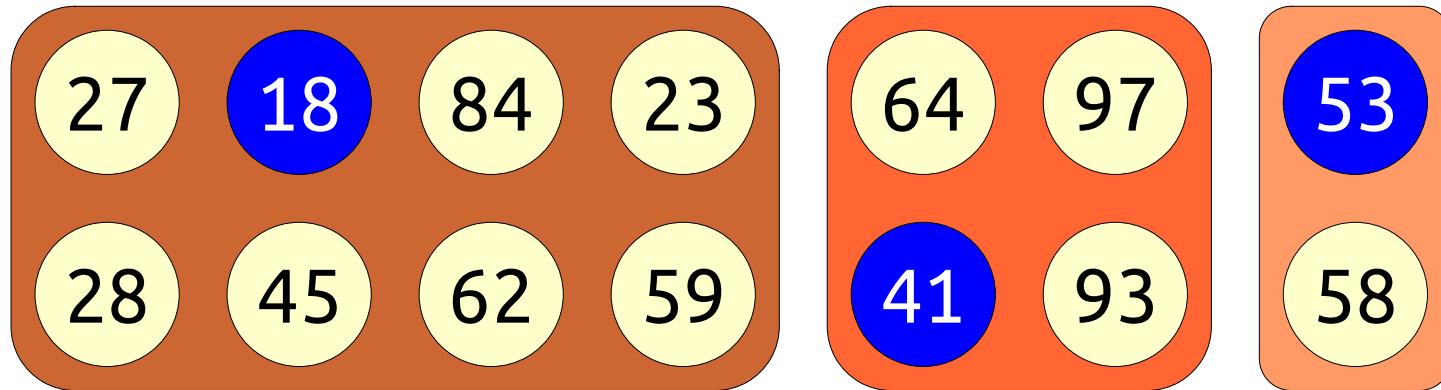
Deleting the Minimum

- If we have a packet with 2^k elements in it and remove a single element, we are left with $2^k - 1$ remaining elements.
- **Fun fact:** $2^k - 1 = 2^0 + 2^1 + 2^2 + \dots + 2^{k-1}$.
- **Idea:** “Fracture” the packet into k smaller packets, then add them back in.



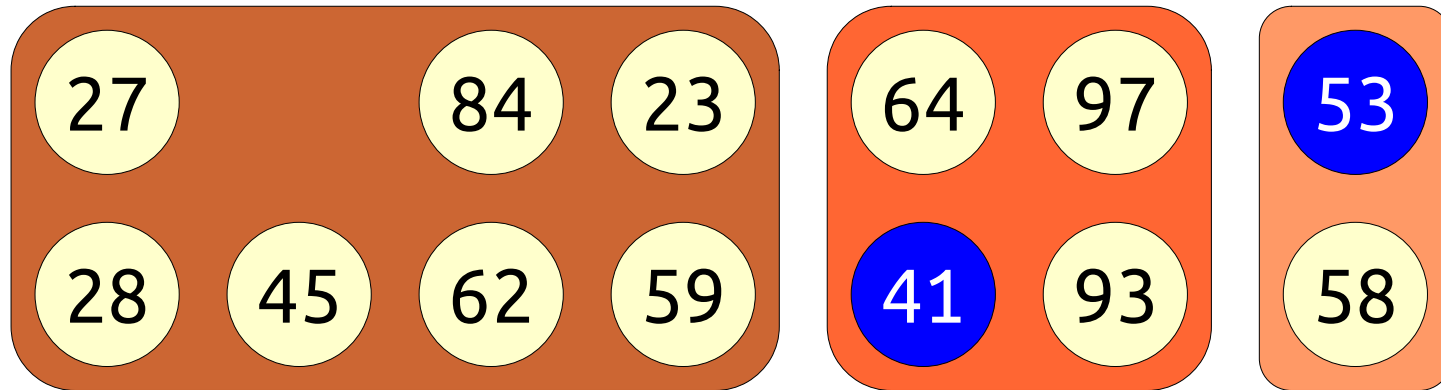
Fracturing Packets

- We can *extract-min* by fracturing the packet containing the minimum and adding the fragments back in.



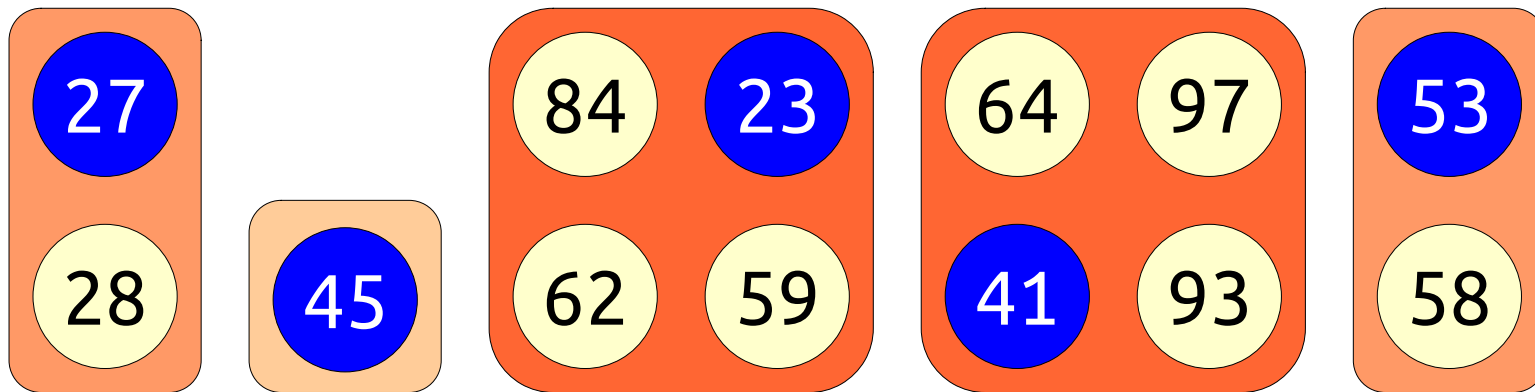
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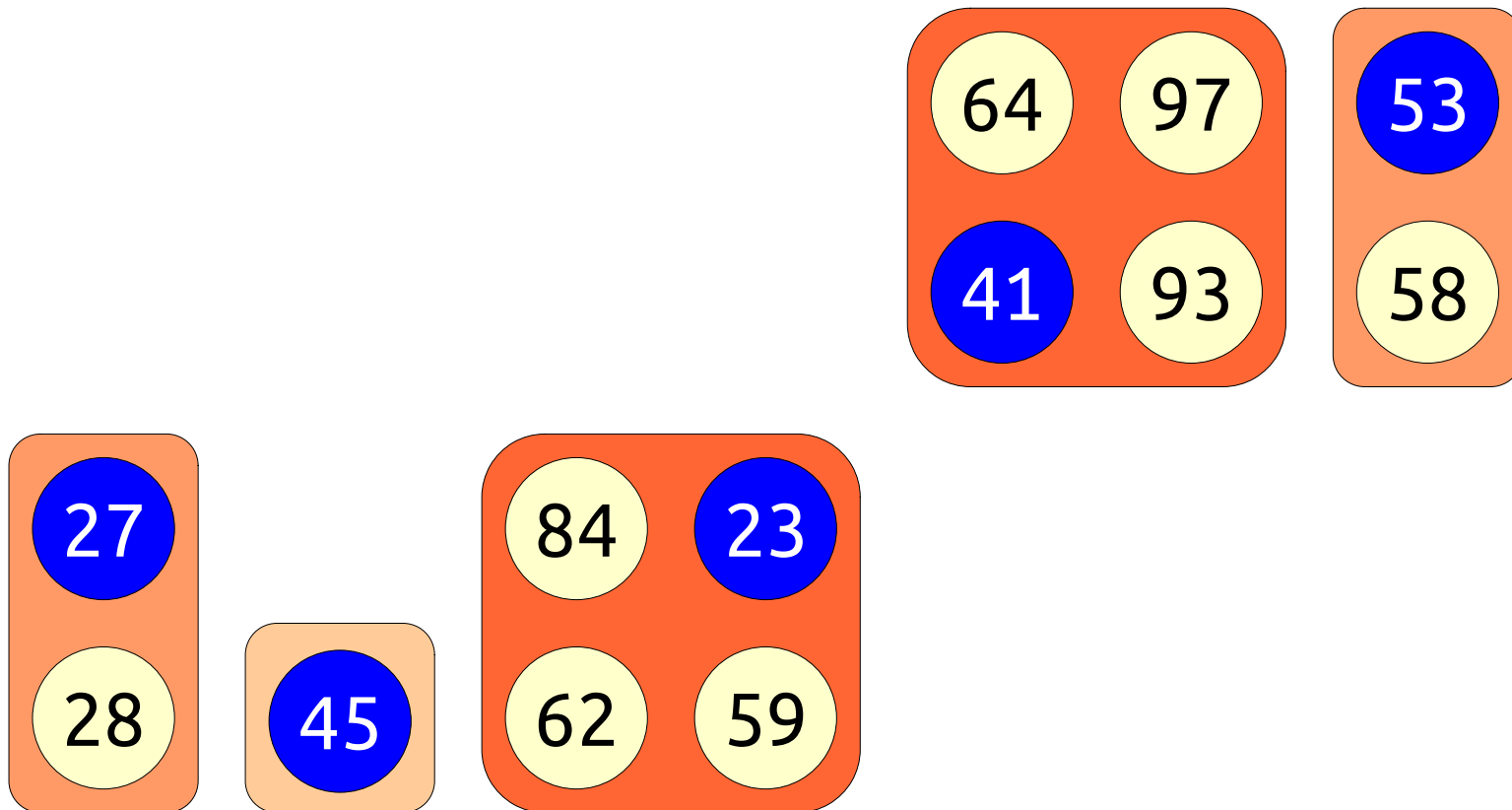
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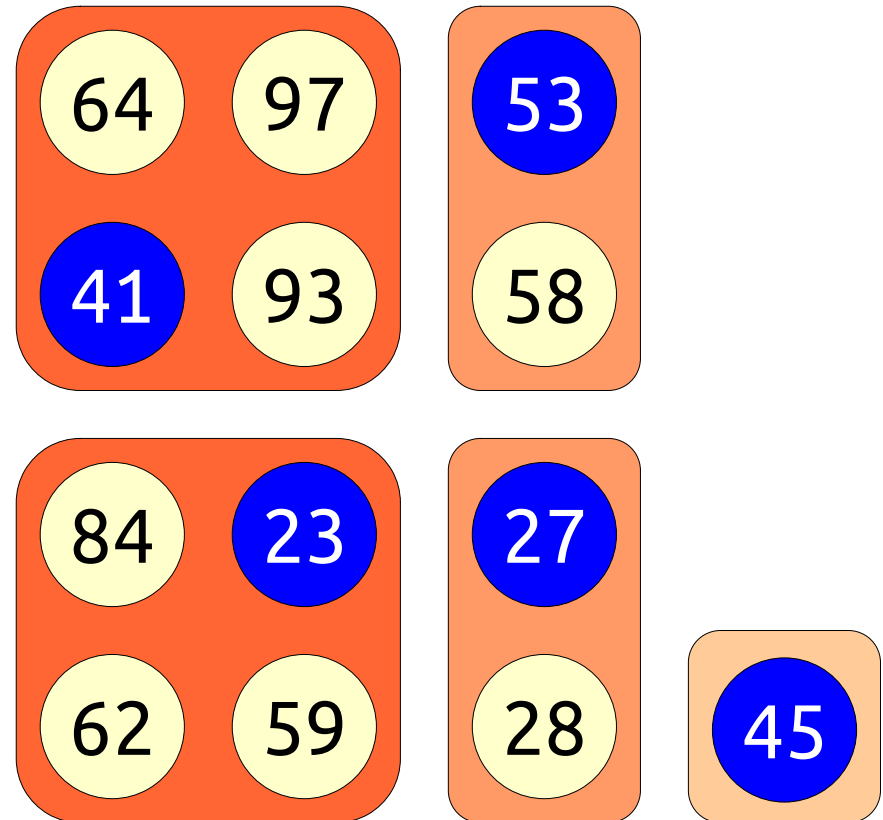
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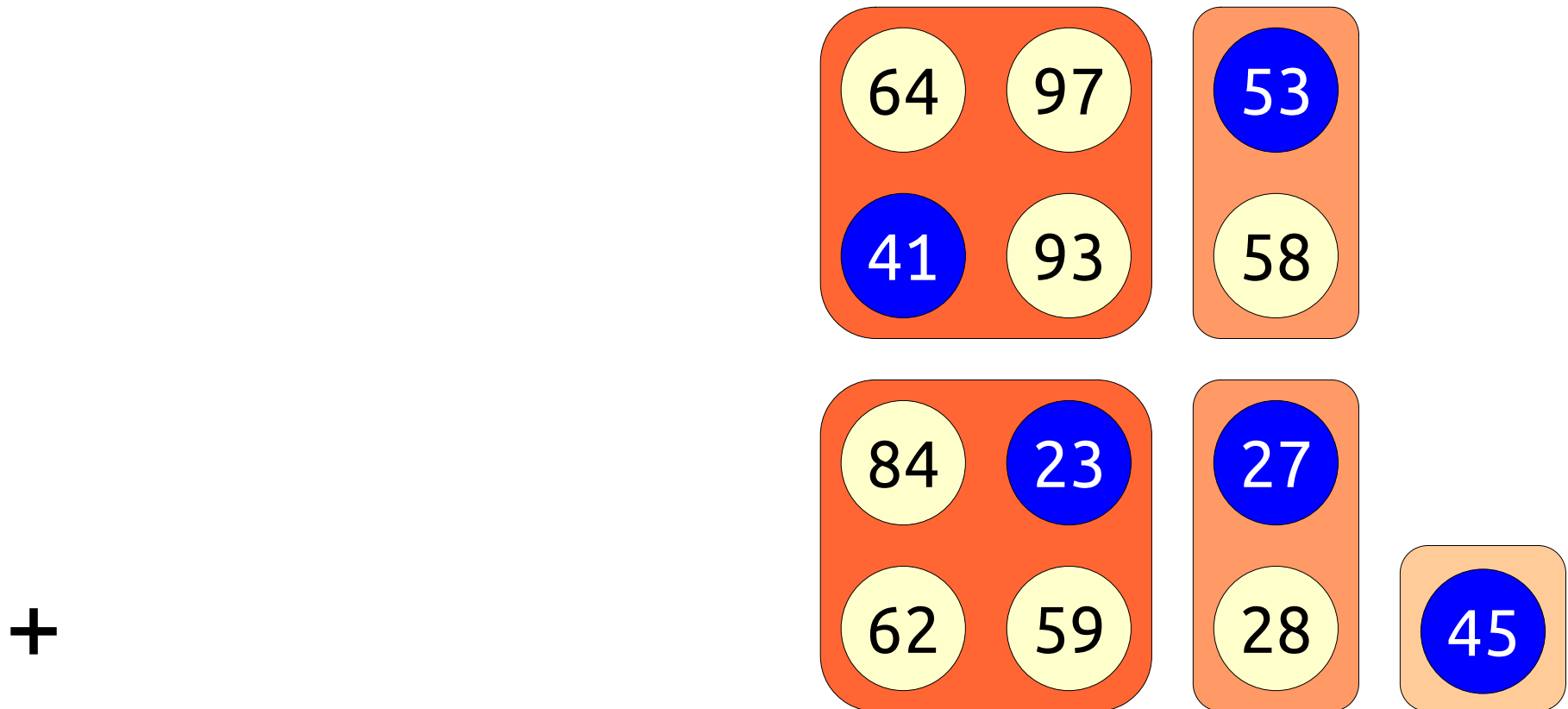
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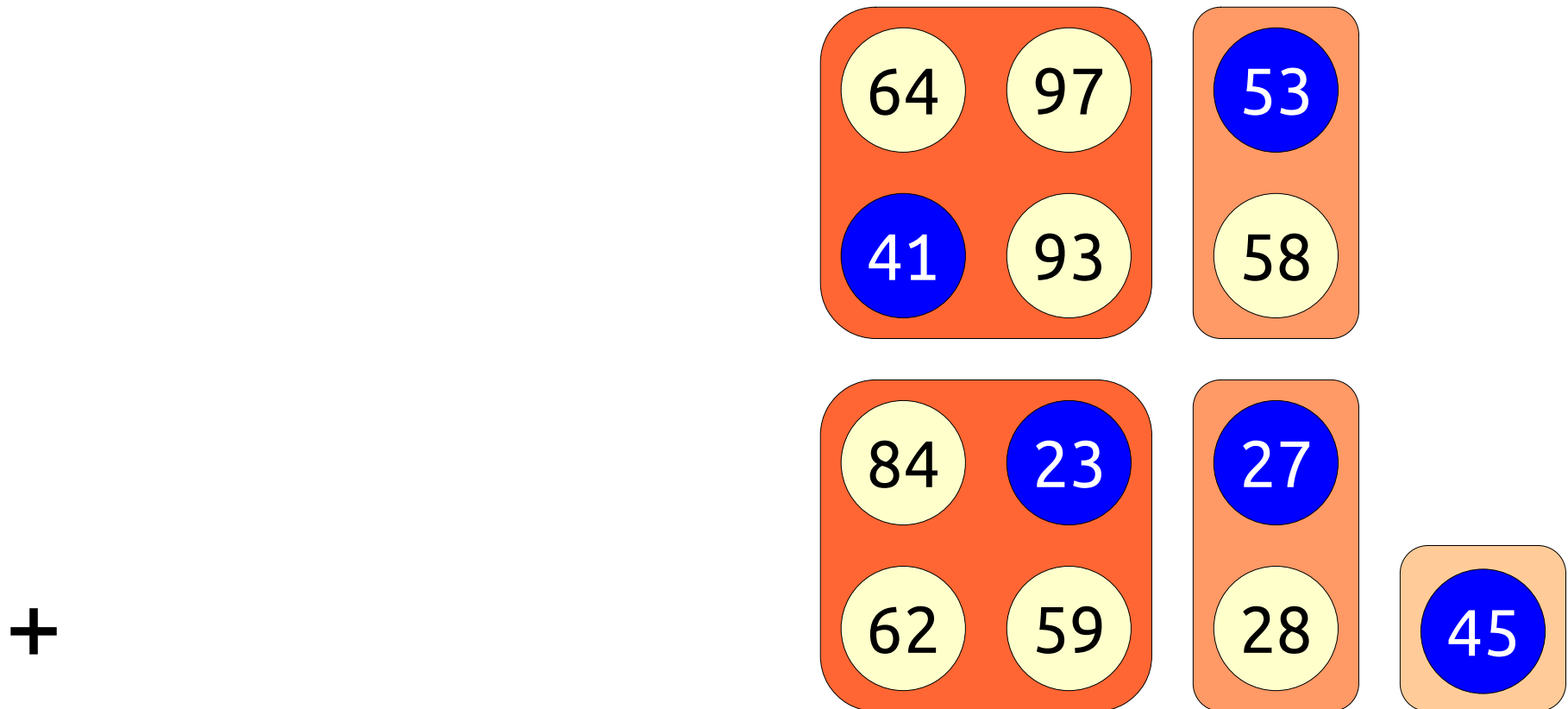
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Fracturing Packets

- We can *extract-min* by fracturing the packet containing the minimum and adding the fragments back in.
- Runtime is $O(\log n)$ fuses in *meld*, plus fracture cost.



Building a Priority Queue

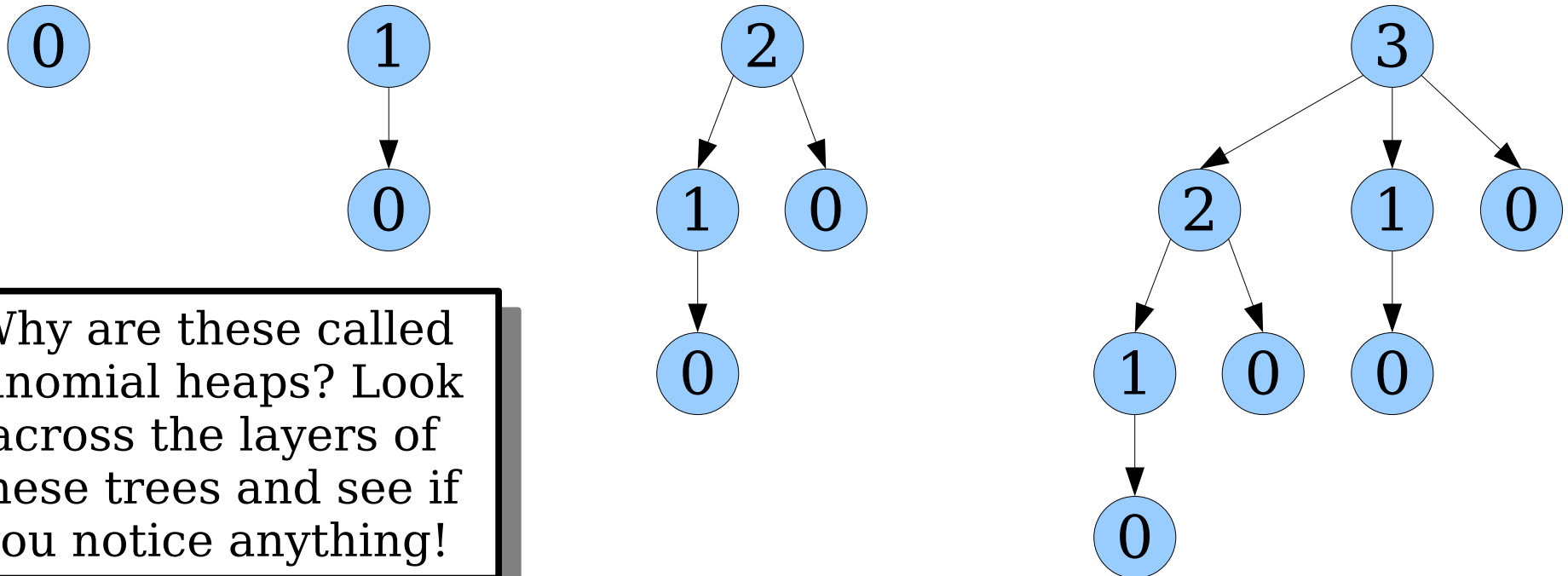
- What properties must our packets have?
 - Size is a power of two.
 - Can efficiently fuse packets of the same size.
 - Can efficiently find the minimum element of each packet.
 - Can efficiently “fracture” a packet of 2^k nodes into packets of $2^0, 2^1, 2^2, 2^3, \dots, 2^{k-1}$ nodes.
- **Question:** How can we represent our packets to support the above operations efficiently?

Binomial Trees

- A **binomial tree of order k** is a type of tree recursively defined as follows:

A binomial tree of order k is a single node whose children are binomial trees of order $0, 1, 2, \dots, k - 1$.

- Here are the first few binomial trees:



Why are these called binomial heaps? Look across the layers of these trees and see if you notice anything!

Binomial Trees

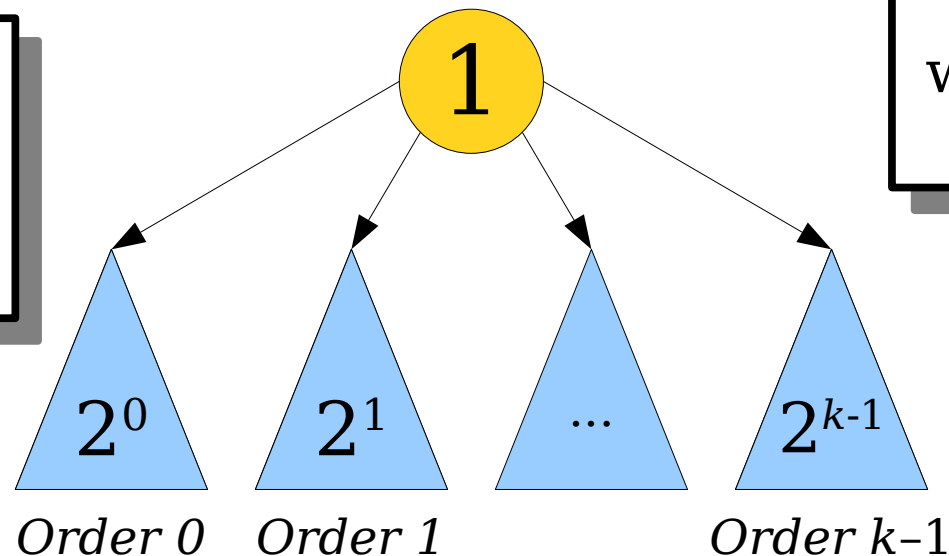
- **Theorem:** A binomial tree of order k has exactly 2^k nodes.
- **Proof:** Induction on k .

Assume that binomial trees of orders $0, 1, \dots, k - 1$ have $2^0, 2^1, \dots, 2^{k-1}$ nodes. The number of nodes in an order- k binomial tree is

$$2^0 + 2^1 + \dots + 2^{k-1} + 1 = 2^k - 1 + 1 = 2^k$$

So the claim holds for k as well. ■

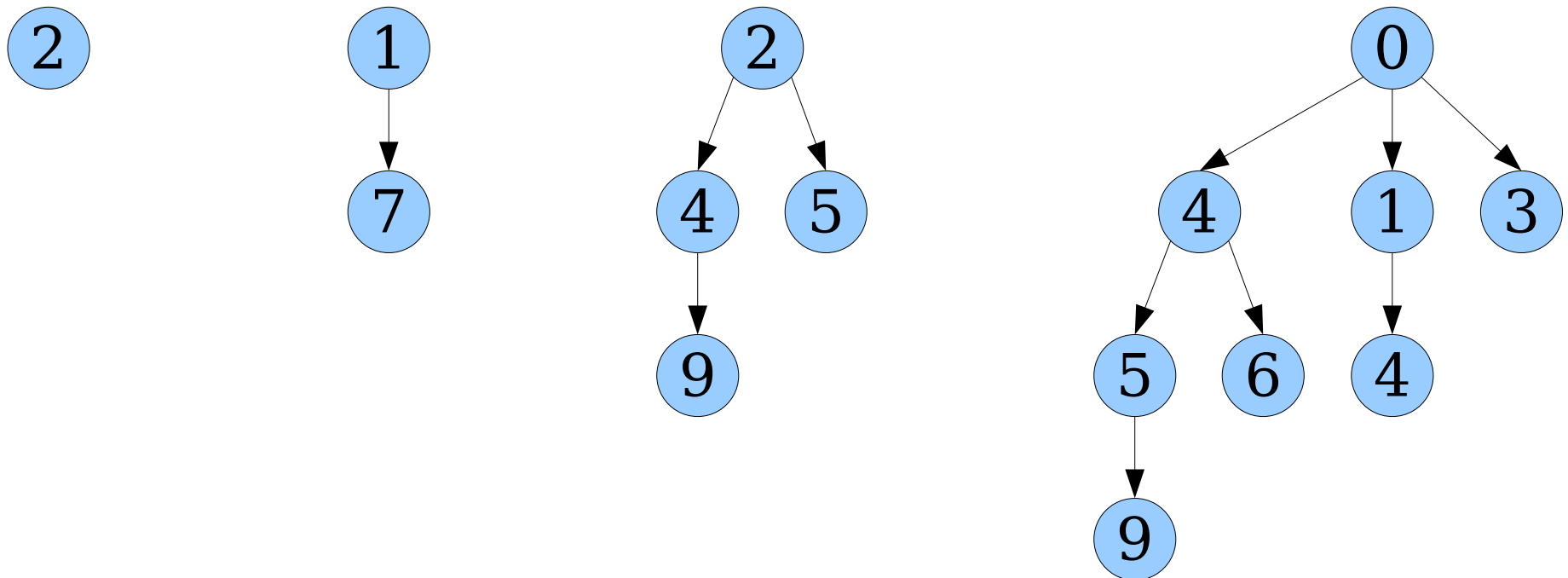
Fun Question:
Why doesn't this inductive proof have a base case?



There's another way to show this. Stay tuned!

Binomial Trees

- A **heap-ordered binomial tree** is a binomial tree whose nodes obey the heap property: all nodes are less than or equal to their descendants.
- We will use heap-ordered binomial trees to implement our “packets.”

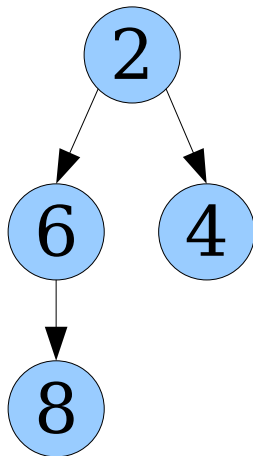


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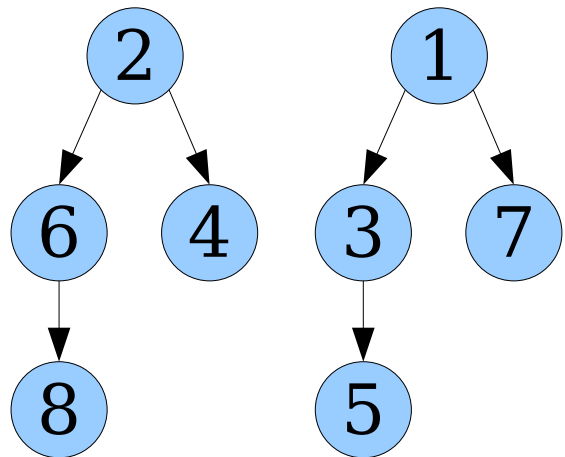
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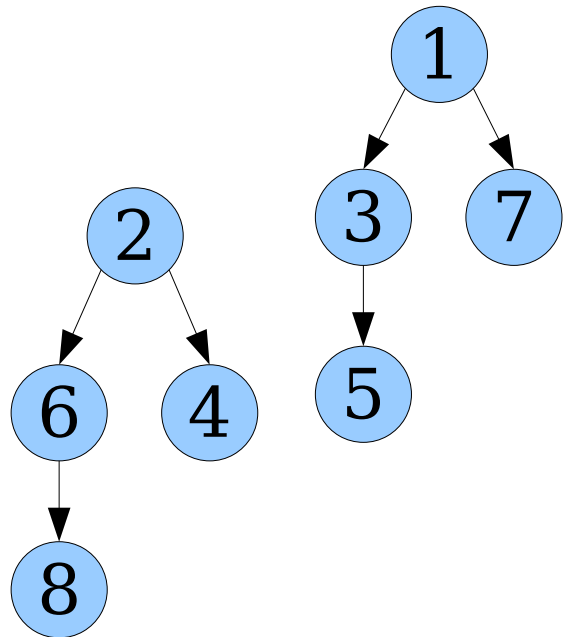
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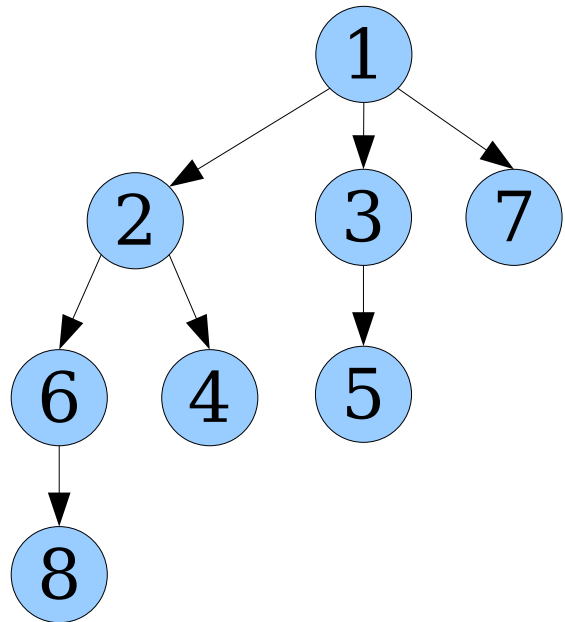
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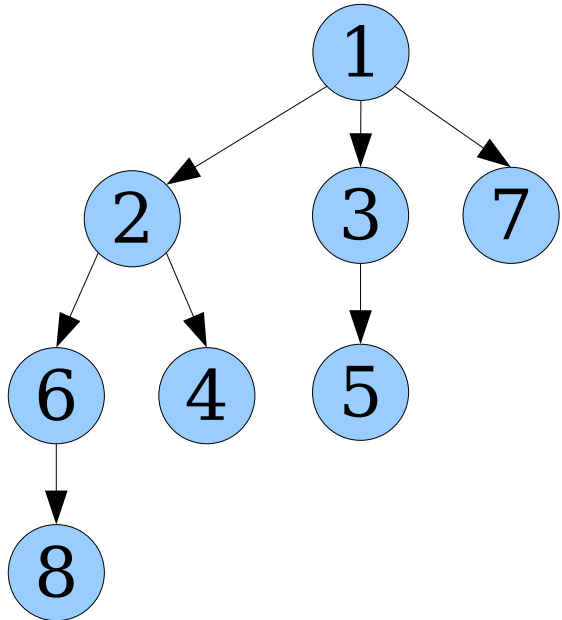
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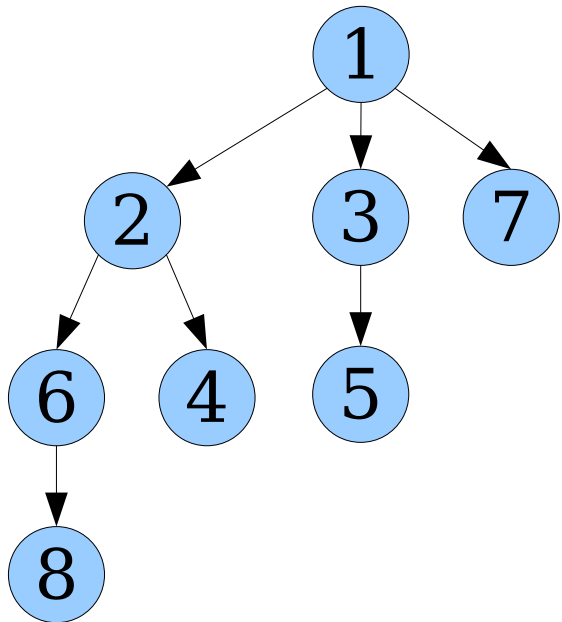
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Make the binomial tree with the larger root the first child of the tree with the smaller root.

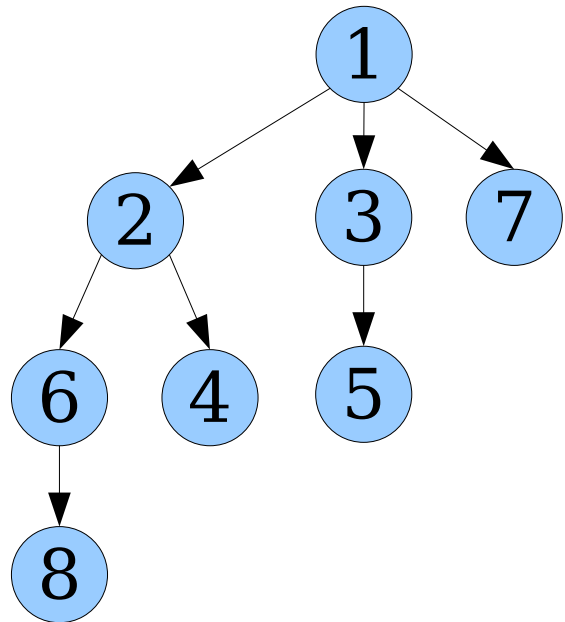
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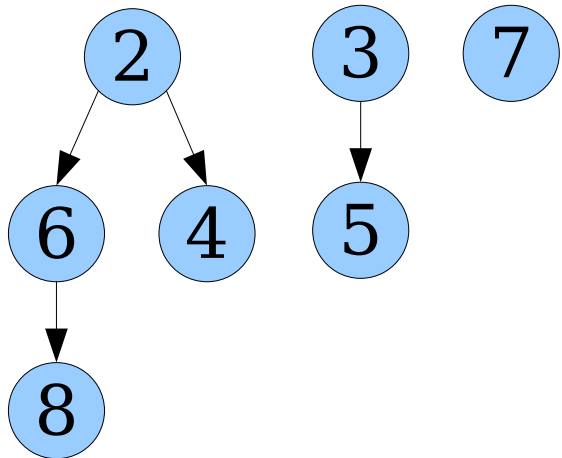
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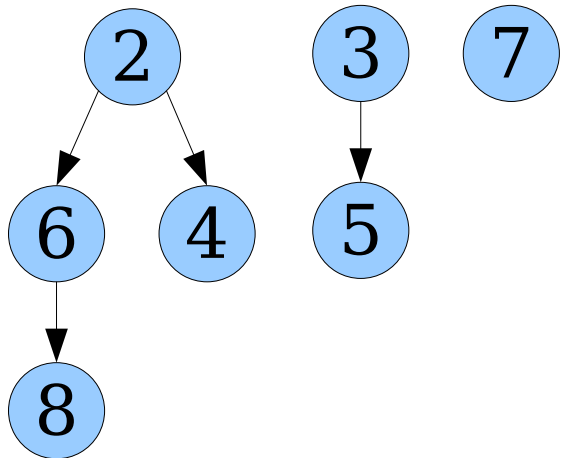
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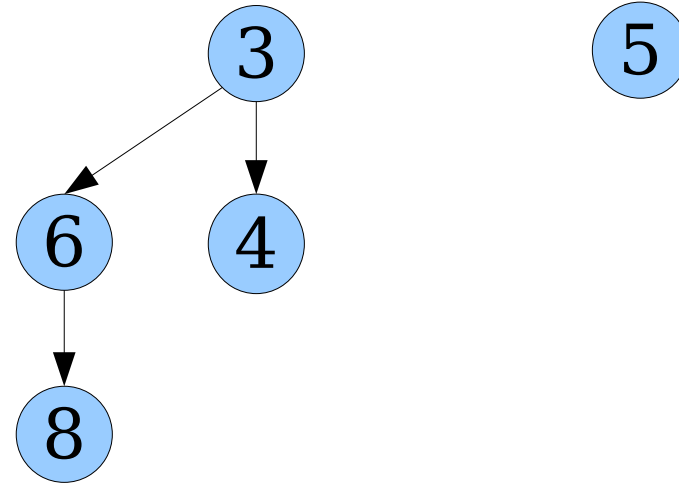
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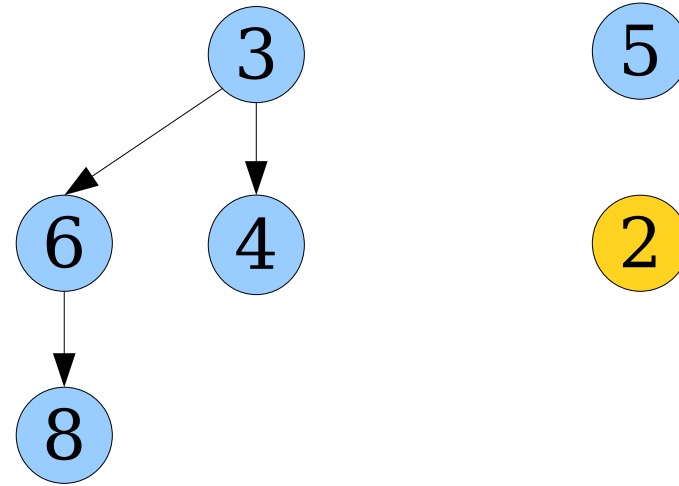
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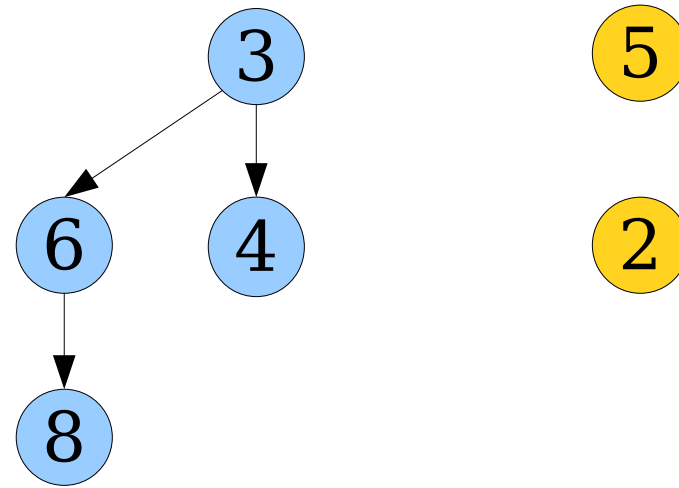


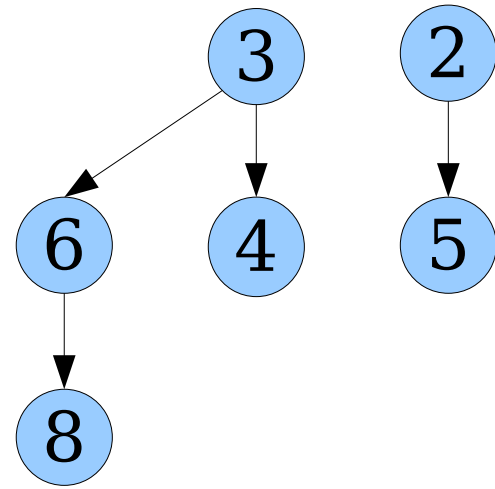
The Binomial Heap

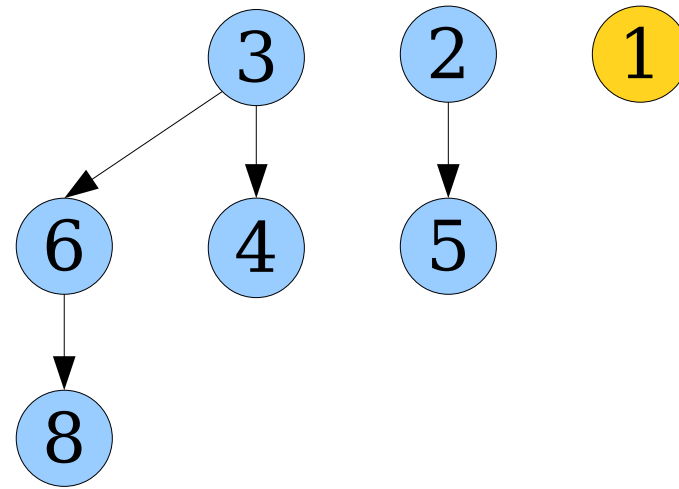
- A **binomial heap** is a collection of binomial trees stored in ascending order of size.
- Operations defined as follows:
 - **meld**(pq_1, pq_2): Use addition to combine all the trees.
 - Fuses $O(\log n + \log m)$ trees. Cost: $O(\log n + \log m)$. Here, assume one binomial heap has n nodes, the other m .
 - pq .**enqueue**(v, k): Meld pq and a singleton heap of (v, k) .
 - Total time: $O(\log n)$.
 - pq .**find-min**(): Find the minimum of all tree roots.
 - Total time: $O(\log n)$.
 - pq .**extract-min**(): Find the min, delete the tree root, then meld together the queue and the exposed children.
 - Total time: $O(\log n)$.

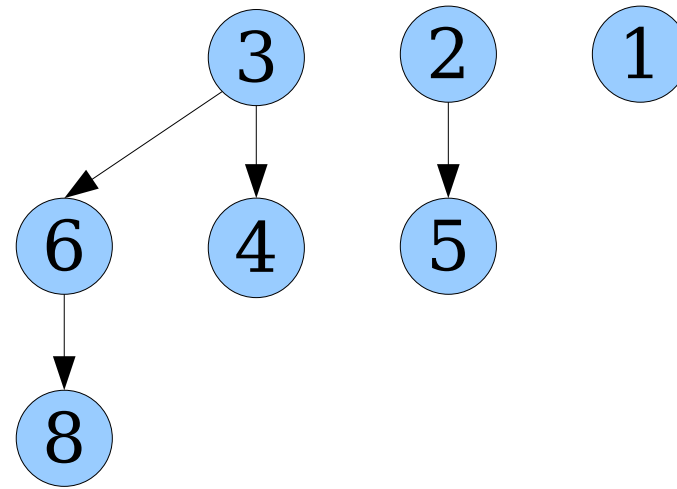


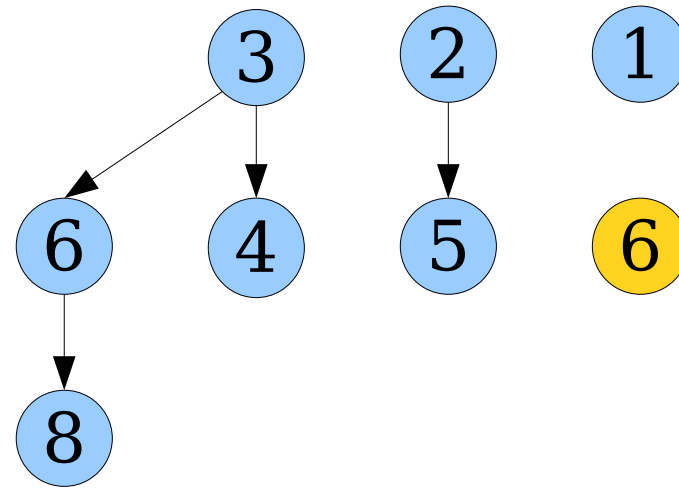


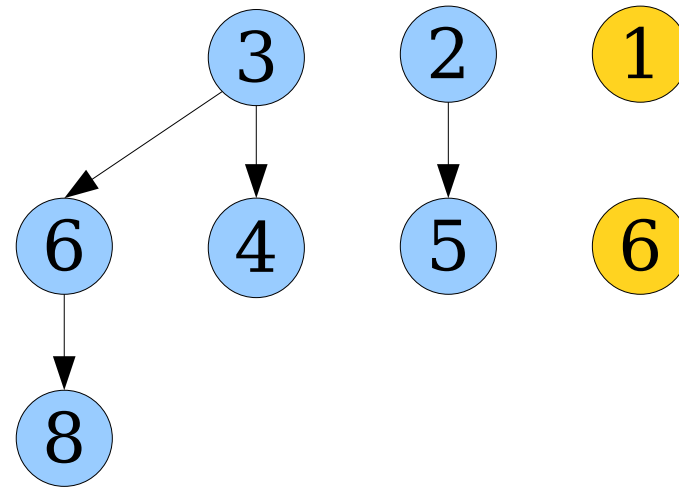


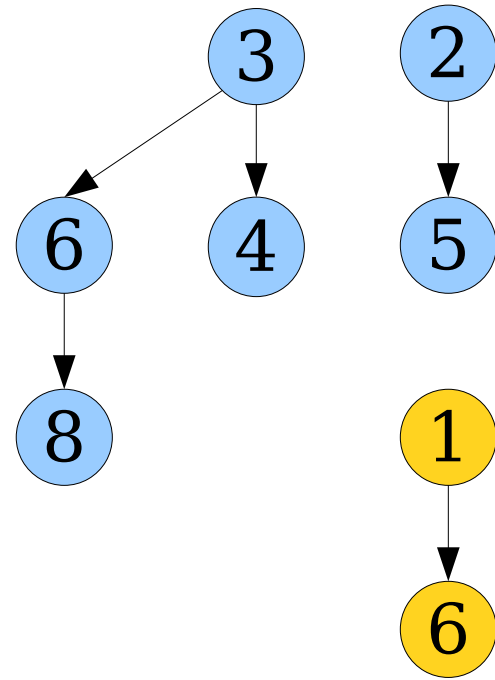


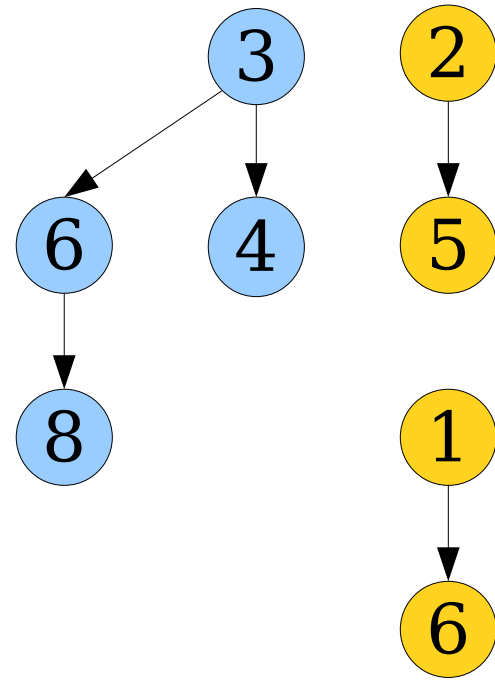


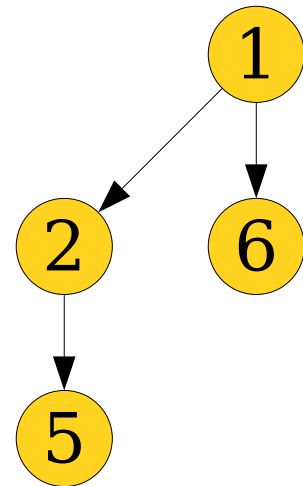
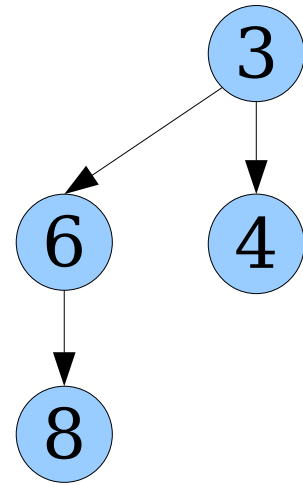


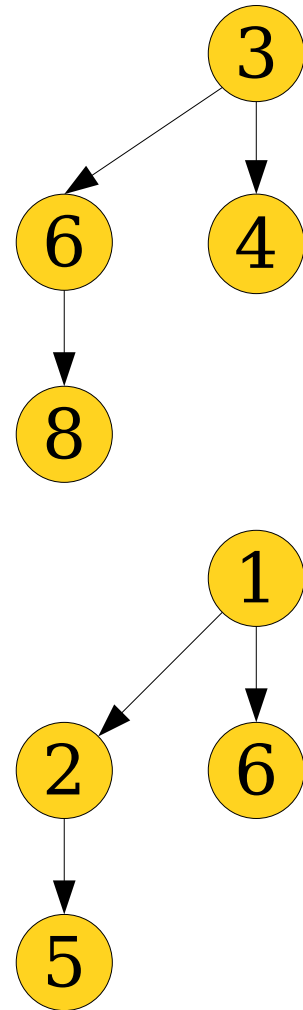


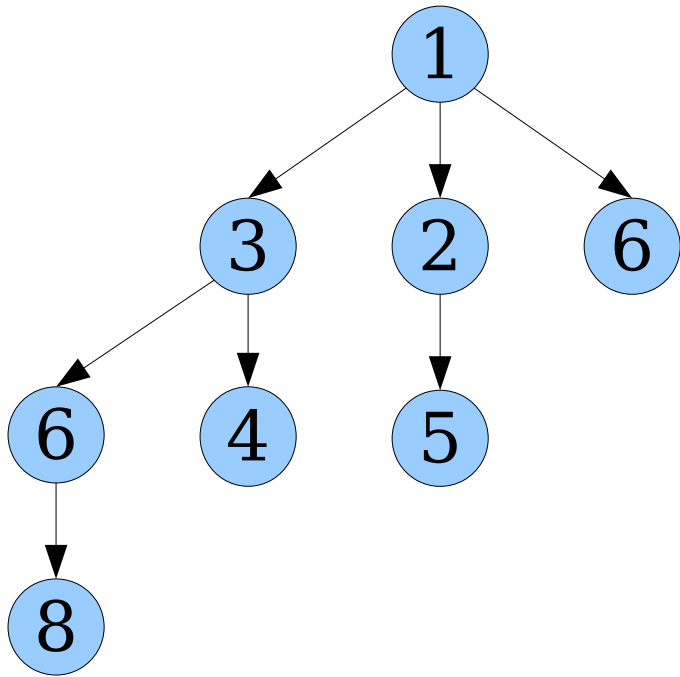


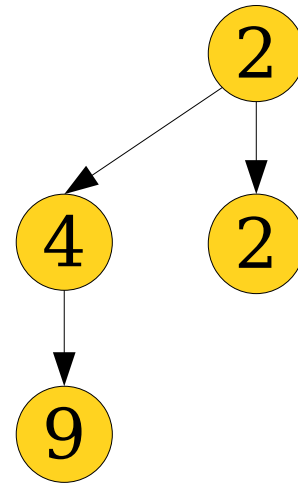
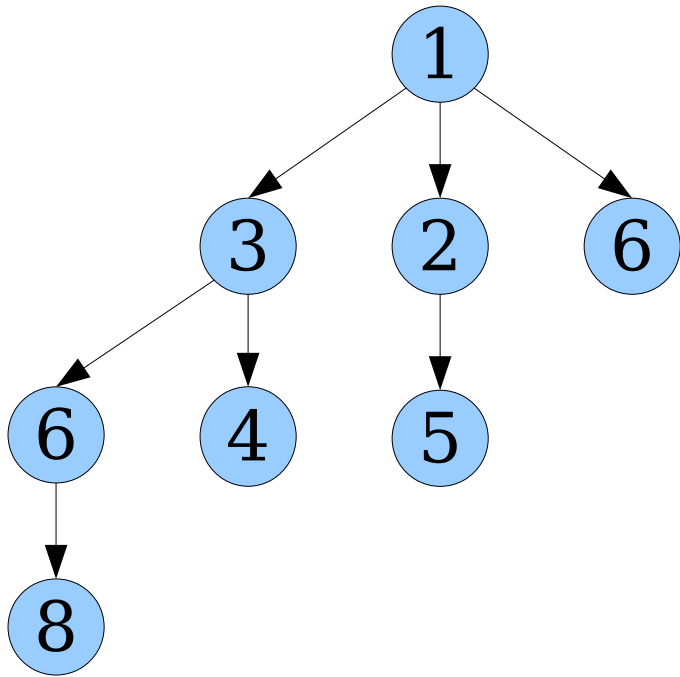


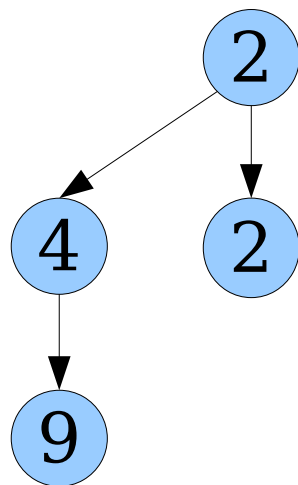
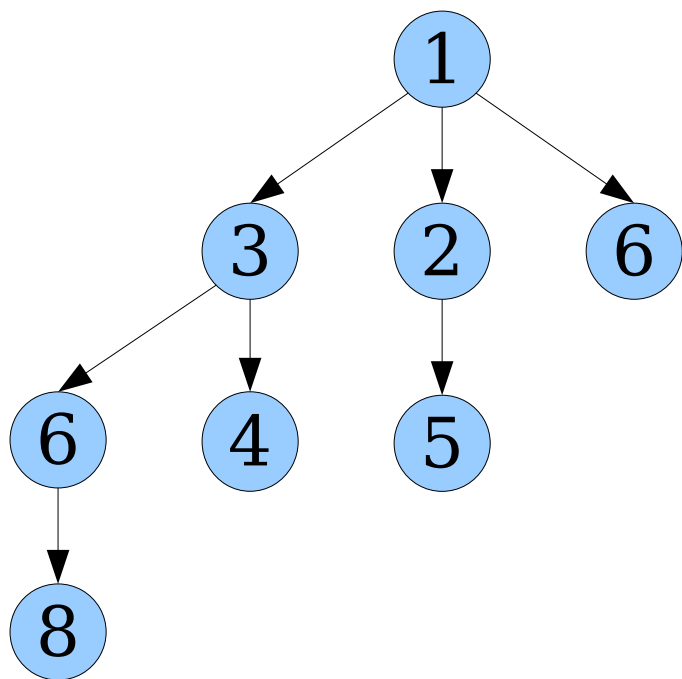


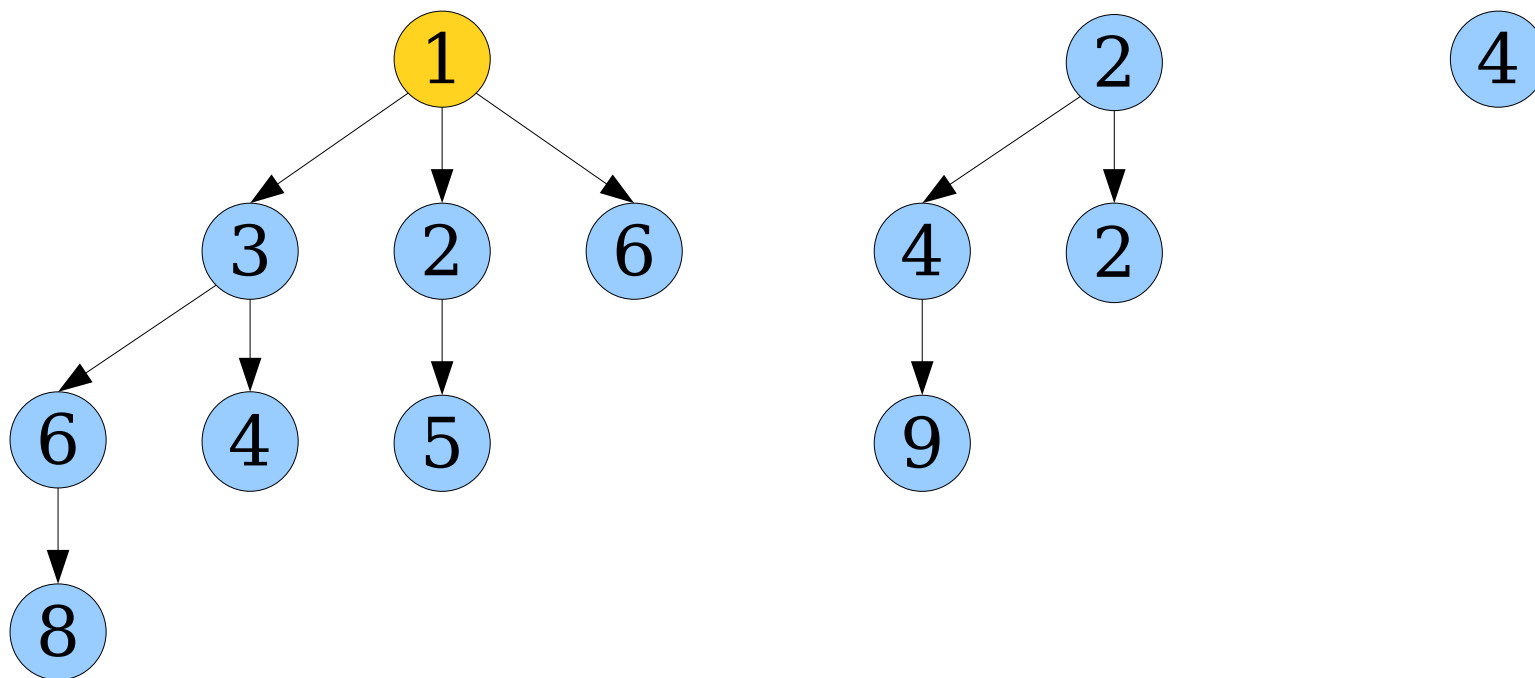


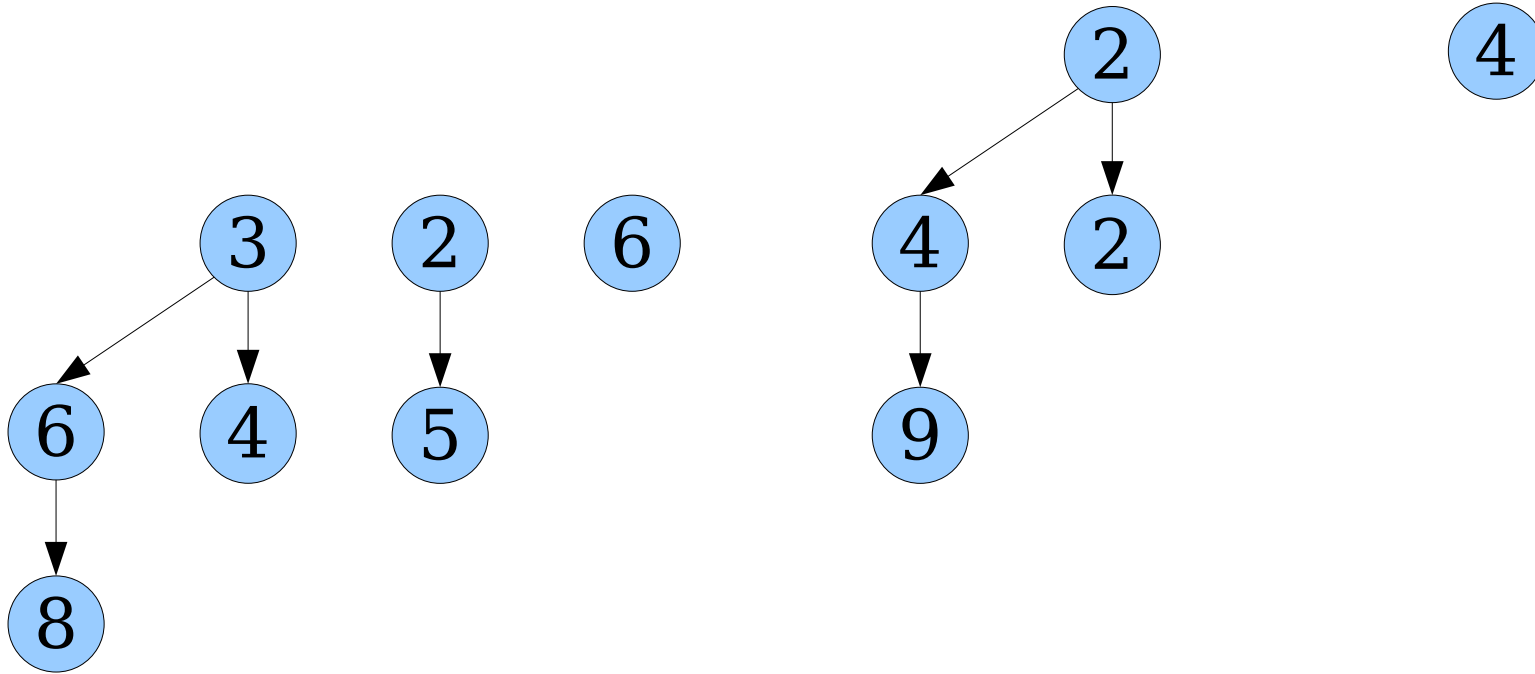


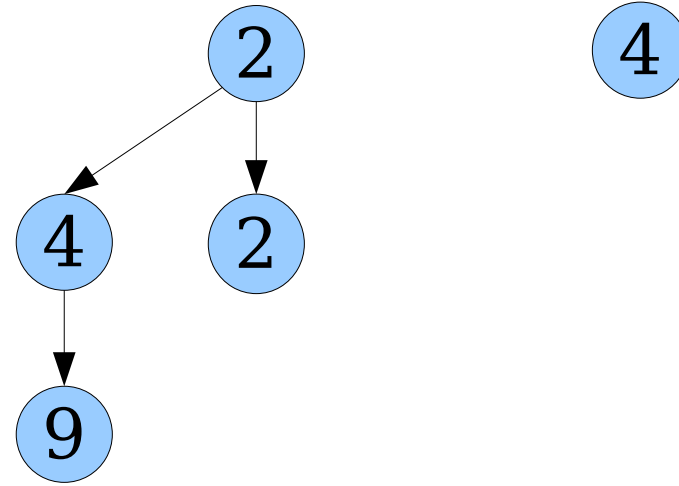
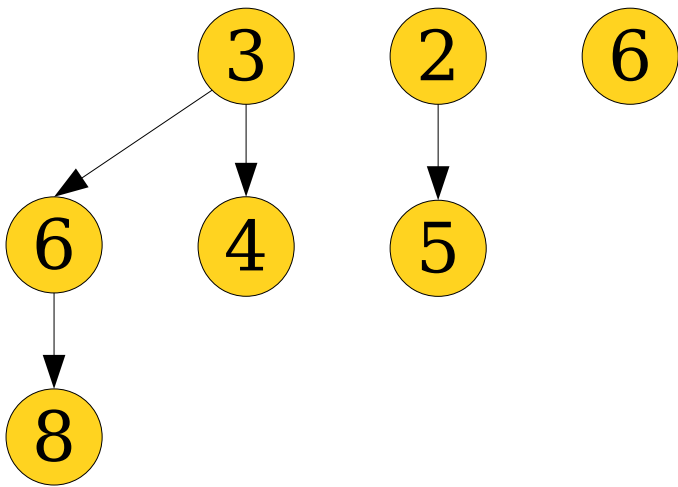


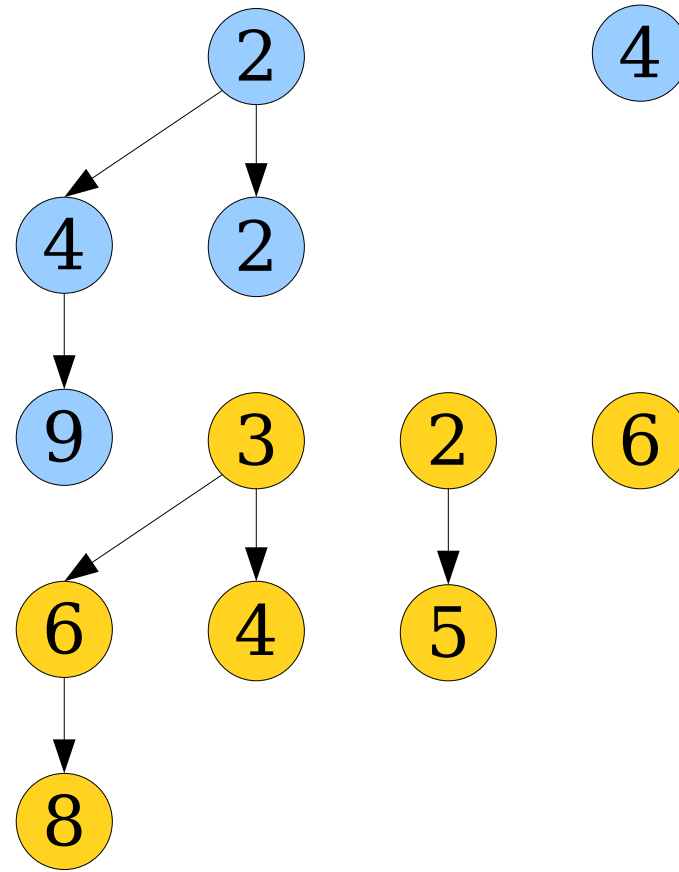


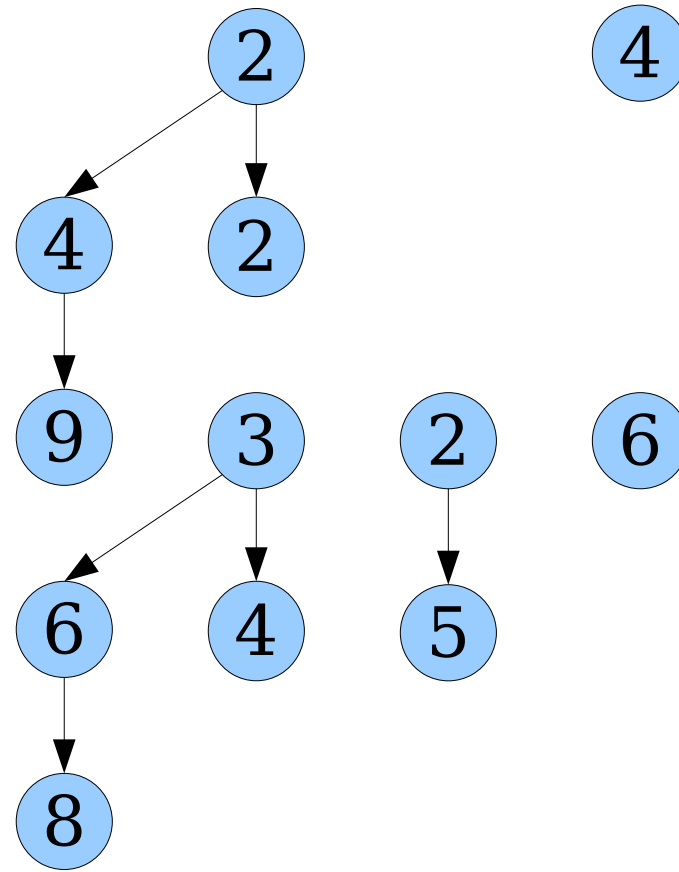


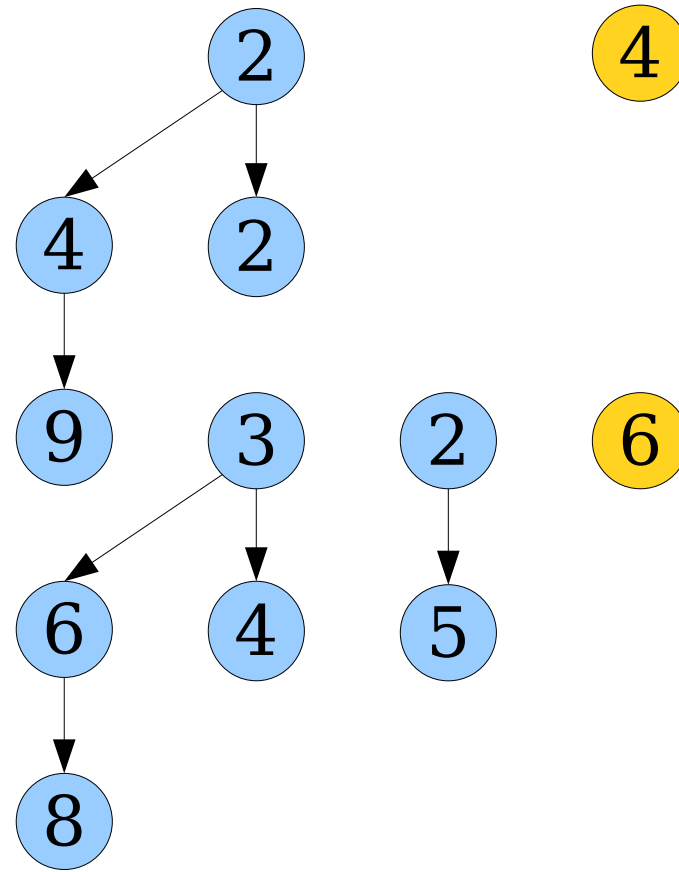


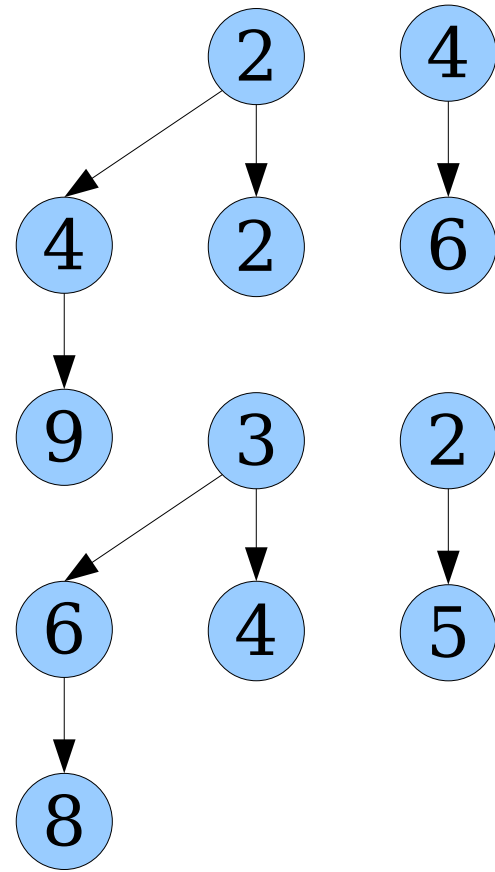


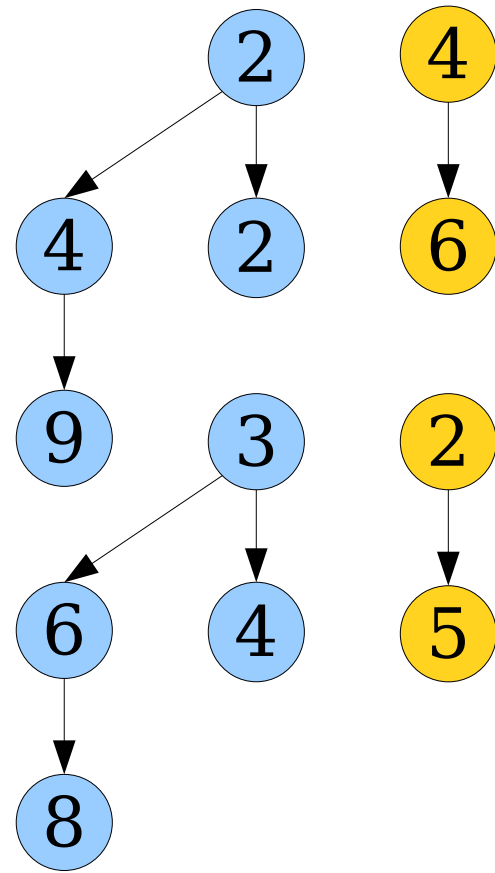


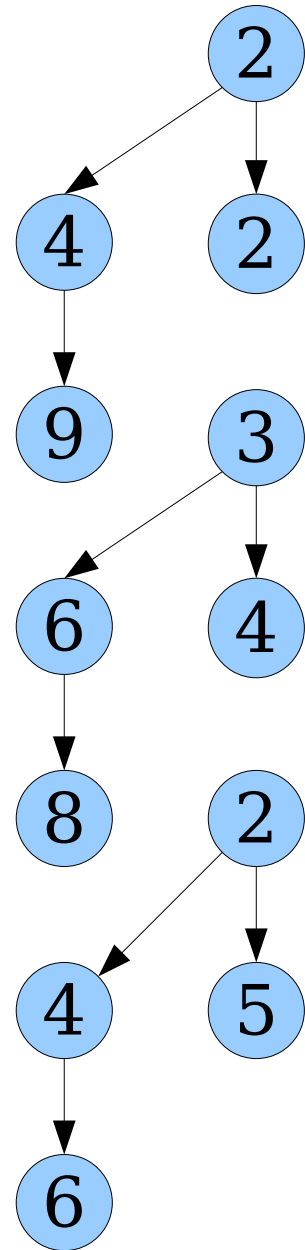


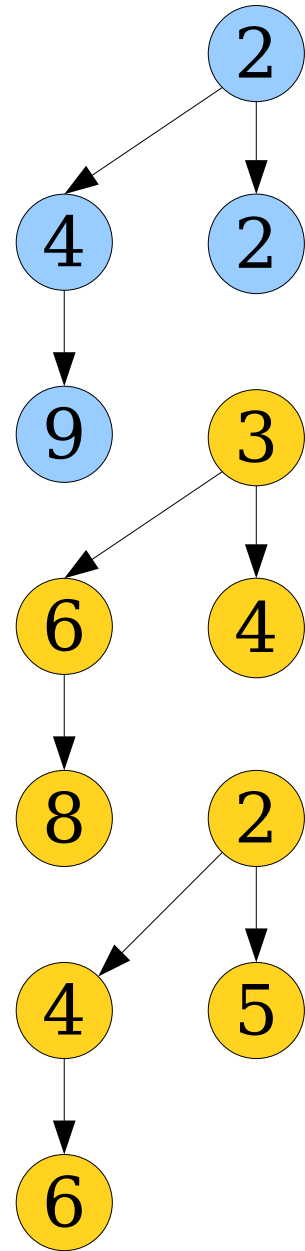


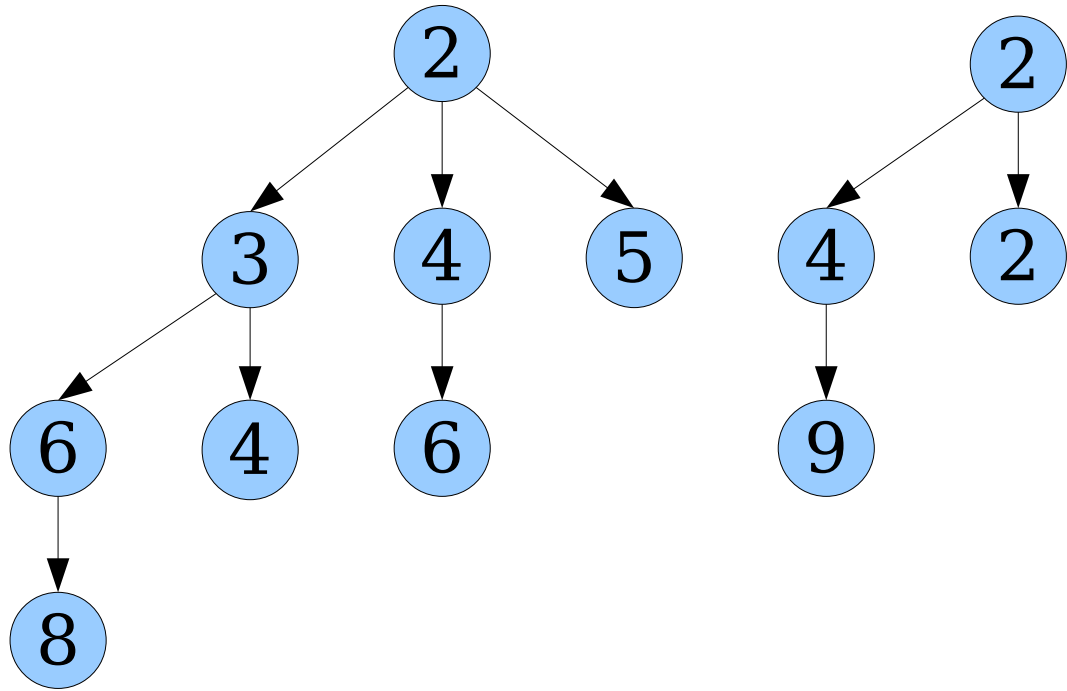












Draw what happens if we *enqueue* the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into a binomial heap.

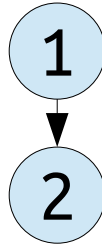
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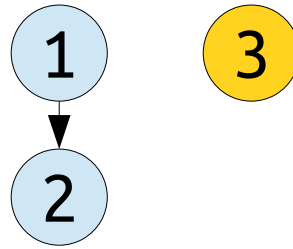
1

2

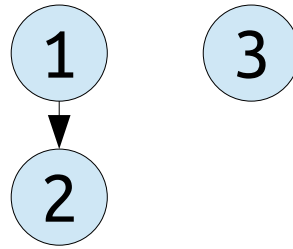
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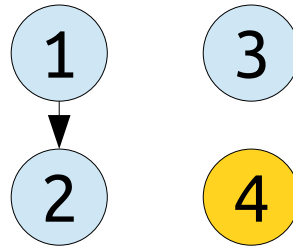
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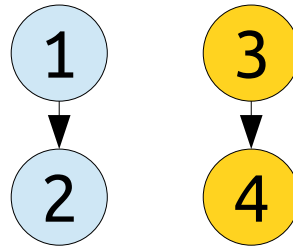
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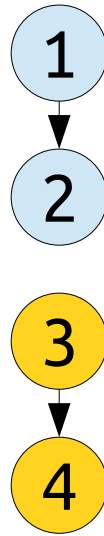
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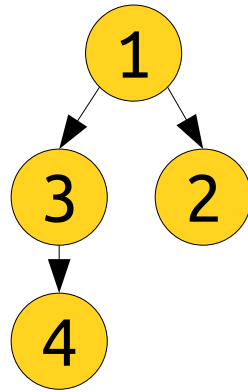
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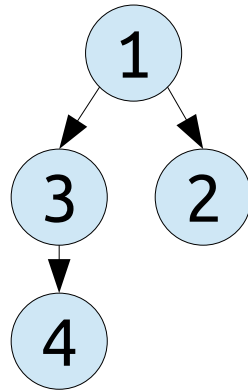
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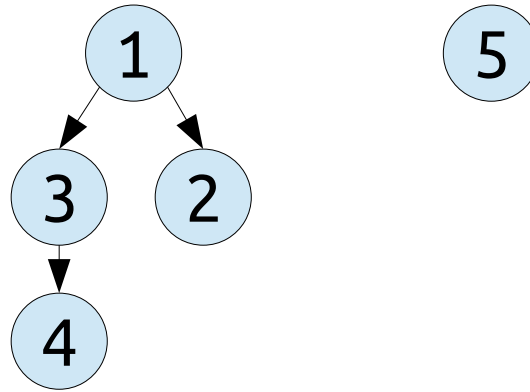
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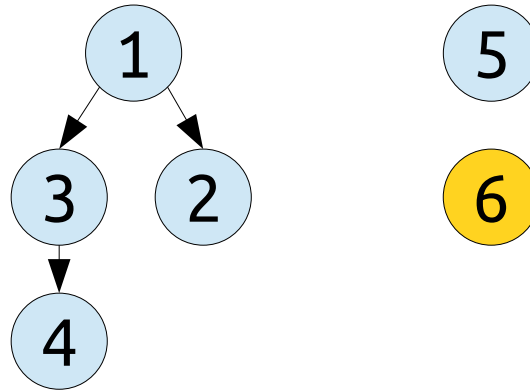
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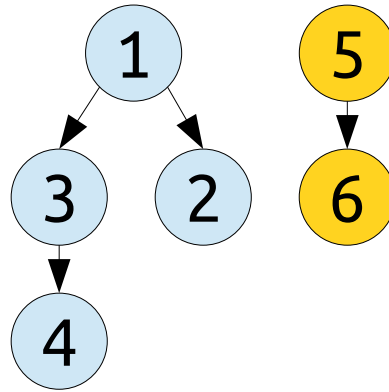
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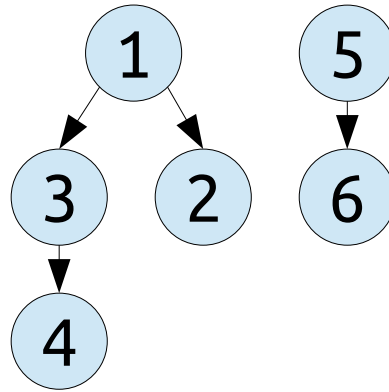
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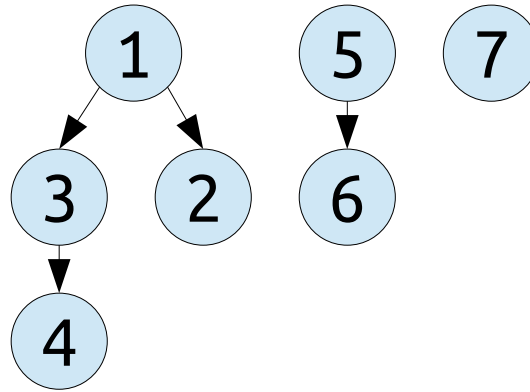
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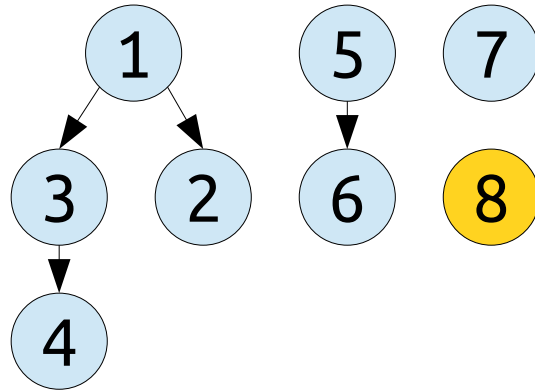
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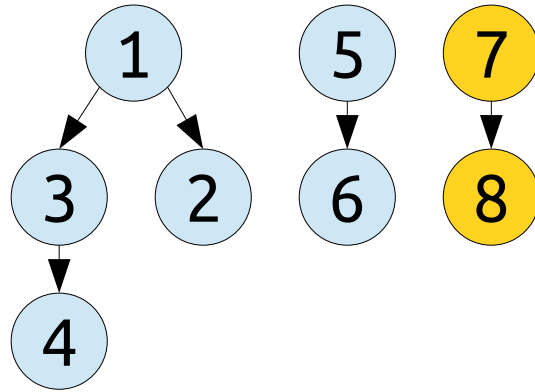
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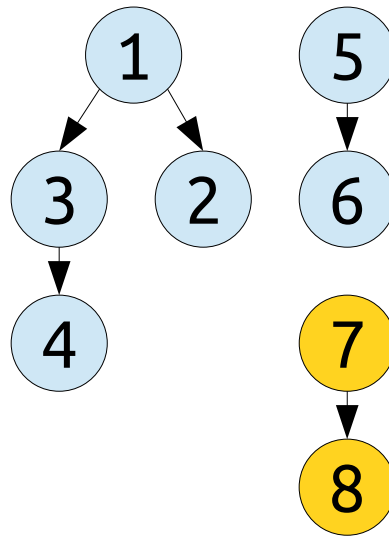
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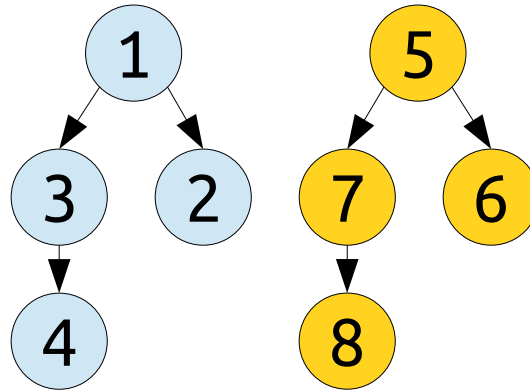
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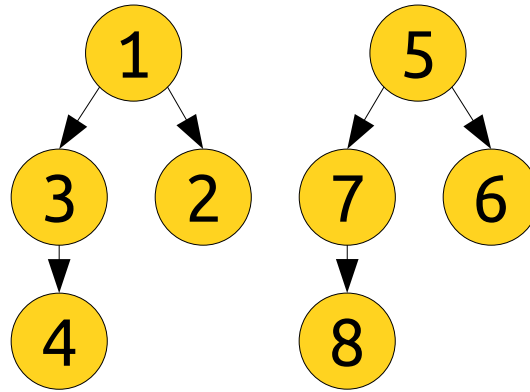
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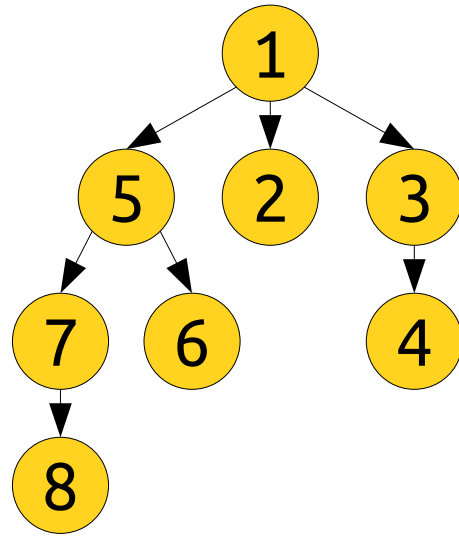
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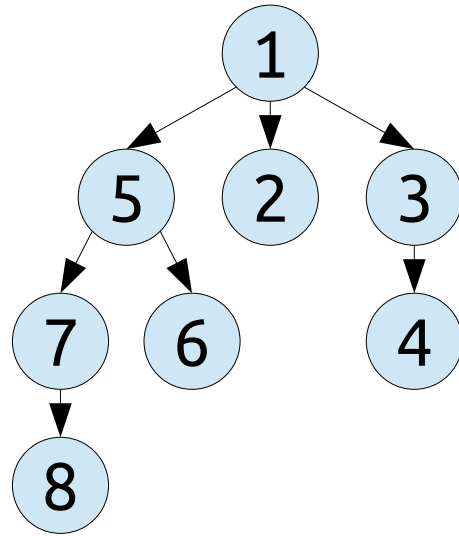
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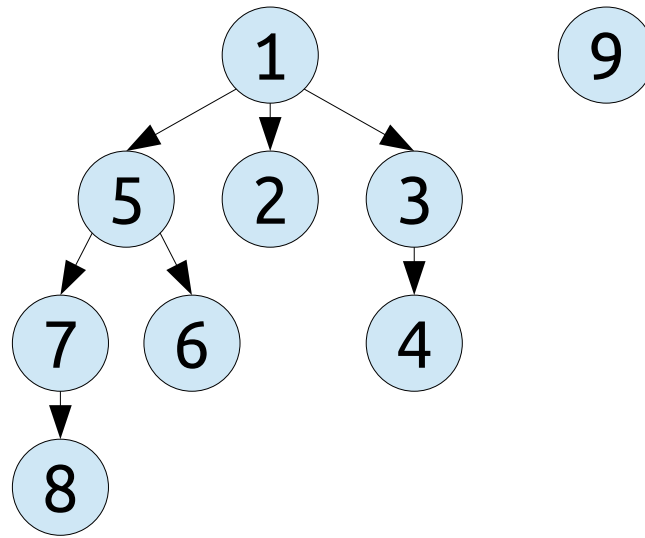
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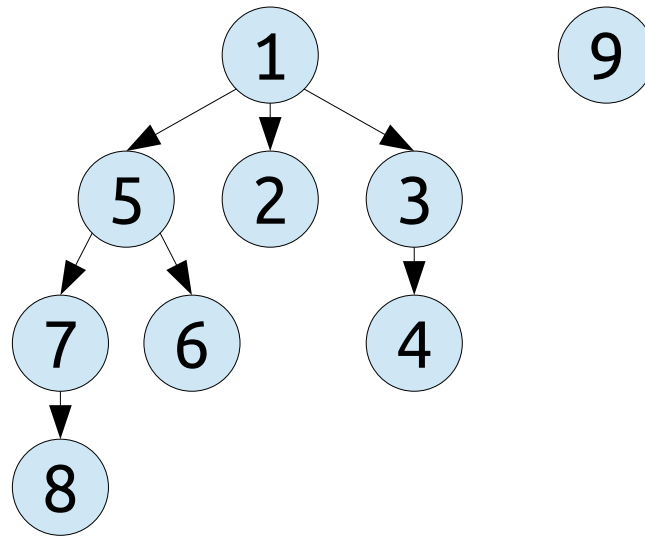
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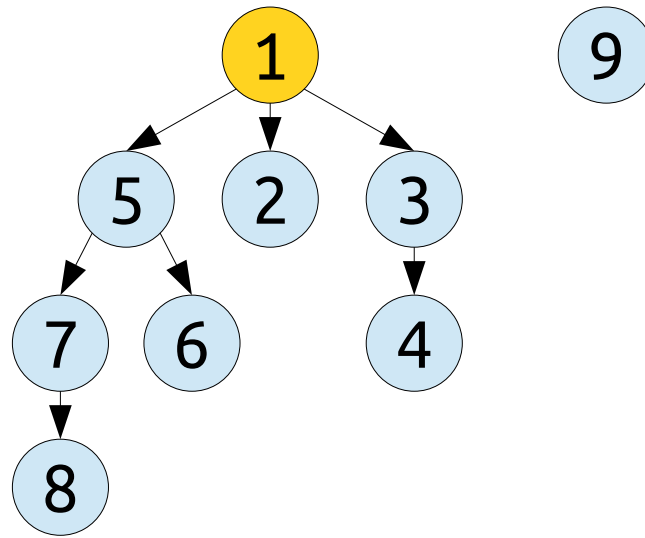
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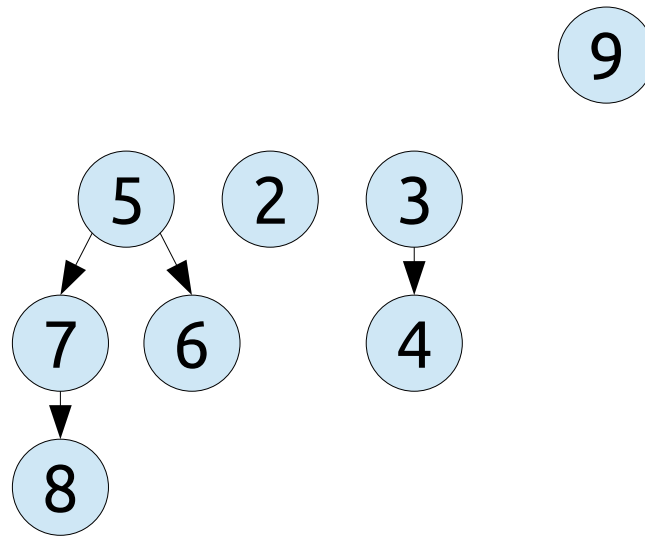
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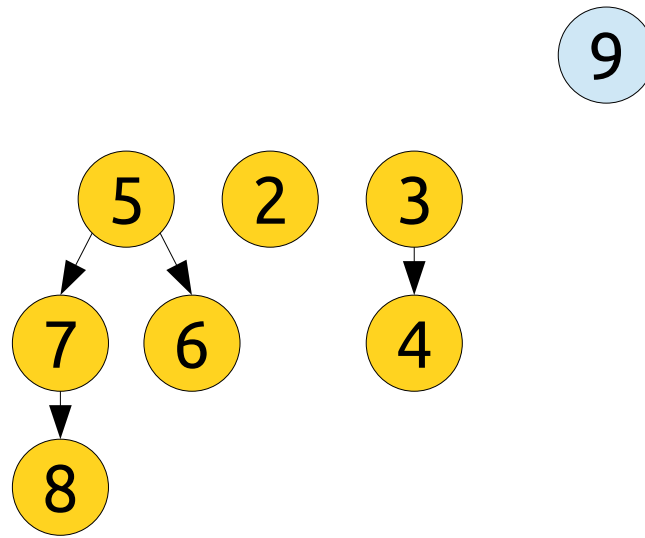
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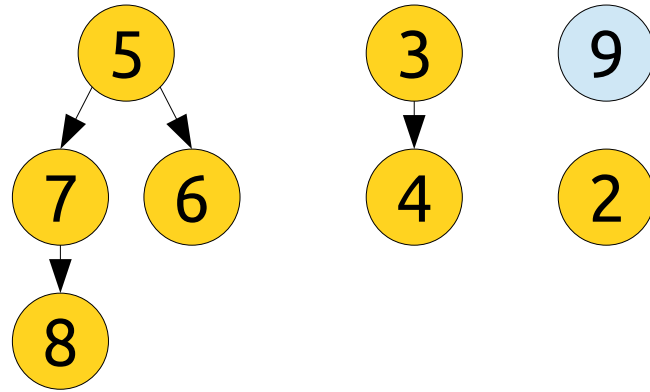
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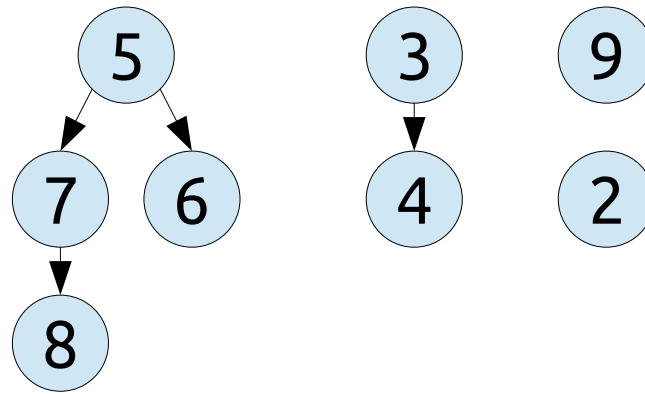
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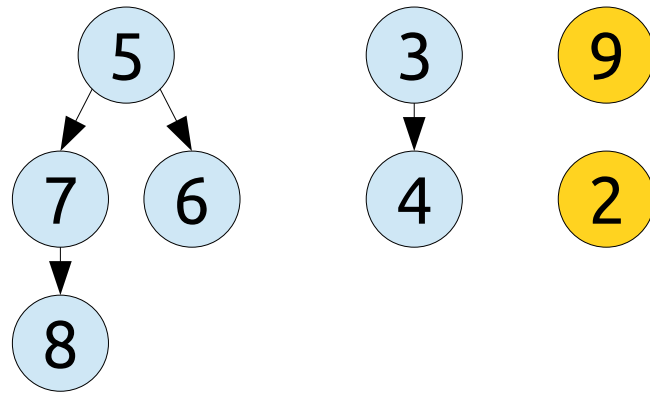
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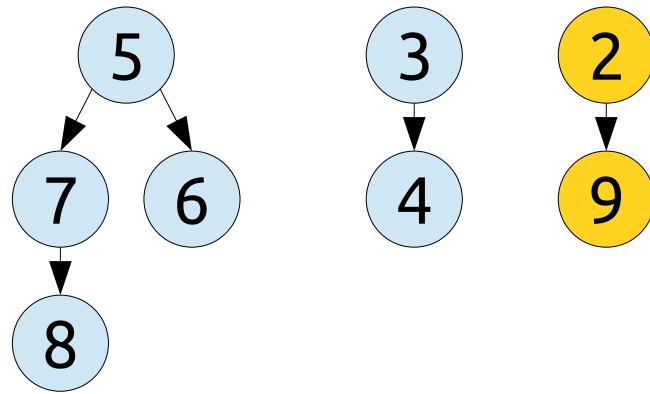
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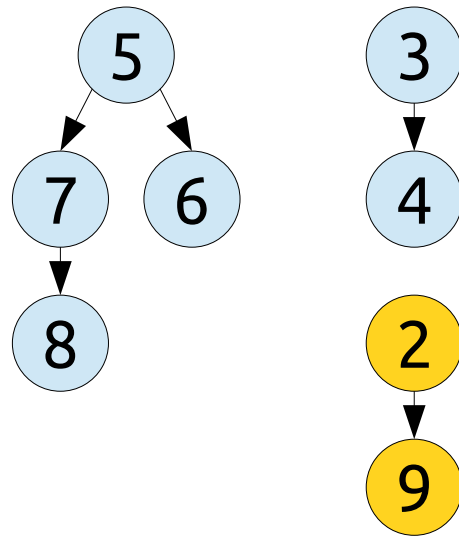
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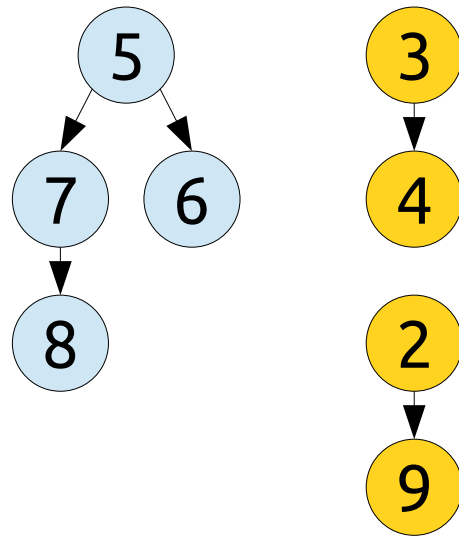
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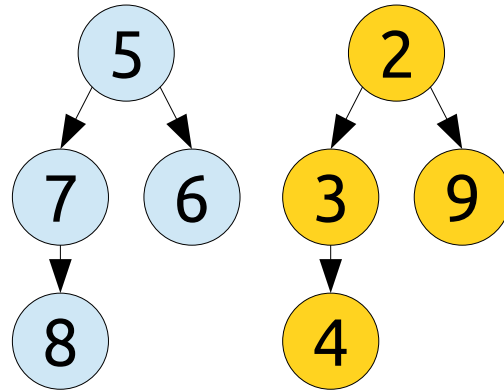
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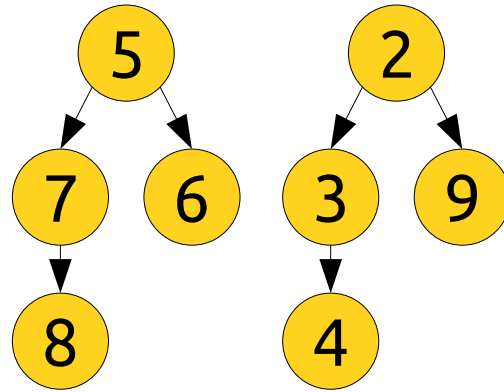
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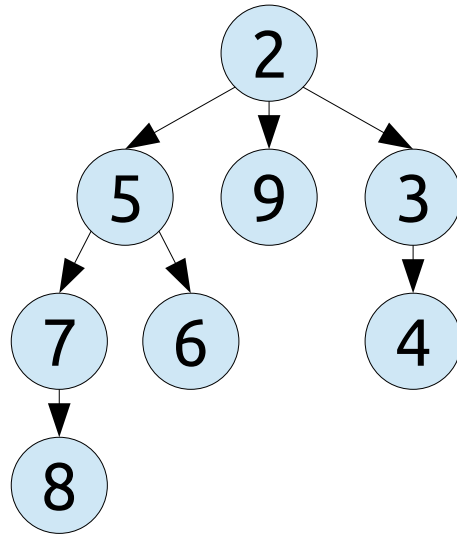
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Draw what happens after performing an ***extract-min*** in this binomial heap.



Draw what happens after performing an *extract-min* in this binomial heap.



Draw what happens after performing an *extract-min* in this binomial heap.

Where We Stand

- Here's the current scorecard for the binomial heap.
- This is a fast, elegant, and clever data structure.
- **Question:** Can we do better?

Binomial Heap

- **enqueue**: $O(\log n)$
- **find-min**: $O(\log n)$
- **extract-min**: $O(\log n)$
- **meld**: $O(\log m + \log n)$.

Where We Stand

- **Theorem:** No comparison-based priority queue structure can have *enqueue* and *extract-min* each take time $o(\log n)$.
- **Proof:** Suppose these operations each take time $o(\log n)$. Then we could sort n elements by perform n *enqueues* and then n *extract-mins* in time $o(n \log n)$. This is impossible with comparison-based algorithms. ■

Binomial Heap

- *enqueue*: $O(\log n)$
- *find-min*: $O(\log n)$
- *extract-min*: $O(\log n)$
- *meld*: $O(\log m + \log n)$.

Where We Stand

- We can't make both *enqueue* and *extract-min* run in time $o(\log n)$.
- However, we could conceivably make one of them faster.
- **Question:** Which one should we prioritize?
- Probably *enqueue*, since we aren't guaranteed to have to remove all added items.
- **Goal:** Make *enqueue* take time $O(1)$.

Binomial Heap

- *enqueue*: $O(\log n)$
- *find-min*: $O(\log n)$
- *extract-min*: $O(\log n)$
- *meld*: $O(\log m + \log n)$.

Where We Stand

- The *enqueue* operation is implemented in terms of *meld*.
- If we want *enqueue* to run in time $O(1)$, we'll need *meld* to take time $O(1)$.
- How could we accomplish this?

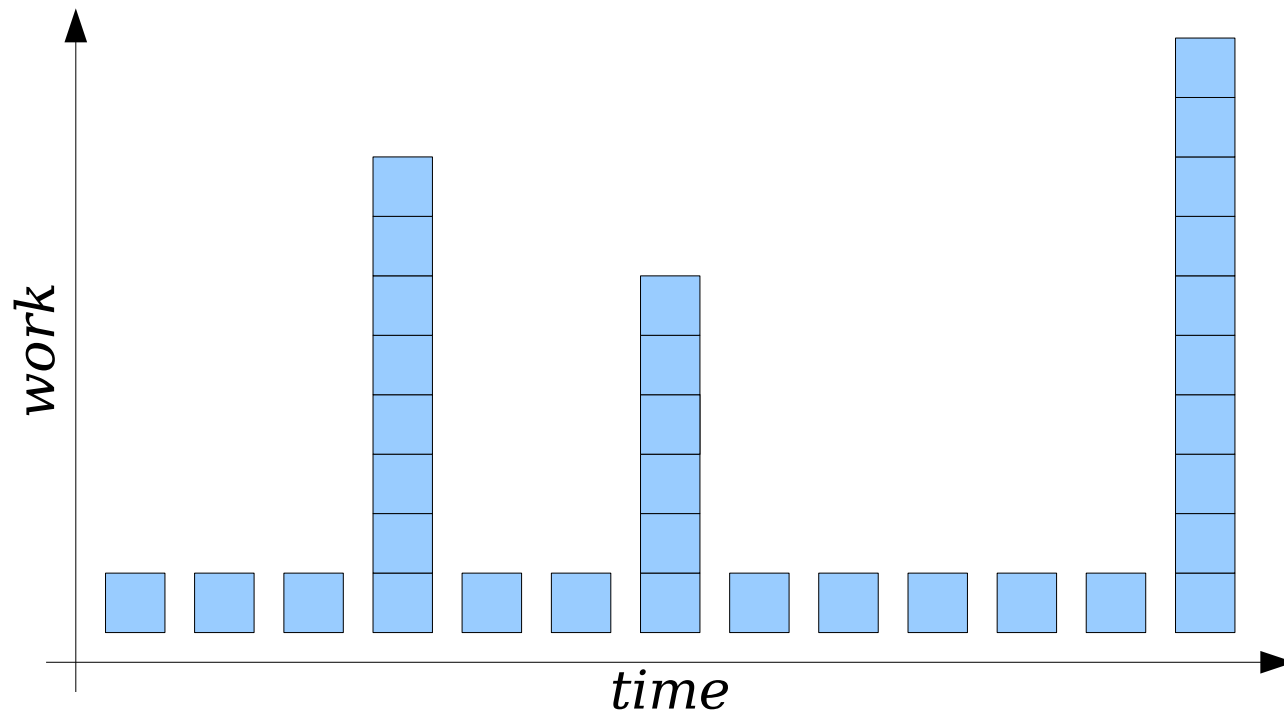
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Thinking With Amortization

Refresher: Amortization

- In an amortized efficient data structure, some operations can take much longer than others, provided that previous operations didn't take too long to finish.
- Think dishwashers: you may have to do a big cleanup at some point, but that's because you did basically no work to wash all the dishes you placed in the dishwasher.

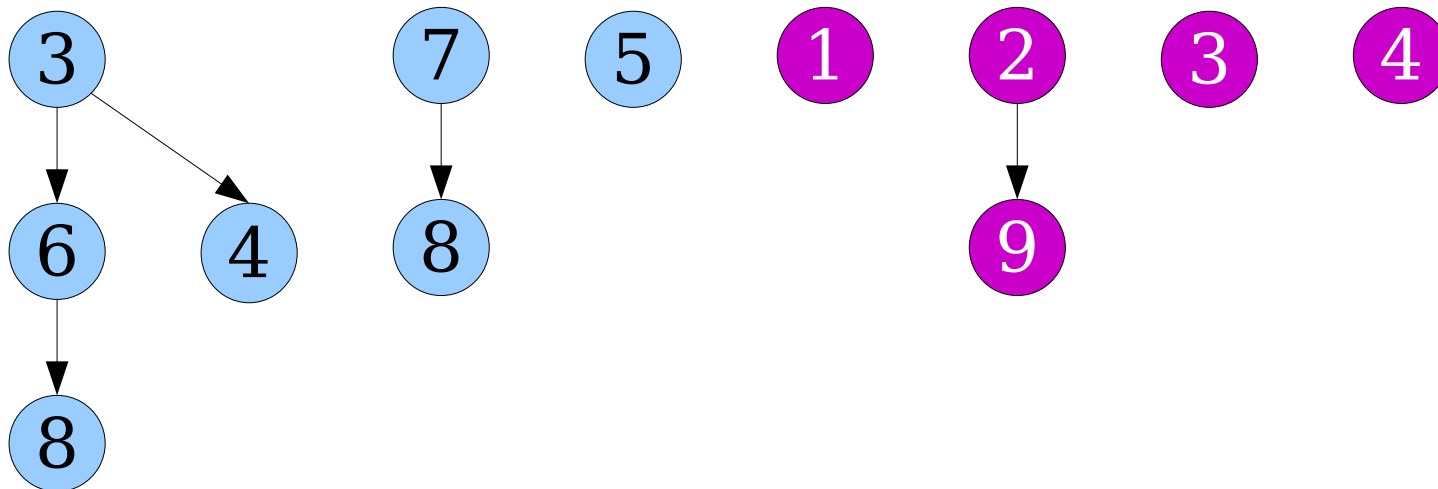


Lazy Melding

- Consider the following lazy *melding* approach:

To meld together two binomial heaps, just combine the two sets of trees together.

- Intuition:** Why do any work to organize keys if we're not going to do an *extract-min*? We'll worry about cleanup then.

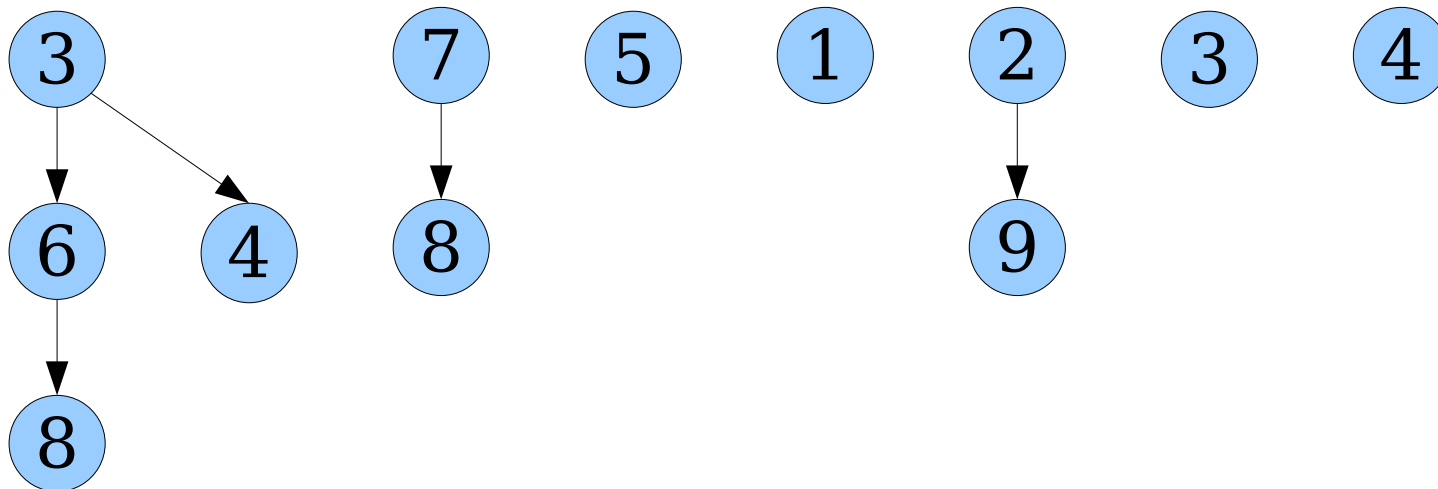


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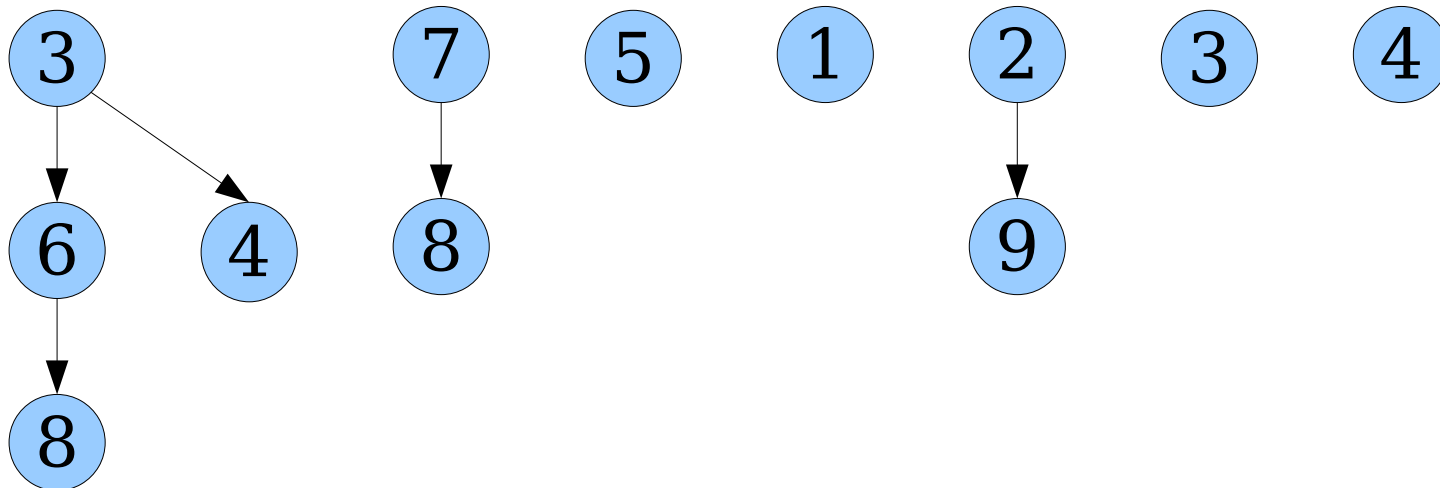
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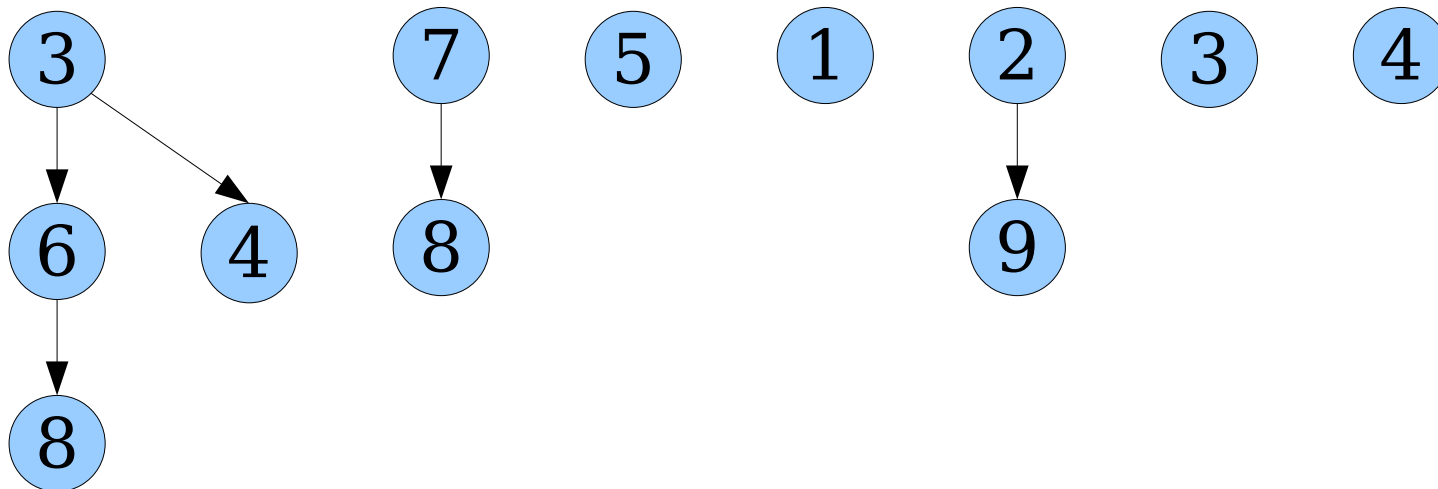
Lazy Melding

- If we store our list of trees as circularly, doubly-linked lists, we can concatenate tree lists in time $O(1)$.
 - Cost of a *meld*: **$O(1)$** .
 - Cost of an *enqueue*: **$O(1)$** .
- If it sounds too good to be true, it probably is.



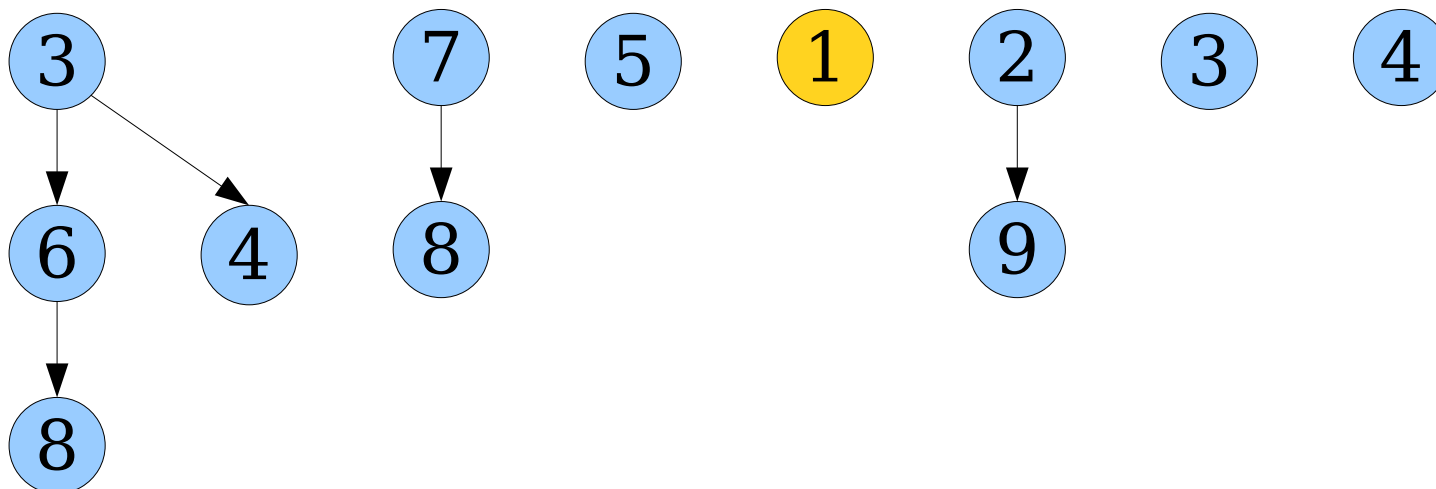
Lazy Melding

- Imagine that we implement *extract-min* the same way as before:
 - Find the packet with the minimum.
 - “Fracture” that packet to expose smaller packets.
 - Meld those packets back in with the master list.
- What happens if we do this with lazy melding?



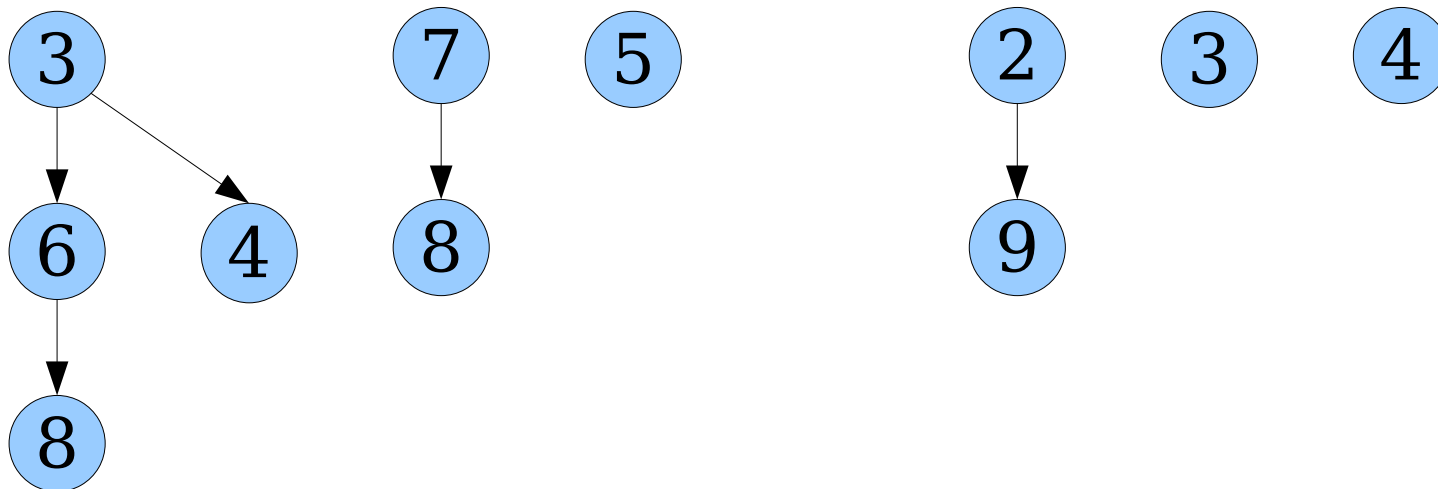
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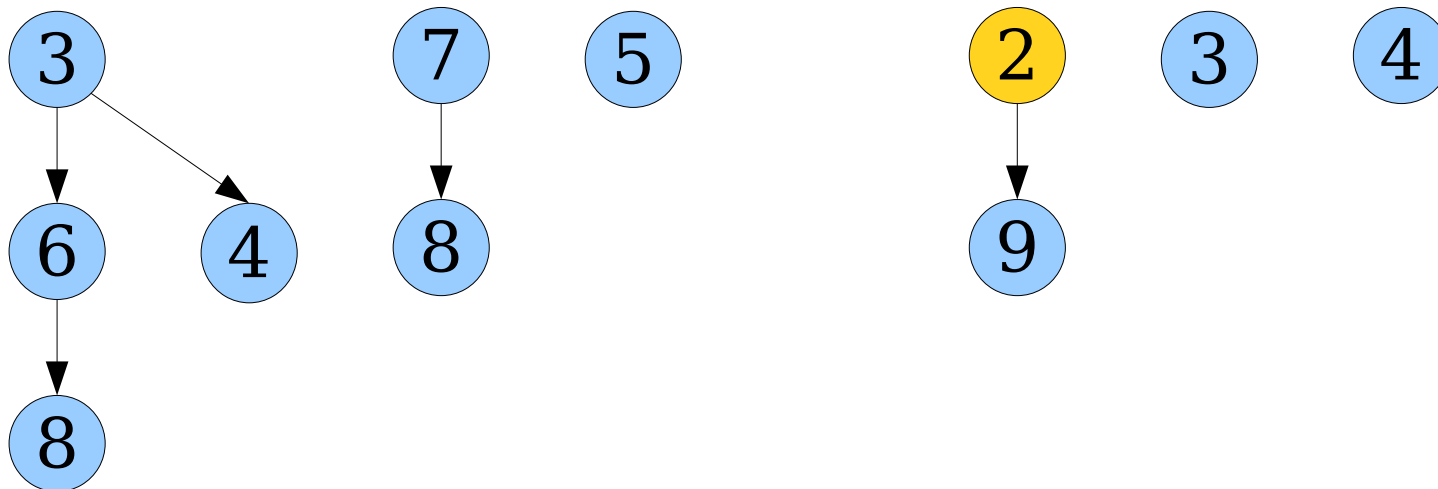
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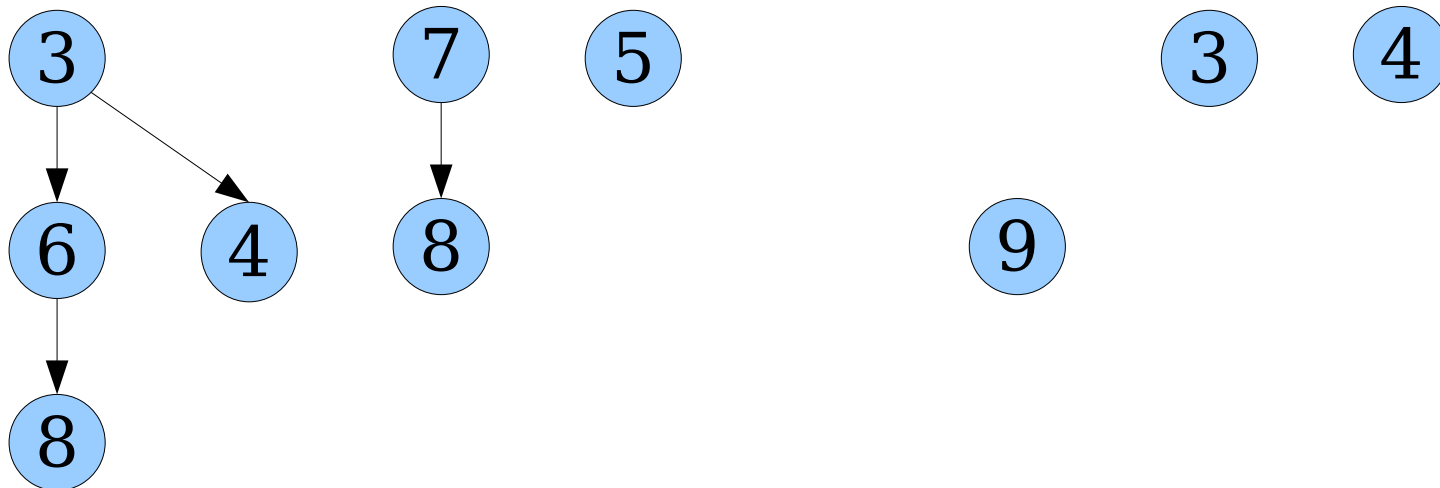
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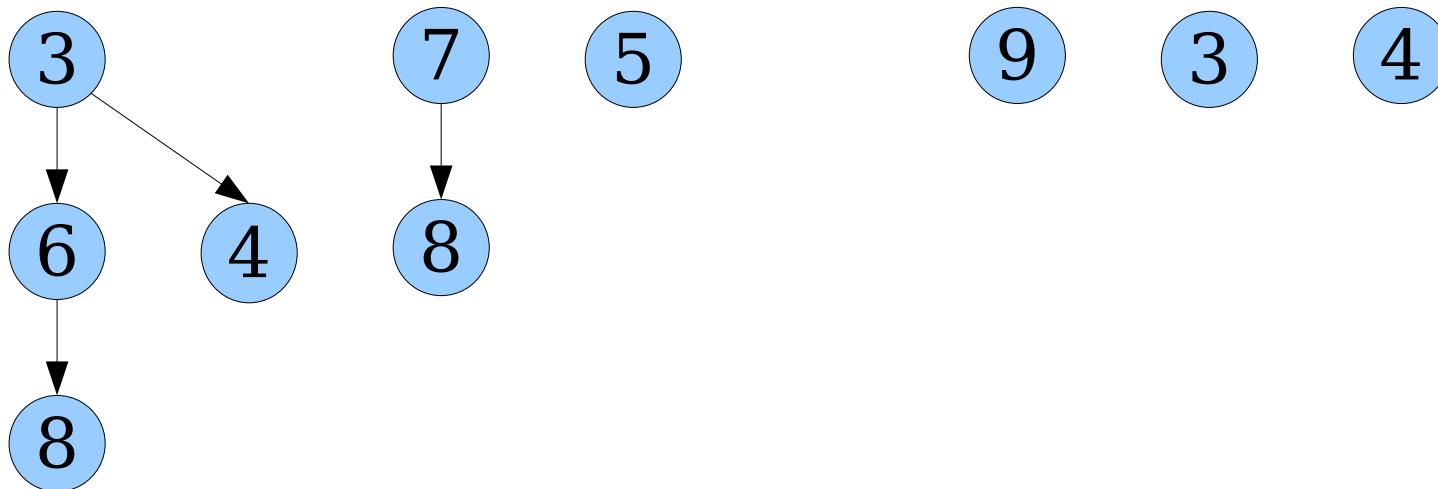
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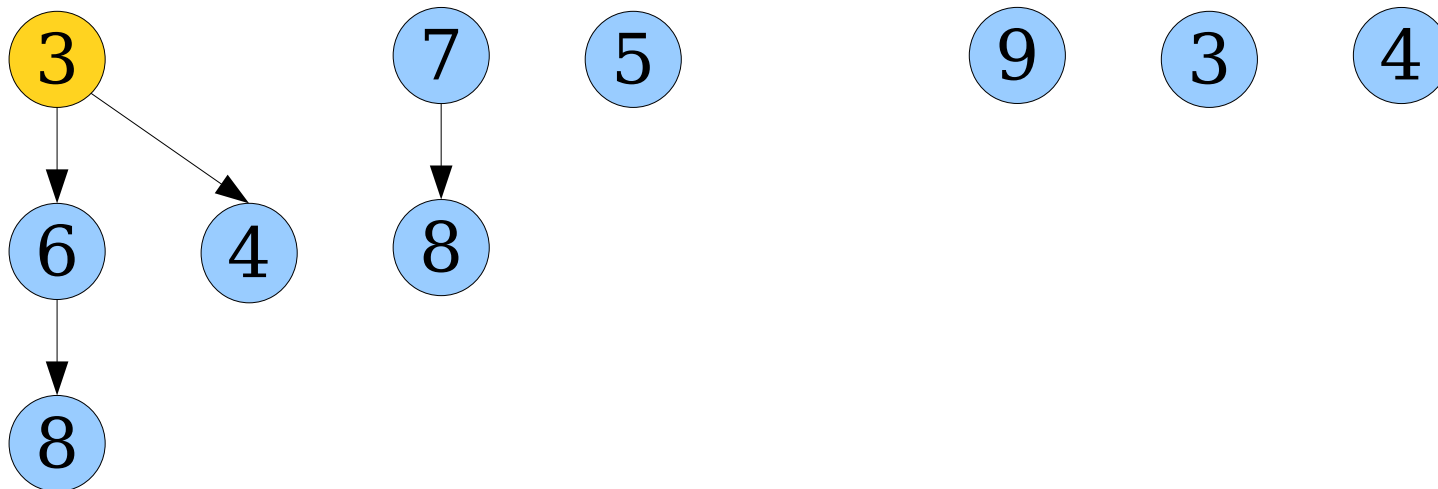
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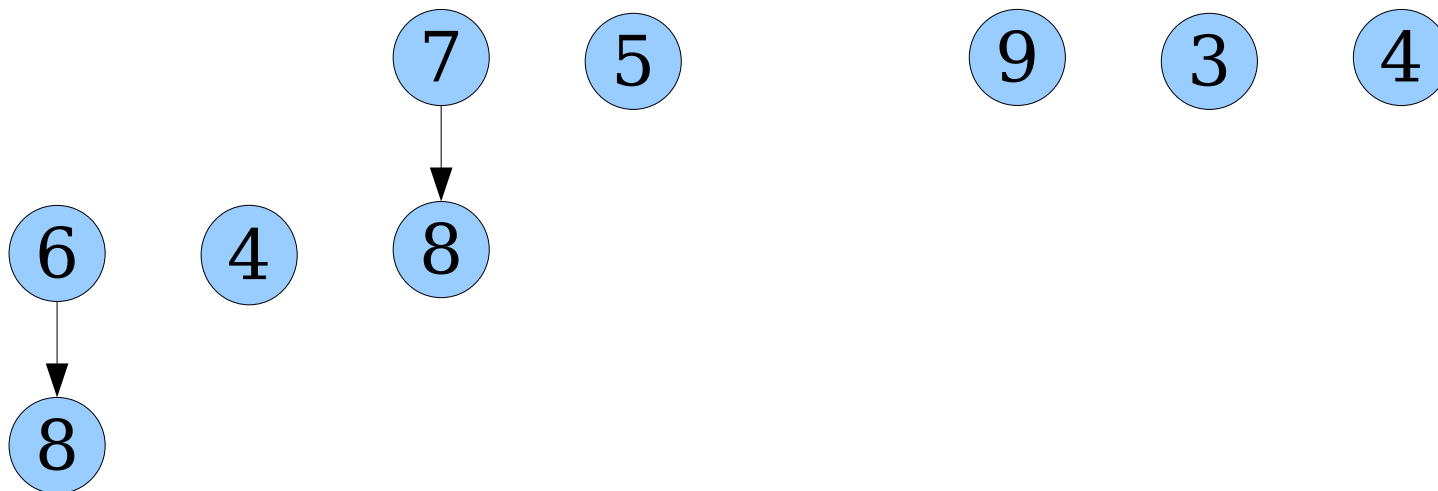
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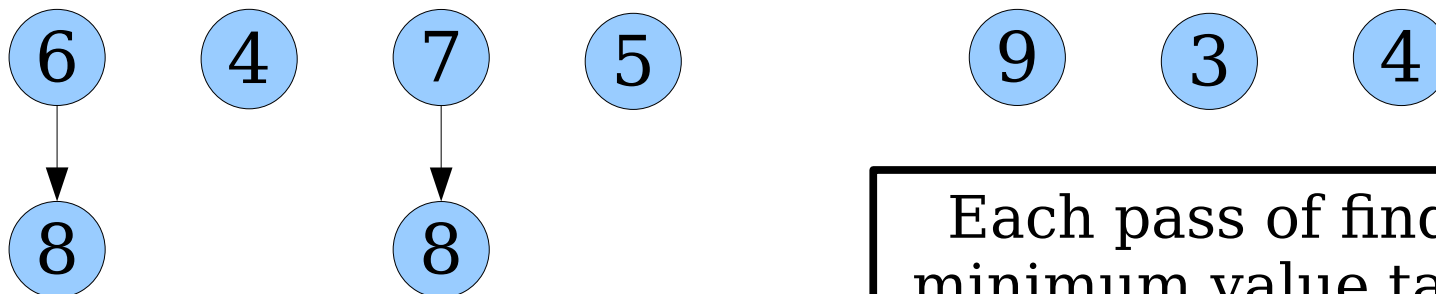
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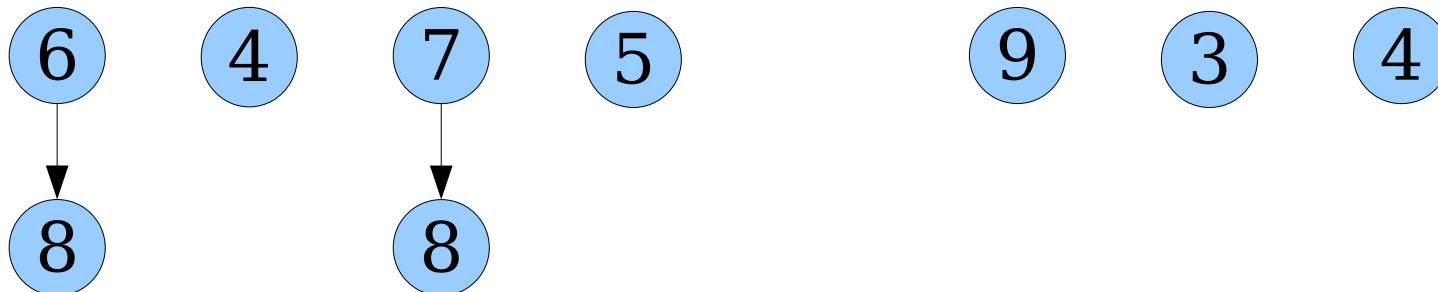
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Each pass of finding the minimum value takes time $\Theta(n)$ in the worst case. We've lost our nice runtime guarantees!

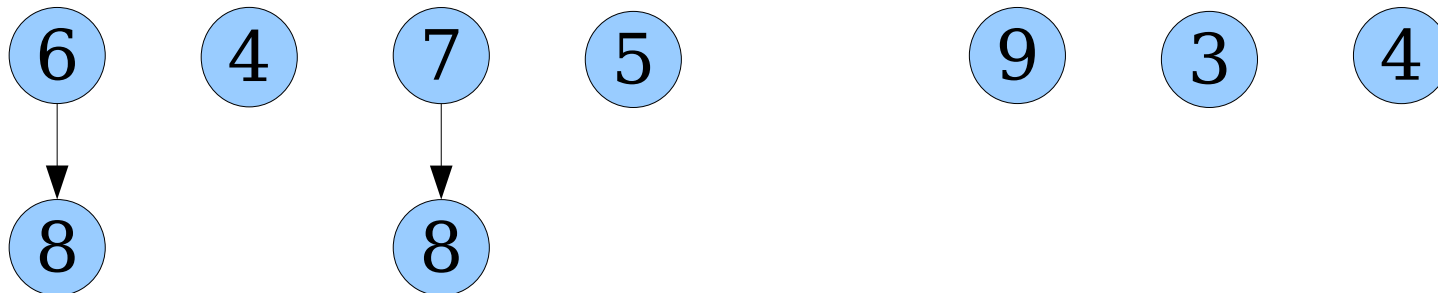
Washing the Dishes

- Every *meld* (and *enqueue*) creates some “dirty dishes” (small trees) that we need to clean up later.
- If we never clean them up, then our *extract-min* will be too slow to be usable.
- **Idea:** Change *extract-min* to “wash the dishes” and make things look nice and pretty again.
- **Question:** What does “wash the dishes” mean here?



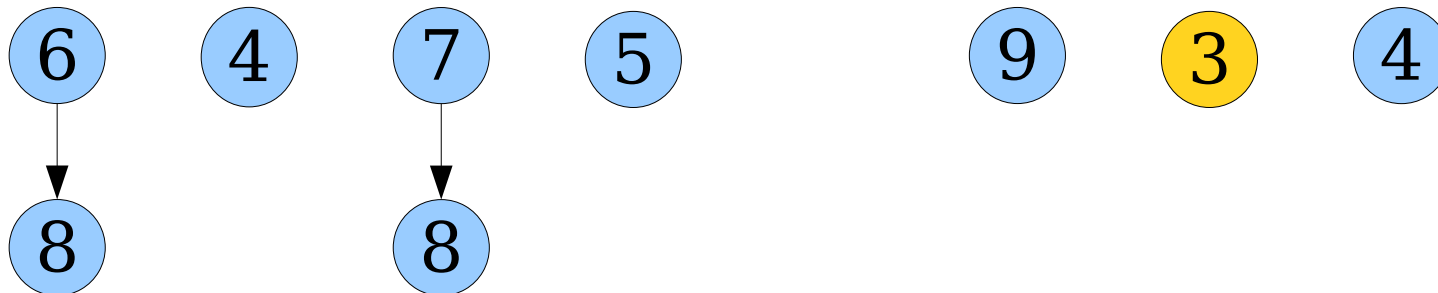
Washing the Dishes

- With our eager *meld* (and *enqueue*) strategy, our priority queue never had more than one tree of each order.
- This kept the number of trees low, which is why each operation was so fast.
- **Idea:** After doing an *extract-min*, do a *coalesce step* to ensure there's at most one tree of each order. This gets us to where we would be if we had been doing cleanup as we go.



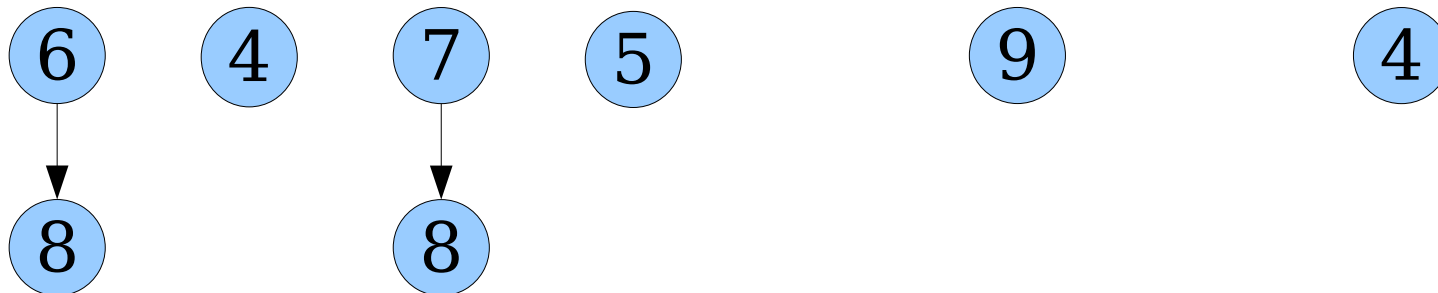
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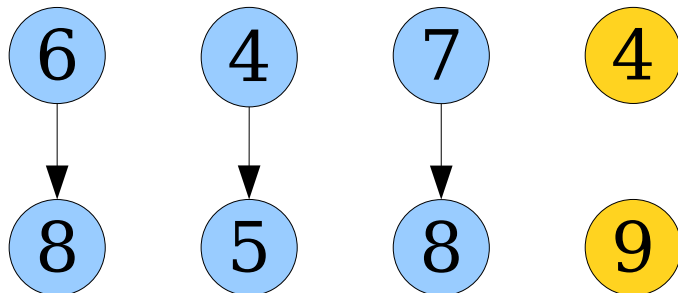
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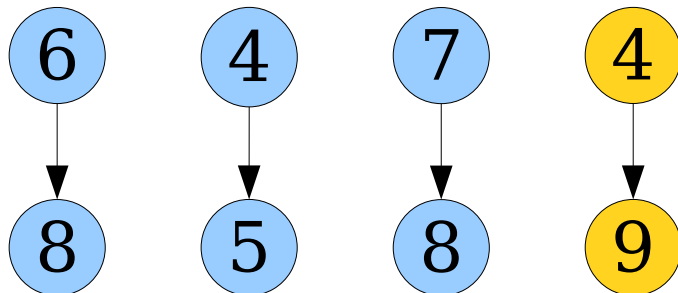
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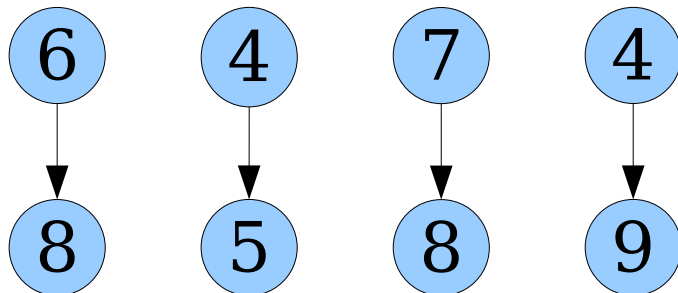
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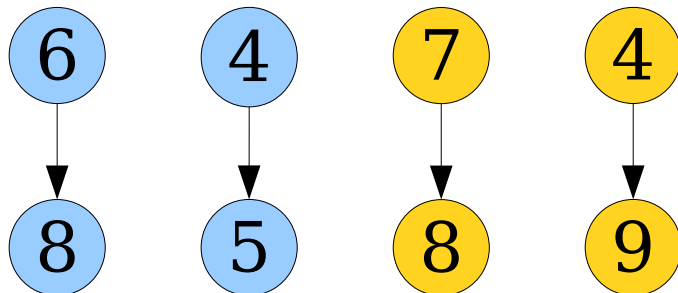
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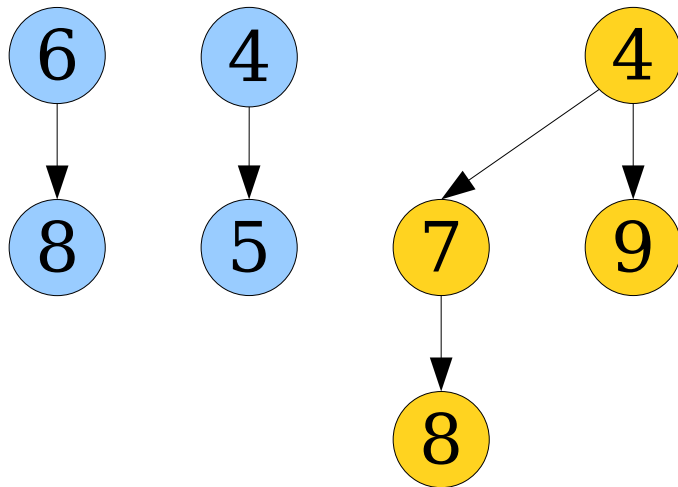
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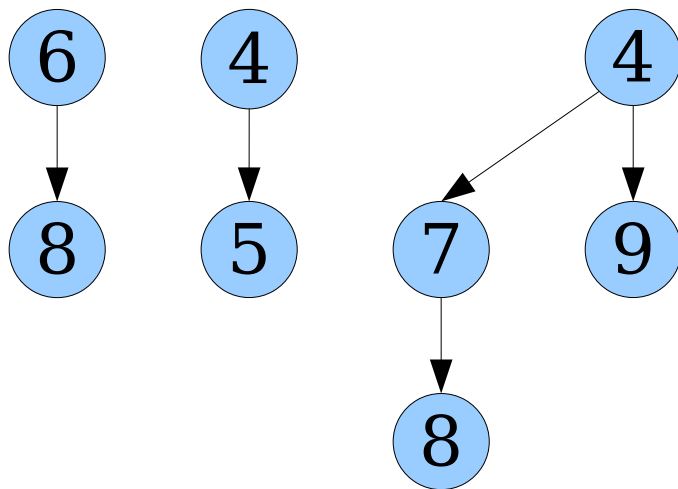
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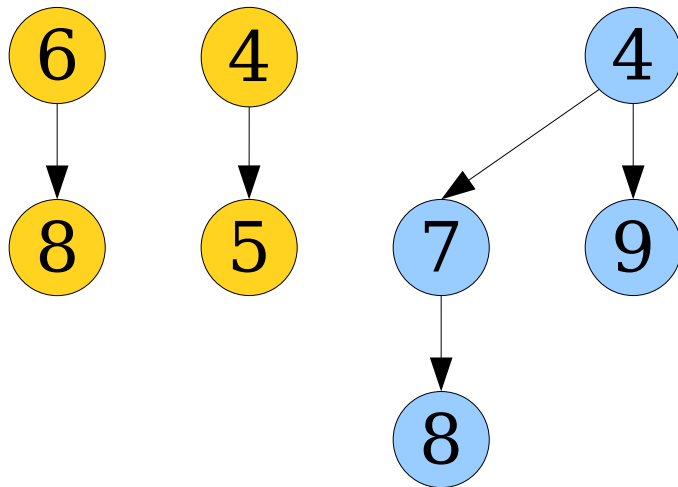
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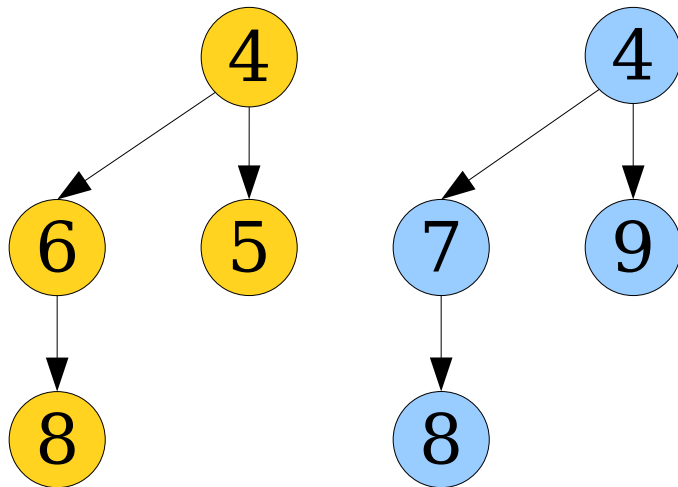
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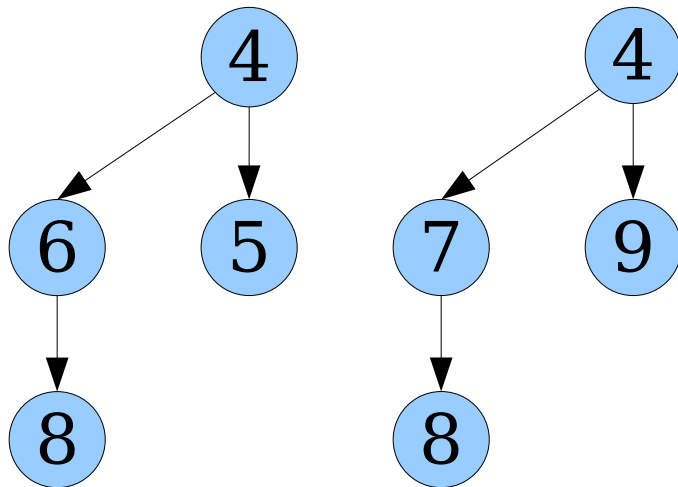
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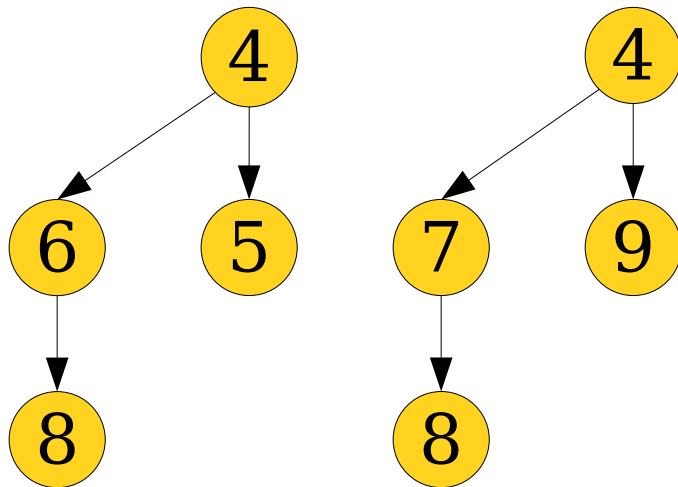
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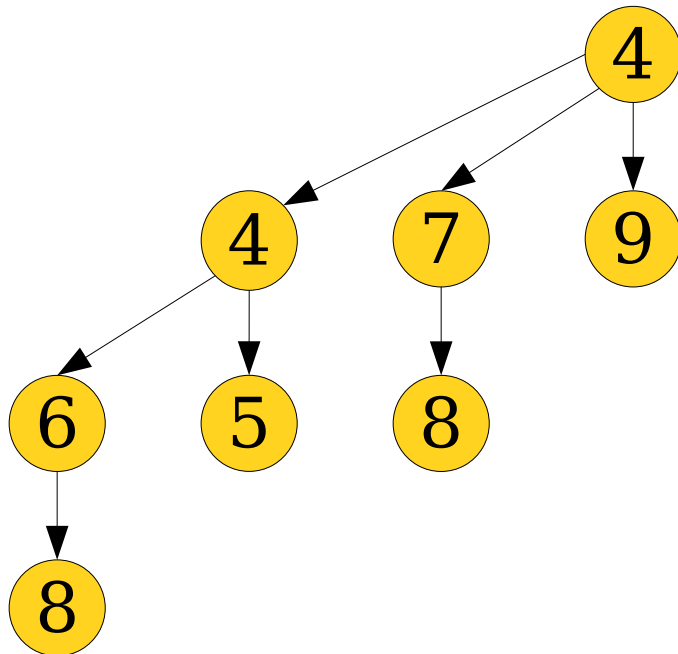
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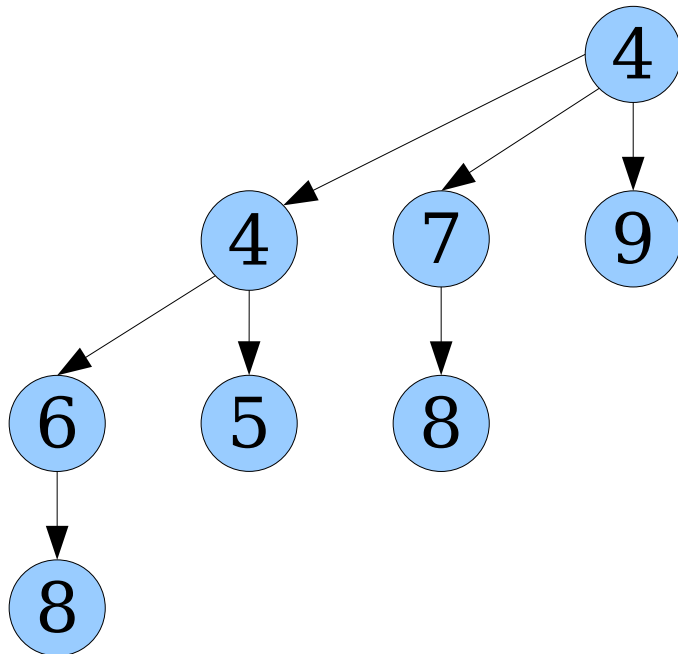
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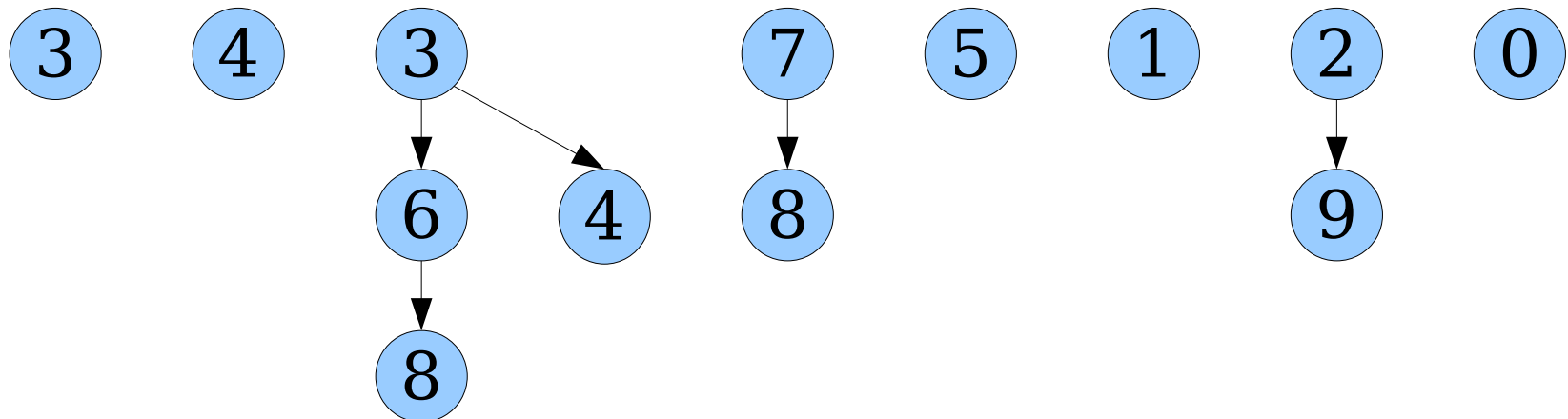
At this point, the mess is cleaned up, and we're left with what we would have had if we had been cleaning up as we go.

Where We're Going

- A **lazy binomial heap** is a binomial heap, modified as follows:
 - The **meld** operation is lazy. It just combines the two groups of trees together.
 - After doing an **extract-min**, we do a **coalesce** to combine together trees until there's at most one tree of each order.
- Intuitively, we'd expect this to amortize away nicely, since the "mess" left by **meld** gets cleaned up later on by a future **extract-min**.
- Questions left to answer:
 - How do we efficiently implement the **coalesce** operation?
 - How efficient is this approach, in an amortized sense?

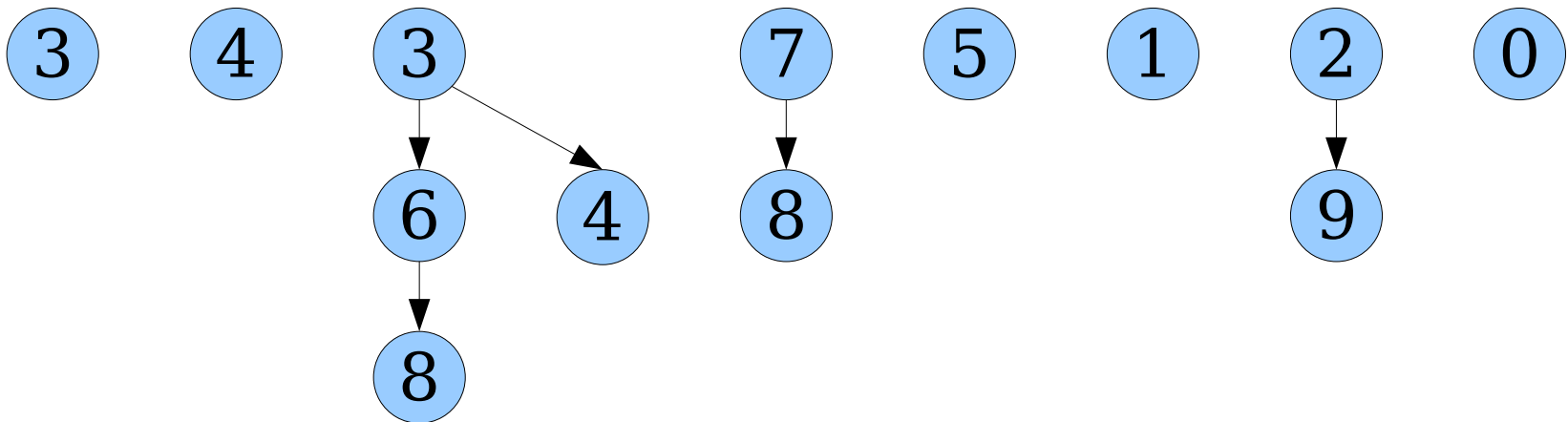
Coalescing Trees

- The *coalesce* step repeatedly combines trees together until there's at most one tree of each order.
- How do we implement this so that it runs quickly?



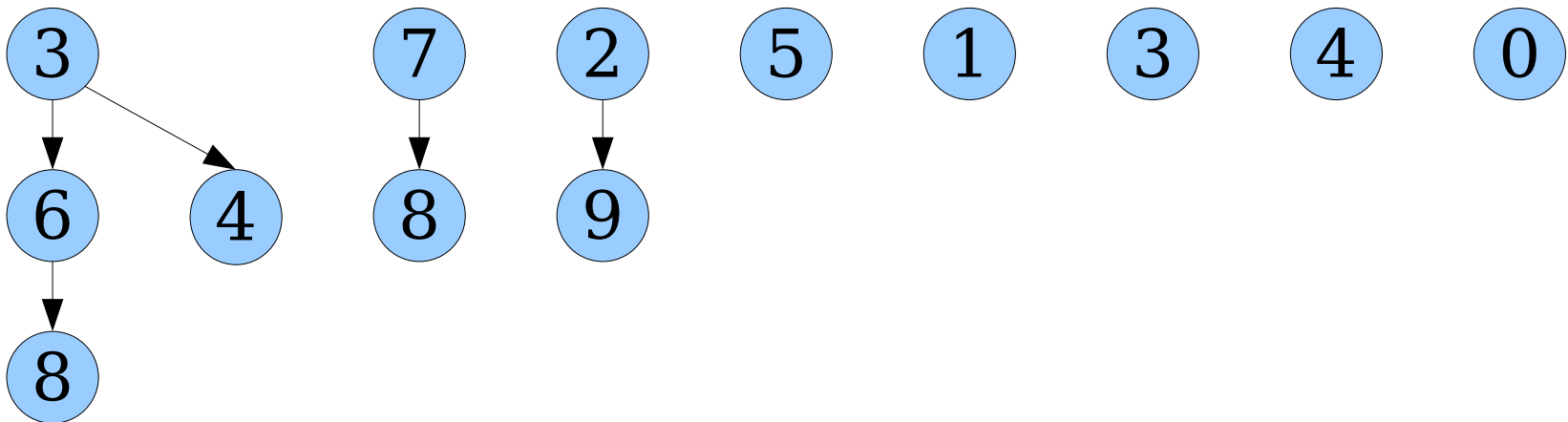
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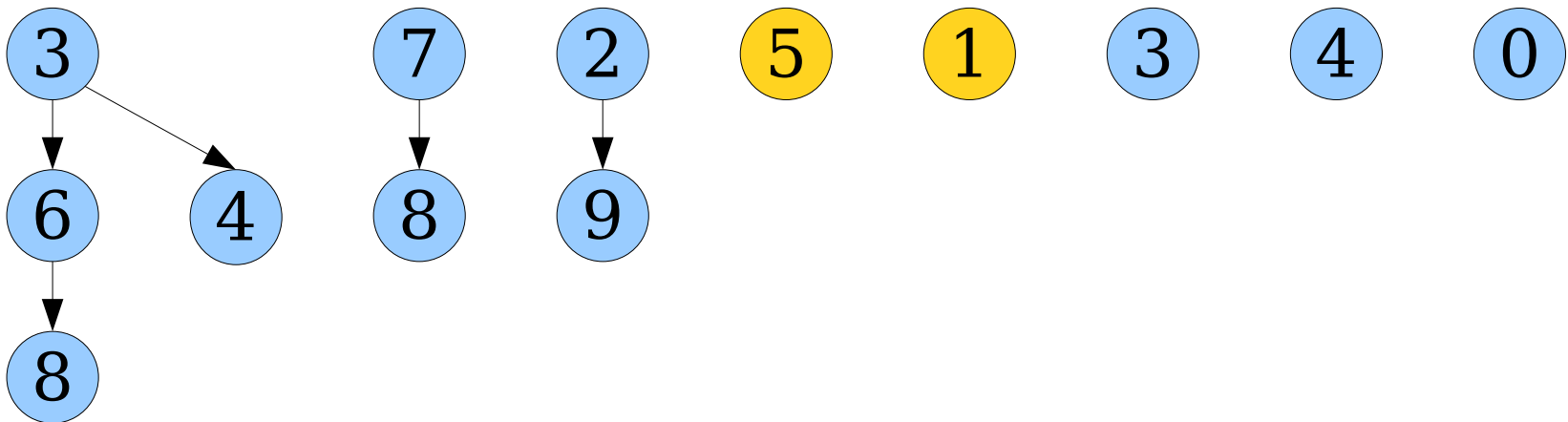
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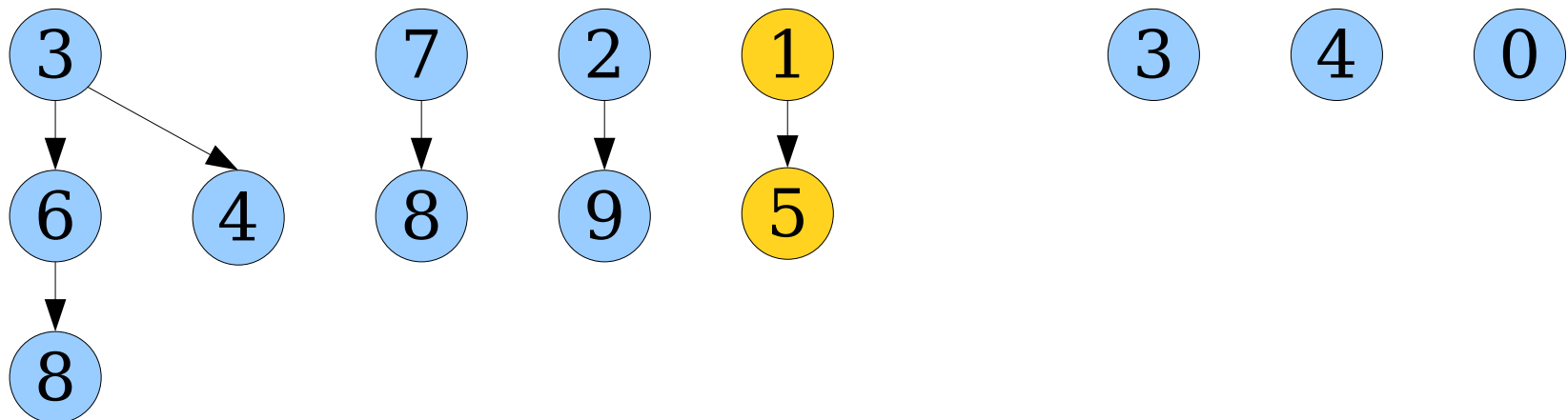
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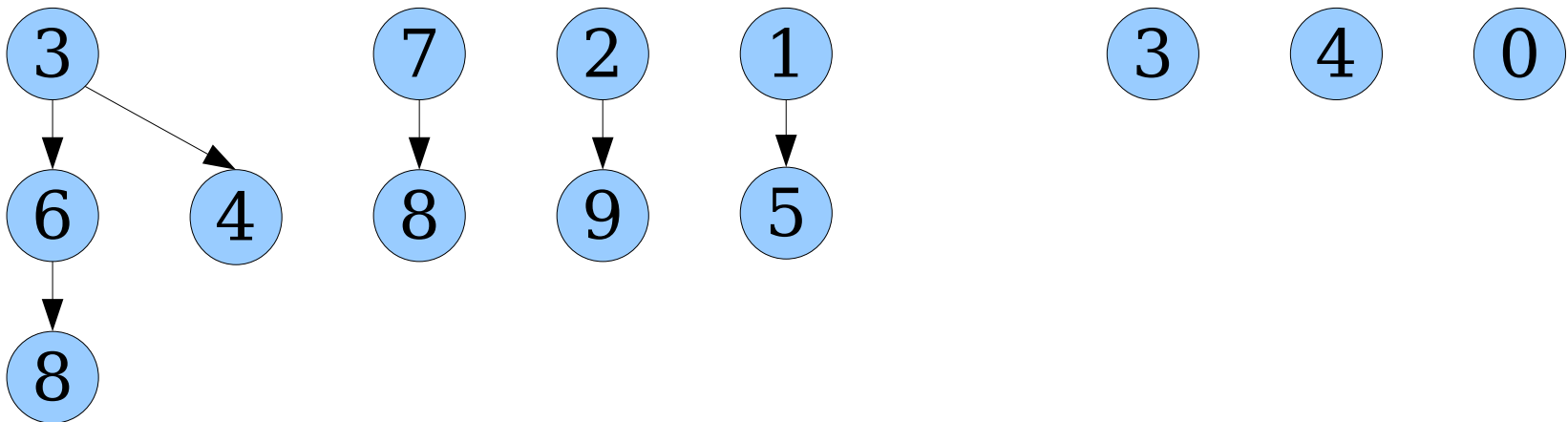
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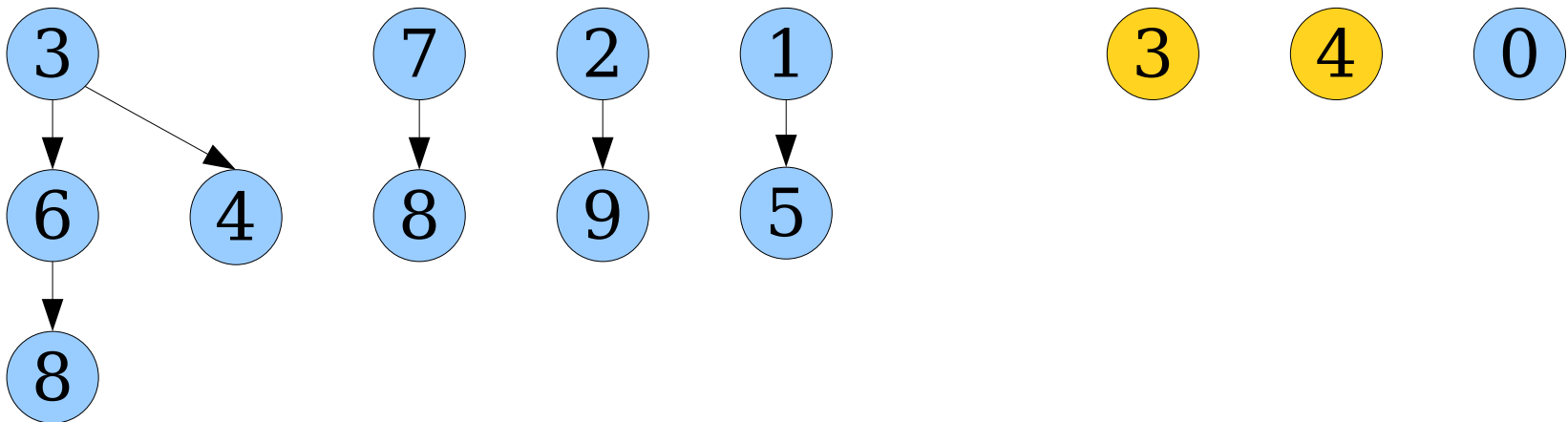
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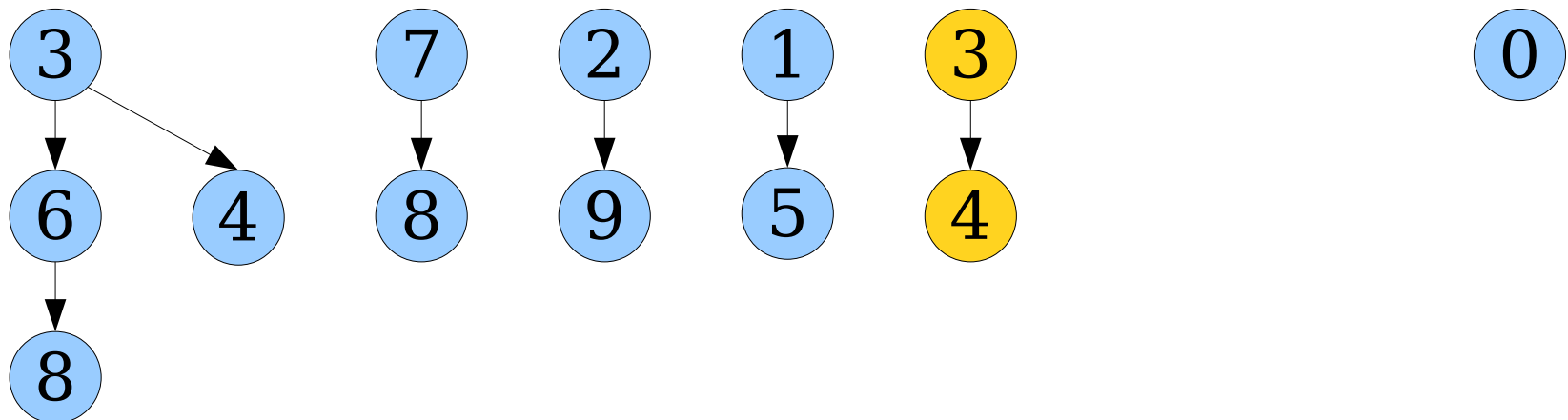
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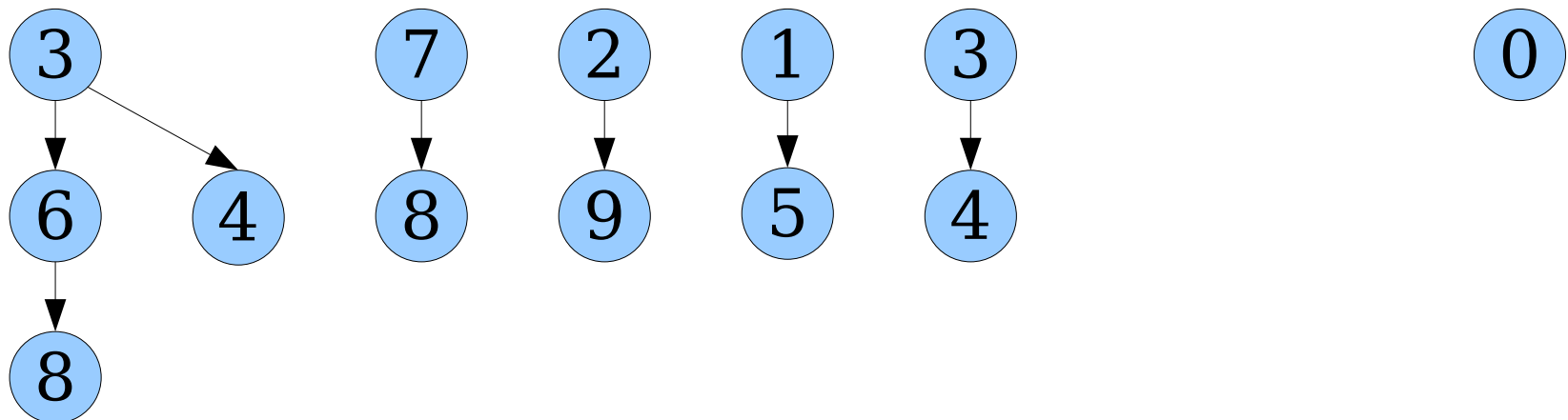
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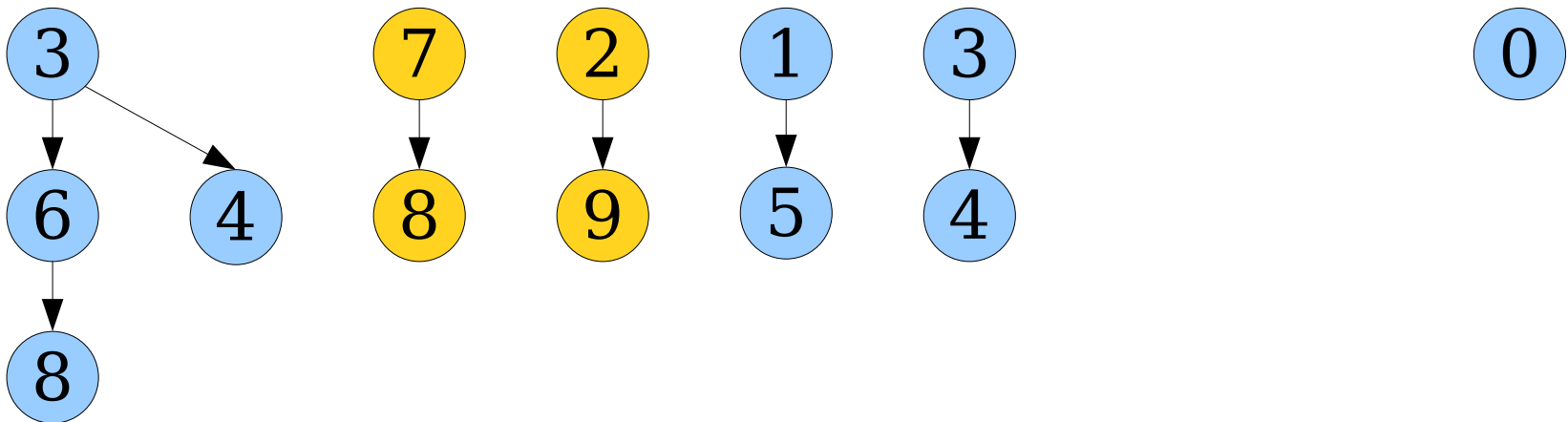
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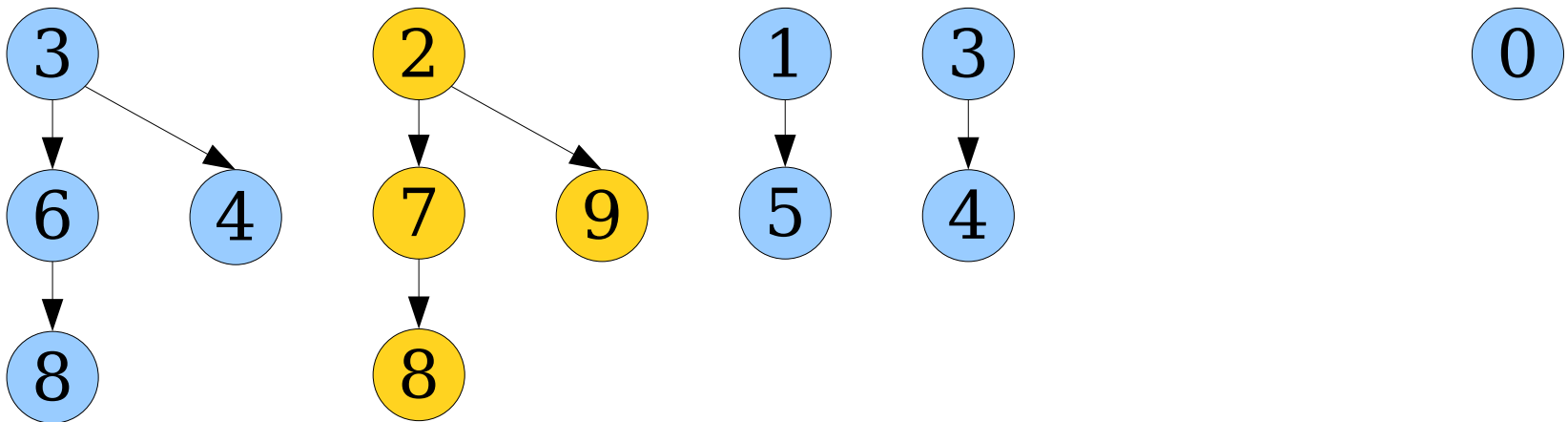
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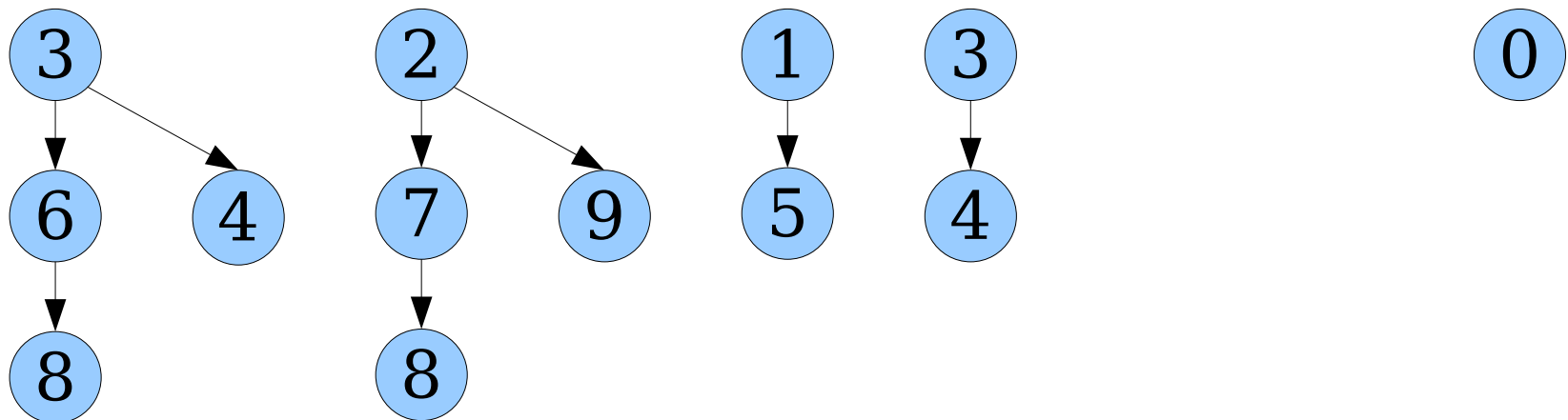
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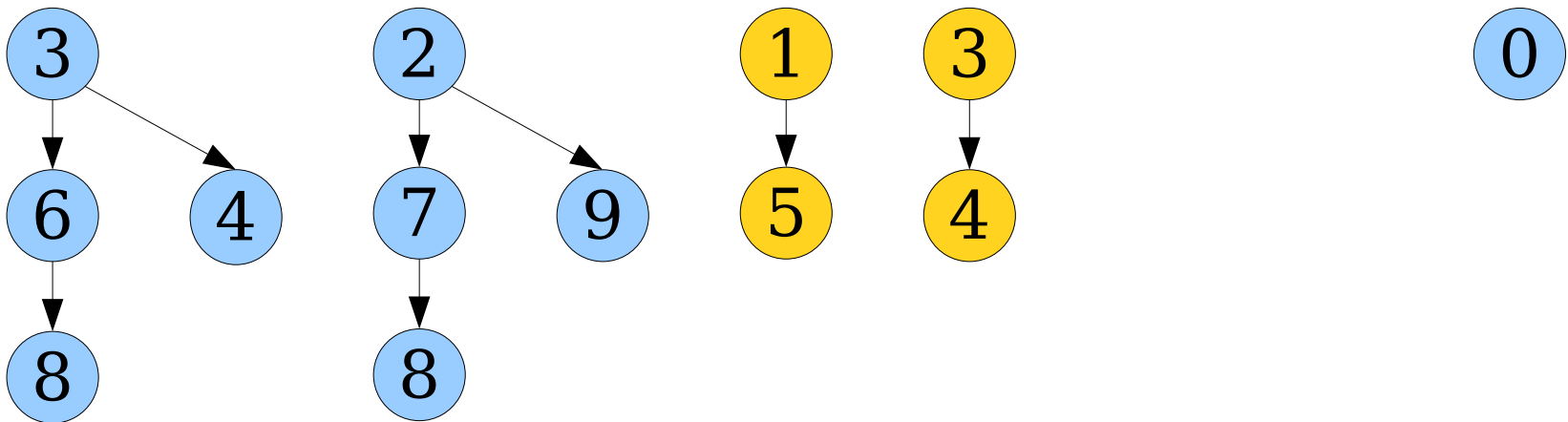
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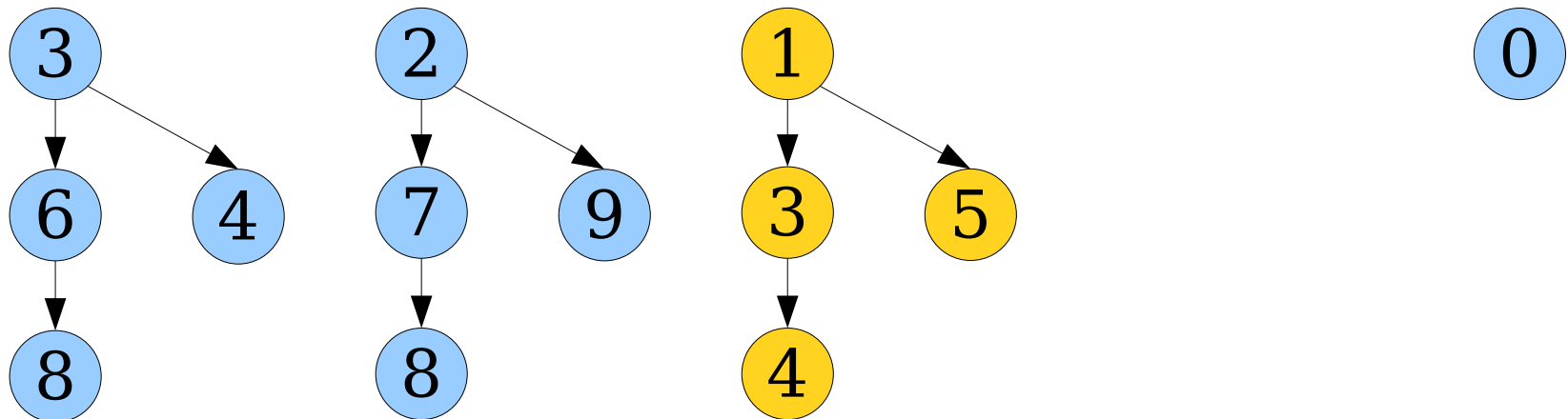
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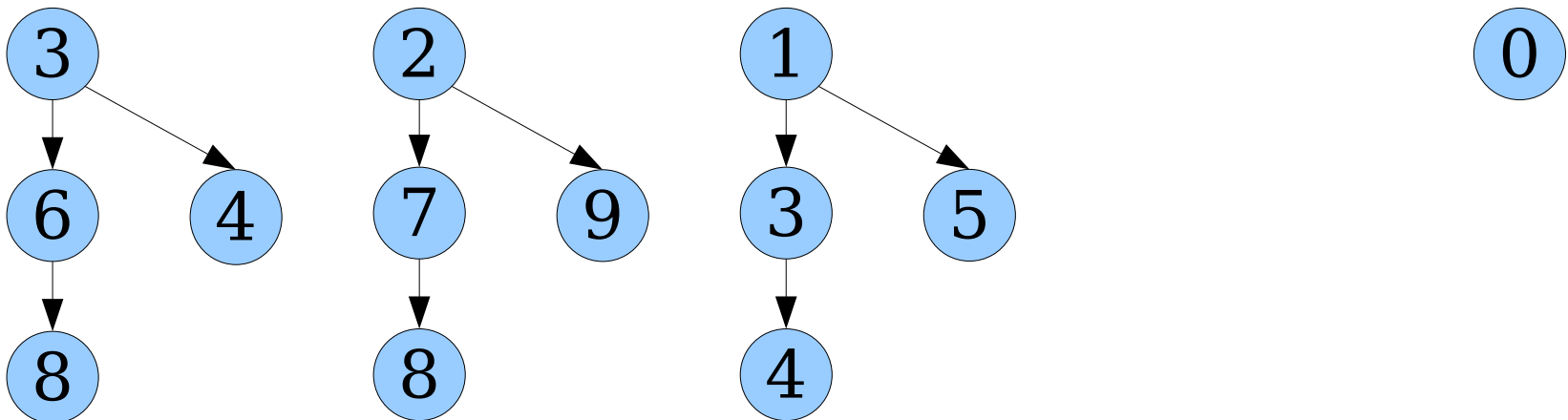
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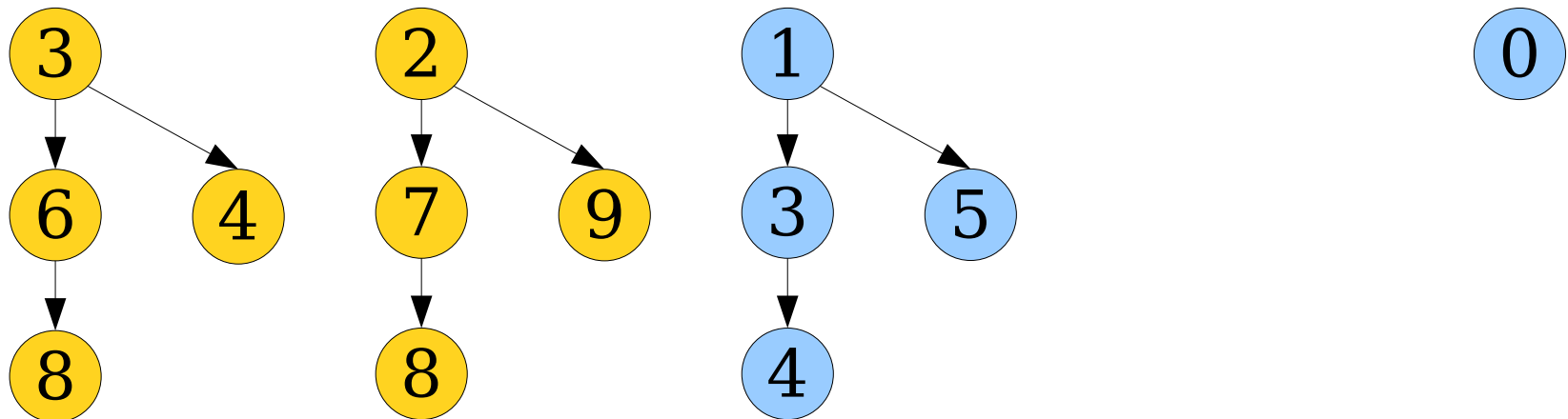
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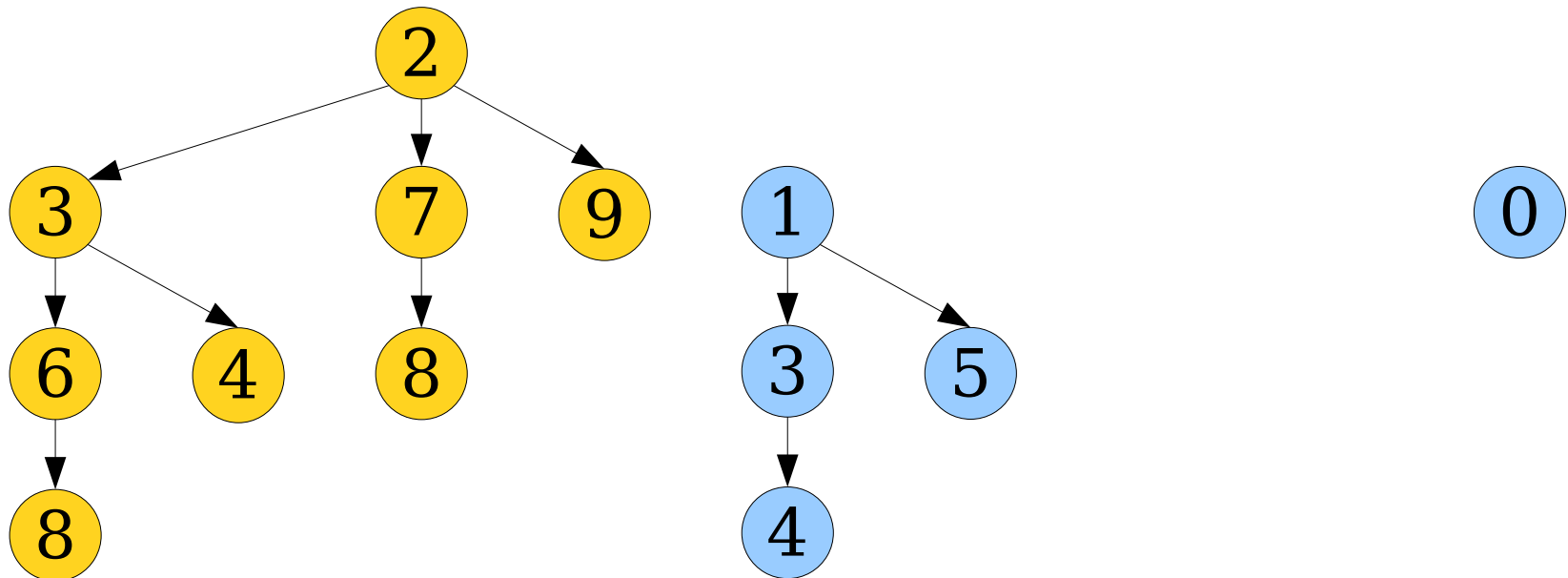
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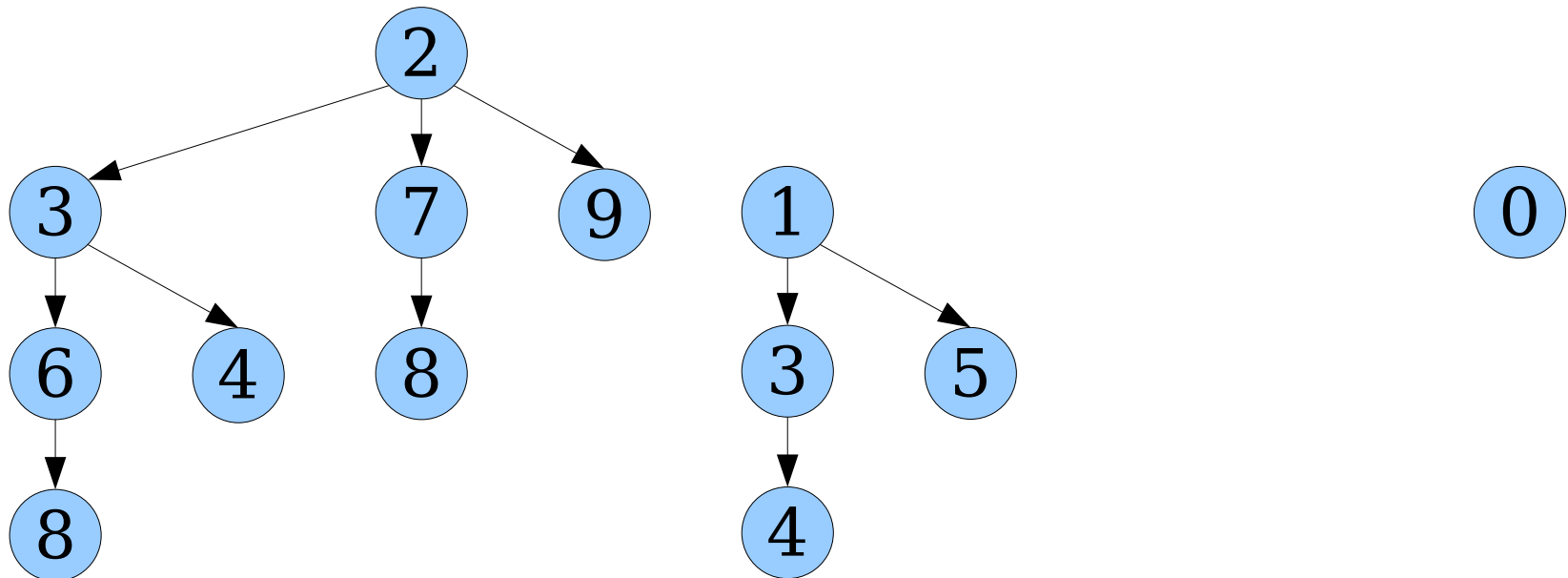
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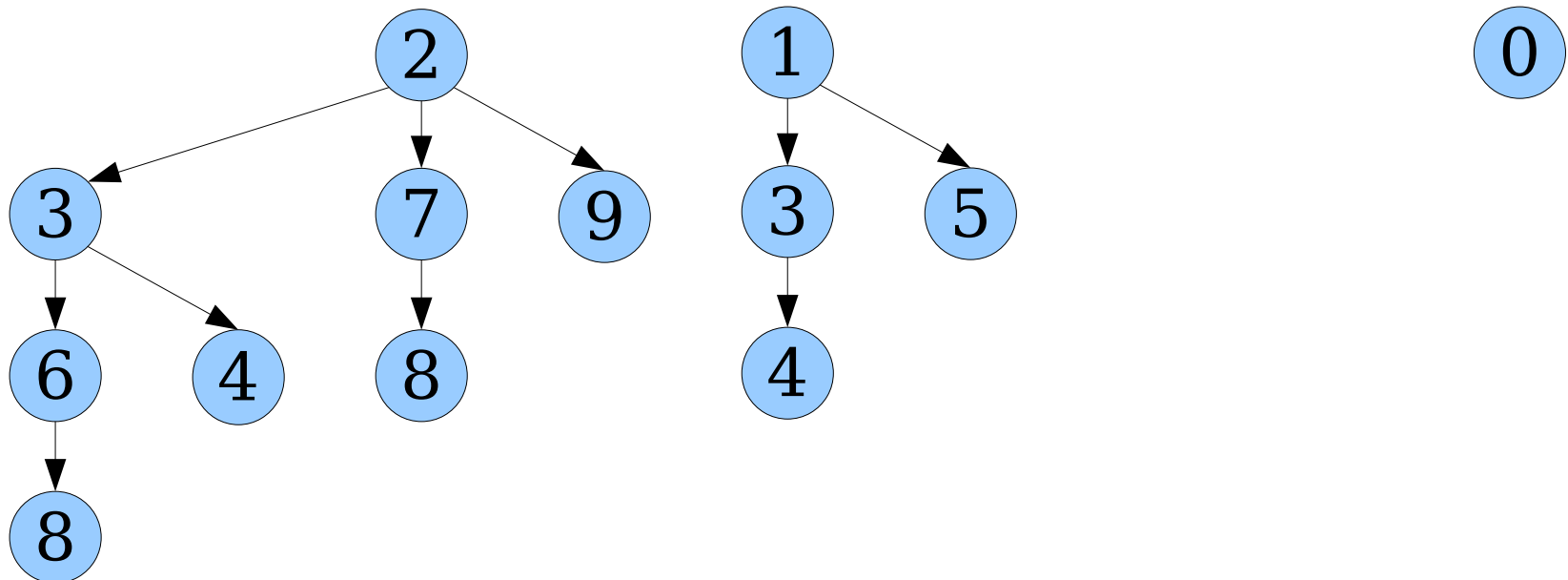
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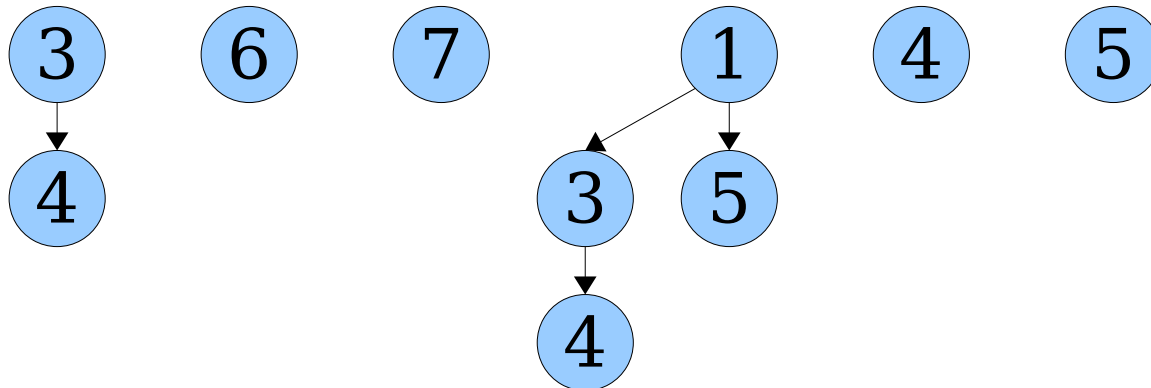


Coalescing Trees

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- We can sort our group of t trees by size in time $O(t \log t)$ using a standard sorting algorithm.
- **Better idea:** All the sizes are small integers. Use counting sort!

Coalescing Trees

- Here is a fast implementation of *coalesce*:
 - Distribute the trees into an array of buckets big enough to hold trees of orders $0, 1, 2, \dots, \lceil \log_2 (n + 1) \rceil$.
 - Start at bucket 0. While there's two or more trees in the bucket, fuse them and place the result one bucket higher.



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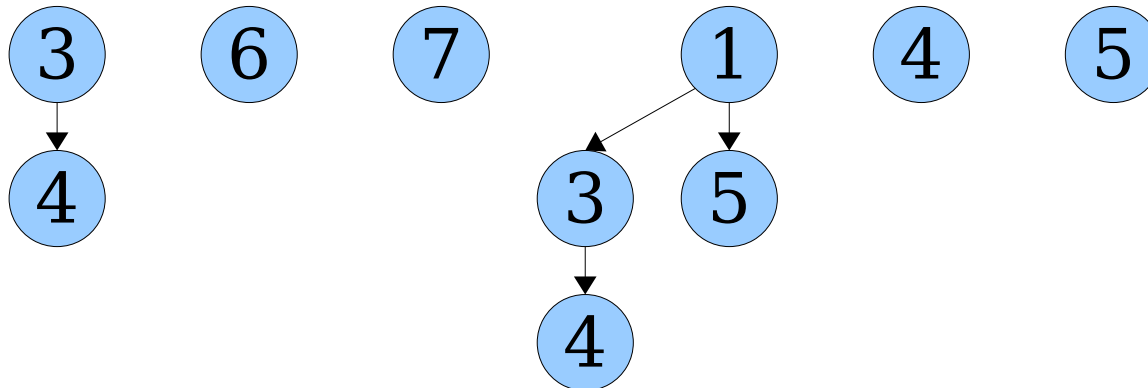
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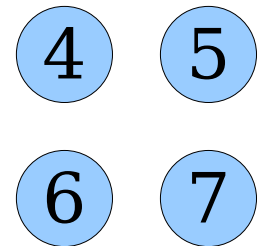
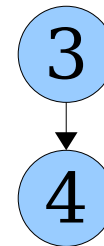
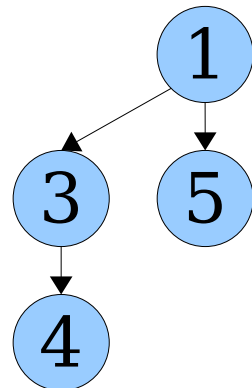
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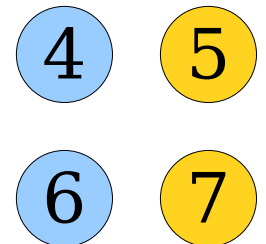
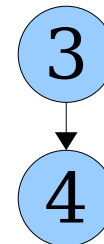
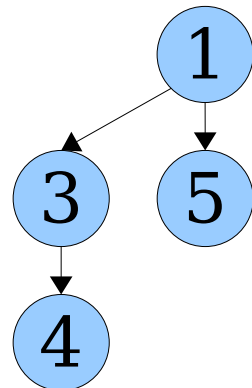
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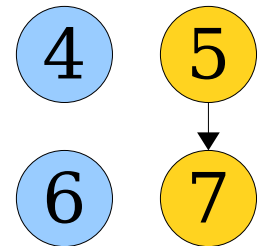
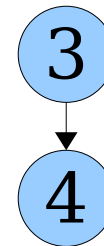
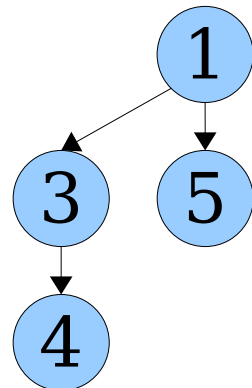
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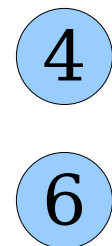
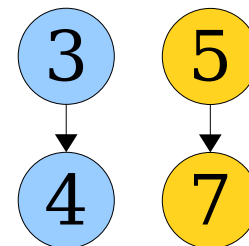
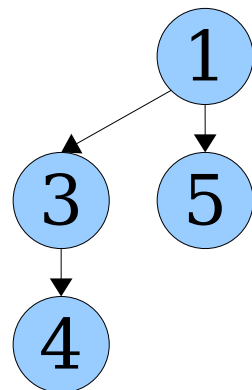
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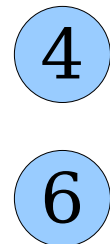
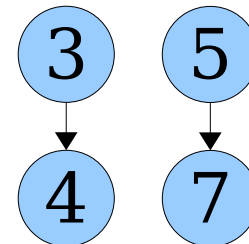
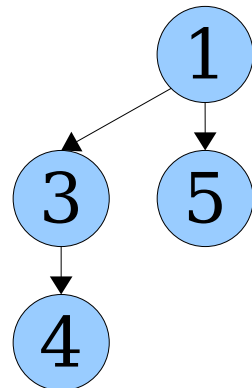
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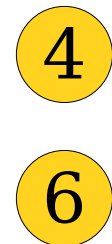
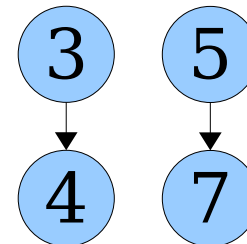
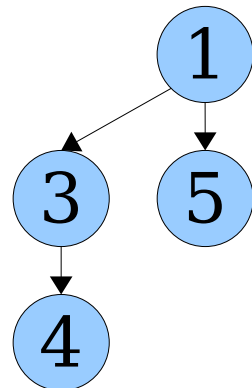
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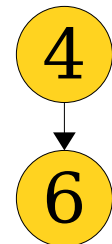
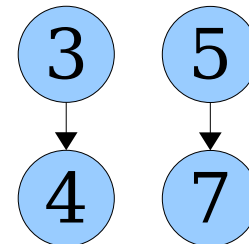
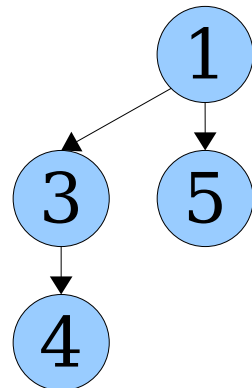
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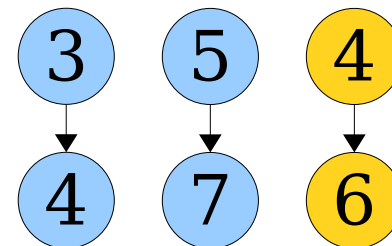
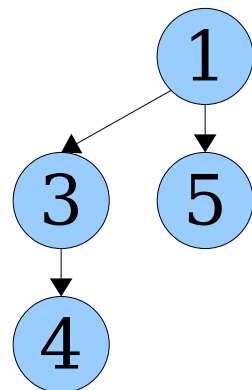
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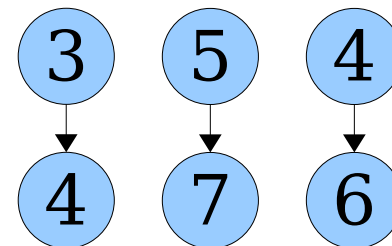
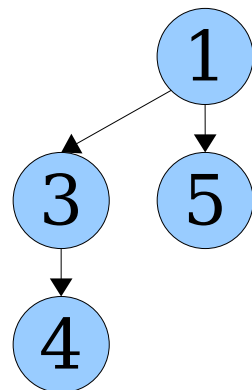
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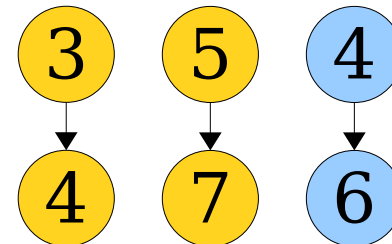
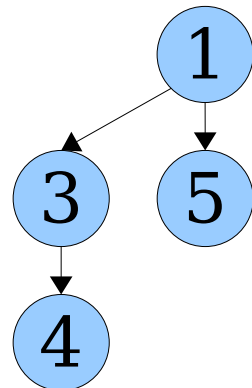
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Order 1

Order 0



Coalescing Trees

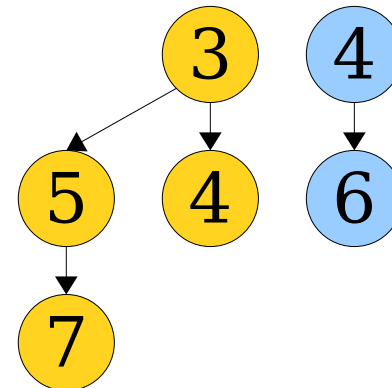
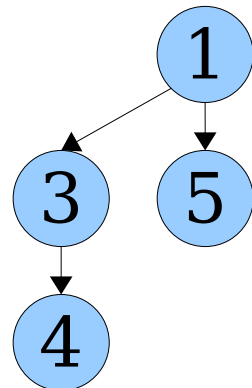
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 - Distribute the trees into an array of buckets big enough to hold trees of orders 0, 1, 2, ..., $\lceil \log_2 (n + 1) \rceil$.
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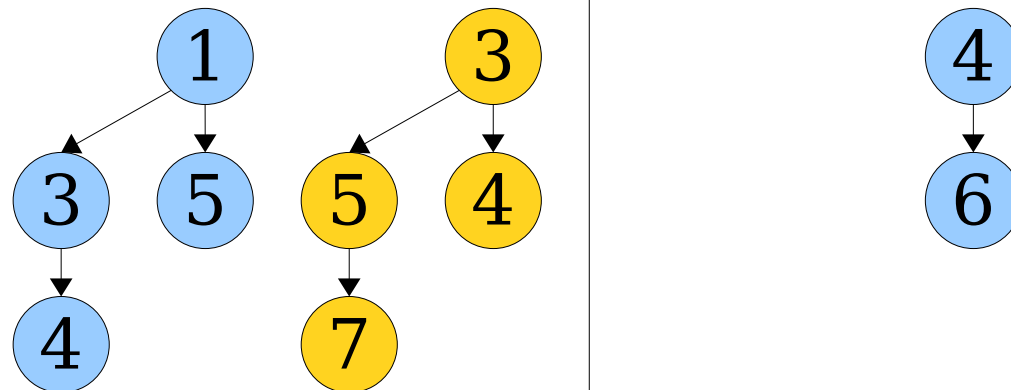
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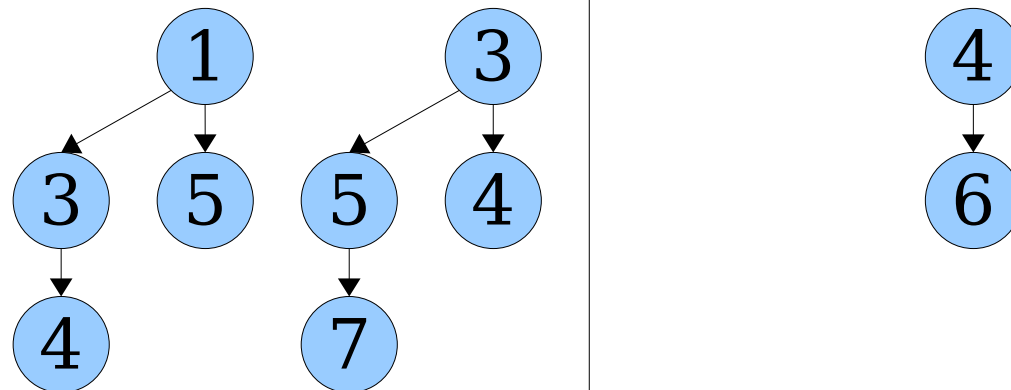
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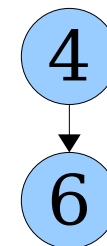
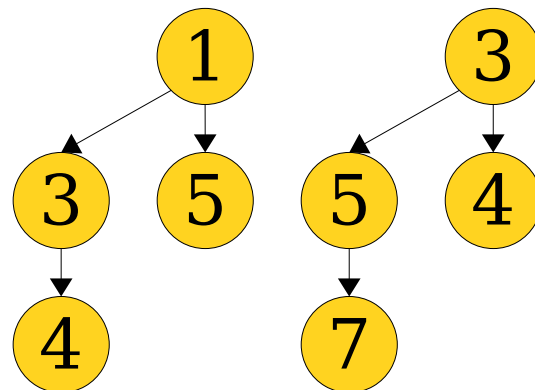
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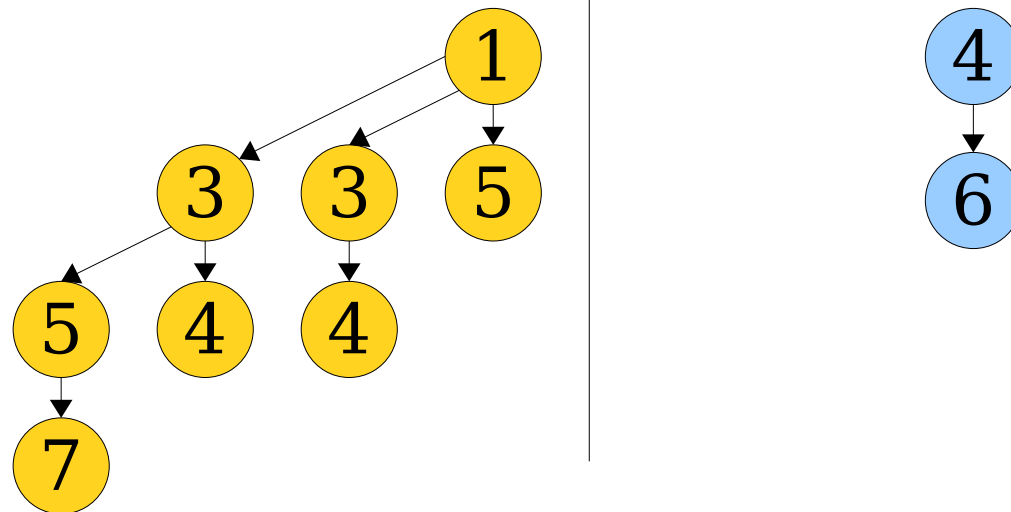
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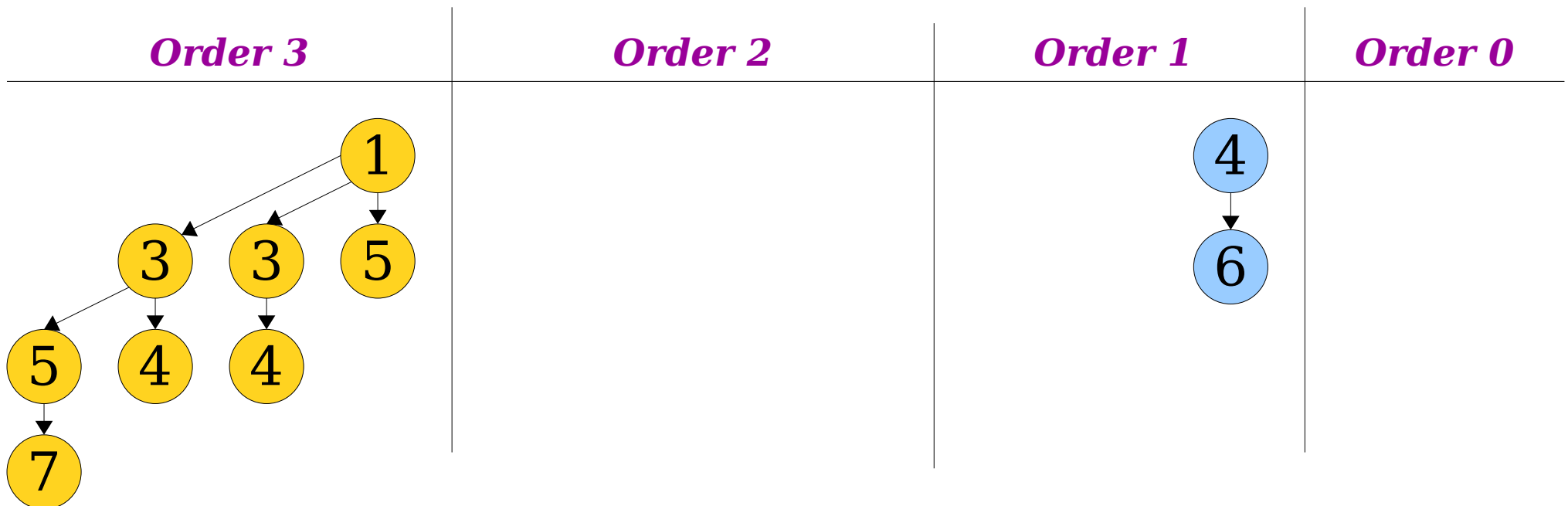
Order 1

Order 0



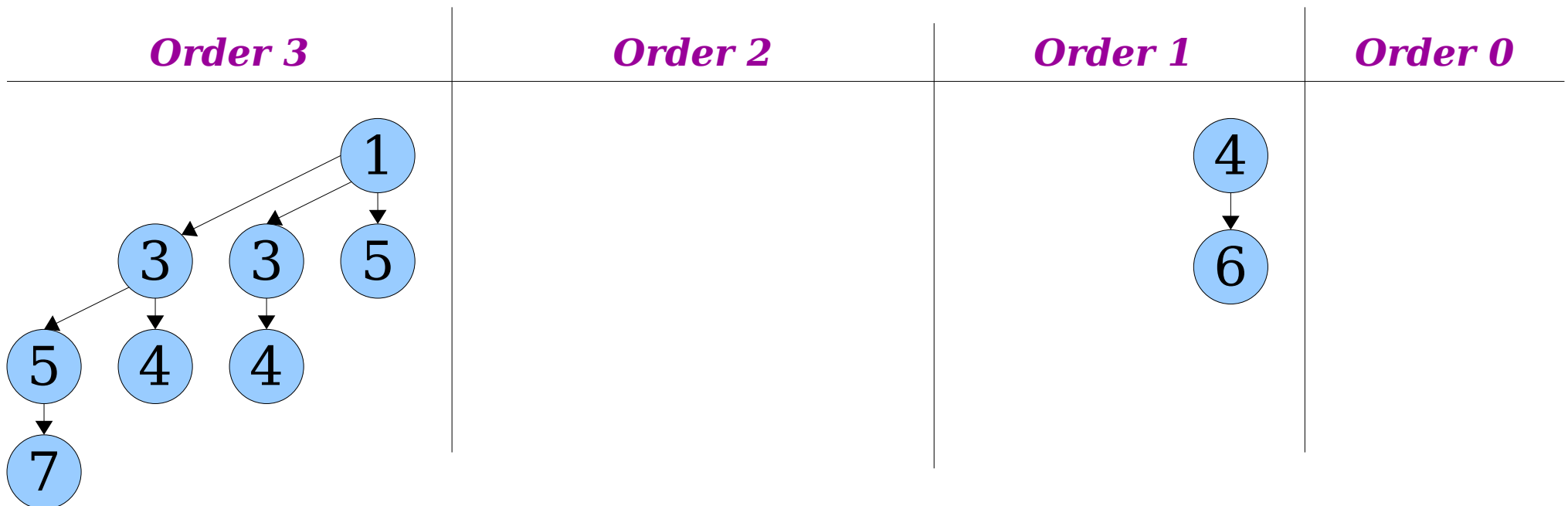
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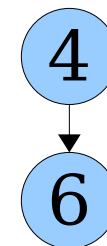
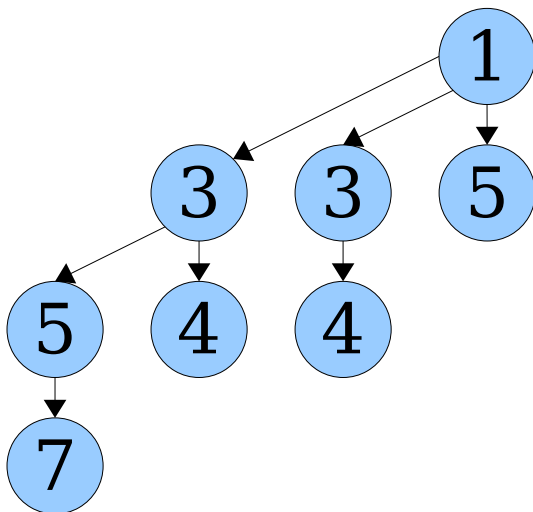
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Coalescing Trees

- Here is a fast implementation of *coalesce*:
 - Distribute the trees into an array of buckets big enough to hold trees of orders $0, 1, 2, \dots, \lceil \log_2 (n + 1) \rceil$.
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Analyzing Coalesce

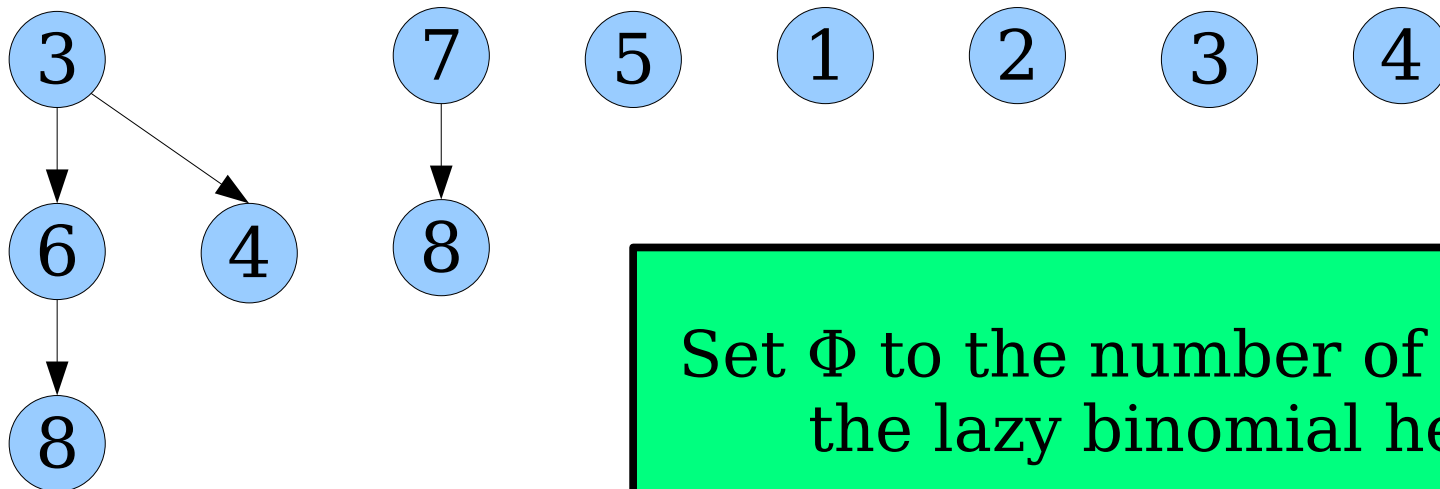
- **Claim:** Coalescing a group of t trees takes time $O(t + \log n)$.
 - Time to create the array of buckets: $O(\log n)$.
 - Time to distribute trees into buckets: $O(t)$.
 - Time to fuse trees: $O(t + \log n)$
 - Number of fuses is $O(t)$, since each fuse decreases the number of trees by one. Cost per fuse is $O(1)$.
 - Need to iterate across $O(\log n)$ buckets.
- Total work done: **$O(t + \log n)$** .
- In the worst case, this is $O(n)$.

The Story So Far

- A binomial heap with lazy melding has these worst-case time bounds:
 - ***enqueue***: $O(1)$
 - ***meld***: $O(1)$
 - ***find-min***: $O(1)$
 - ***extract-min***: $O(n)$.
- But these are *worst-case* time bounds. Intuitively, things should nicely amortize away.
 - The number of trees grows slowly (one per ***enqueue***).
 - The number of trees drops quickly (at most one tree per order) after an ***extract-min***).

An Amortized Analysis

- This is a great spot to use an amortized analysis by defining a potential function Φ .
- In each case, the idea is to clearly mark what “messes” we need to clean up.
- In our case, each tree is a “mess,” since our future *coalesce* operation has to clean it up.



Set Φ to the number of trees in the lazy binomial heap.

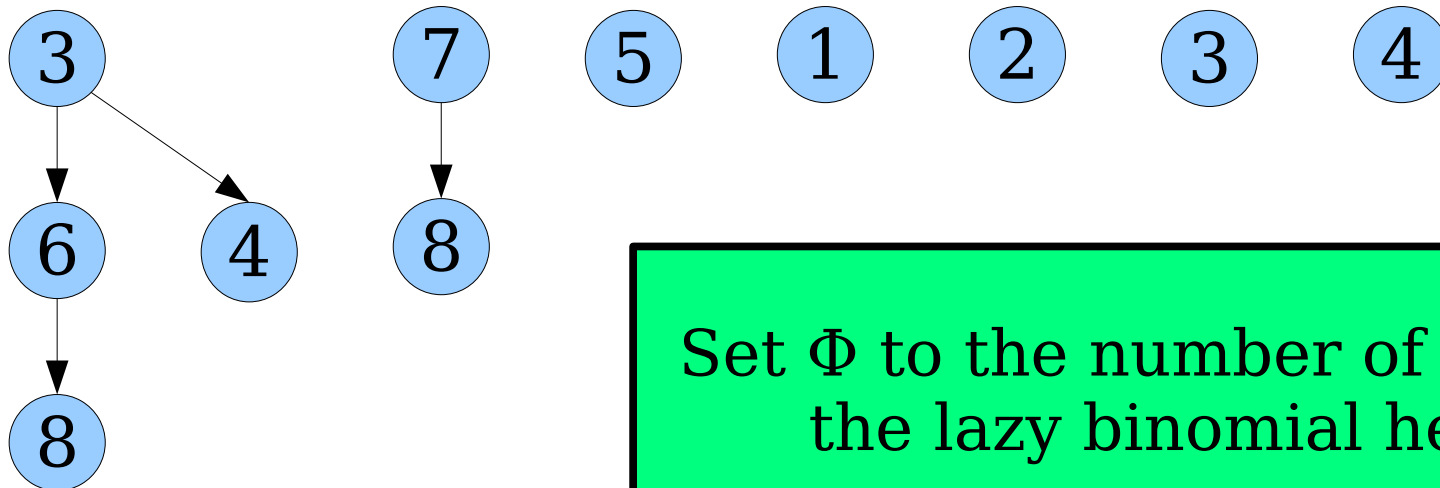
An Amortized Analysis

- **Recall:** We assign amortized costs as

$$\text{amortized-cost} = \text{real-cost} + O(1) \cdot \Delta\Phi,$$

where $\Delta\Phi = \Phi_{\text{after}} - \Phi_{\text{before}}$.

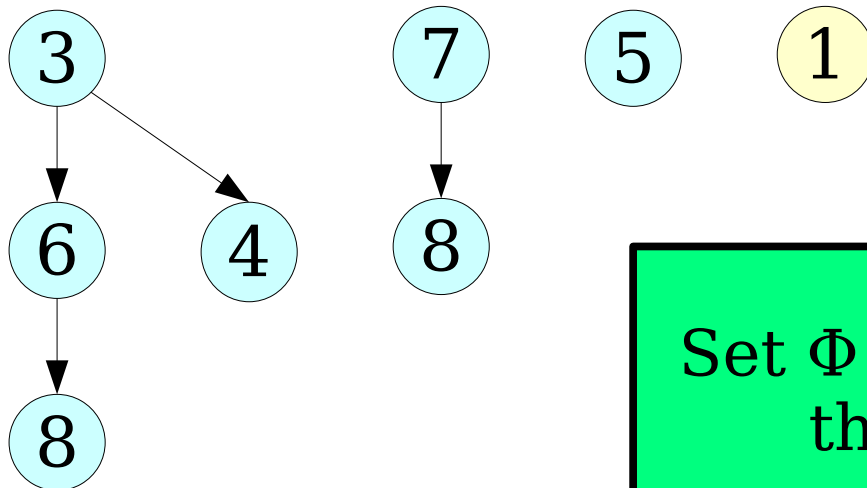
- Increasing Φ (adding more trees) artificially boosts costs.
- Decreasing Φ (removing trees) artificially lowers costs.
- Let's work out the amortized costs of each operation on a lazy binomial heap.



Set Φ to the number of trees in the lazy binomial heap.

Analyzing an Insertion

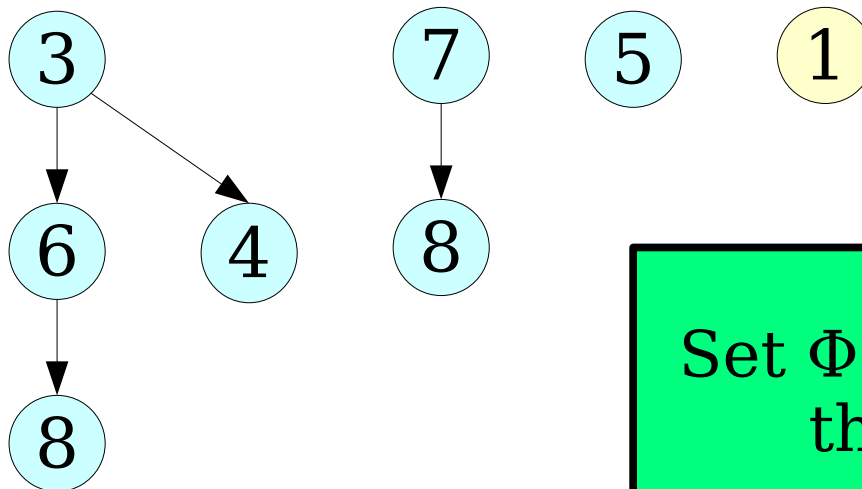
- To *enqueue* a key, we add a new binomial tree to the forest.



Set Φ to the number of trees in the lazy binomial heap.

Analyzing an Insertion

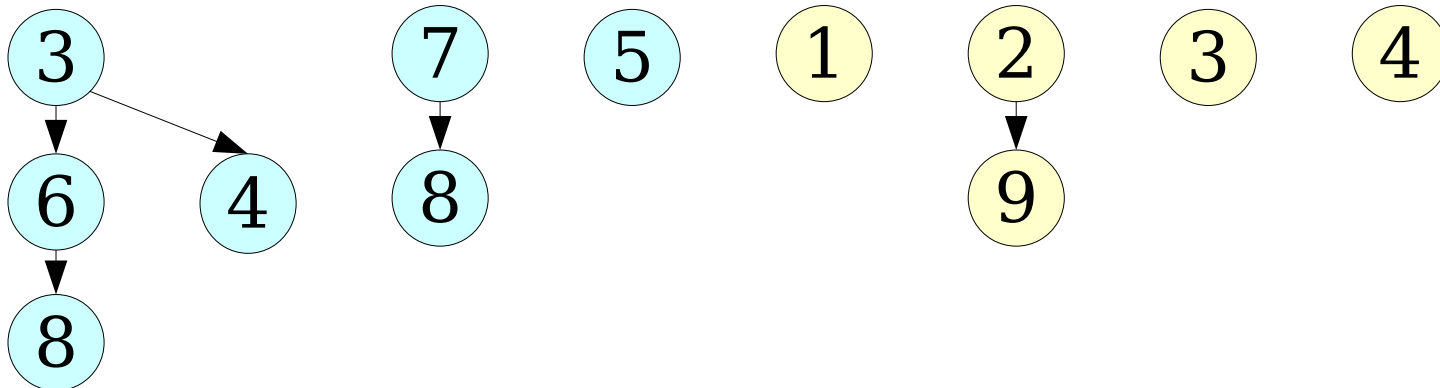
- To *enqueue* a key, we add a new binomial tree to the forest.
- Actual time: $O(1)$. $\Delta\Phi$: $+1$
- Amortized cost: **$O(1)$** .



Set Φ to the number of trees in the lazy binomial heap.

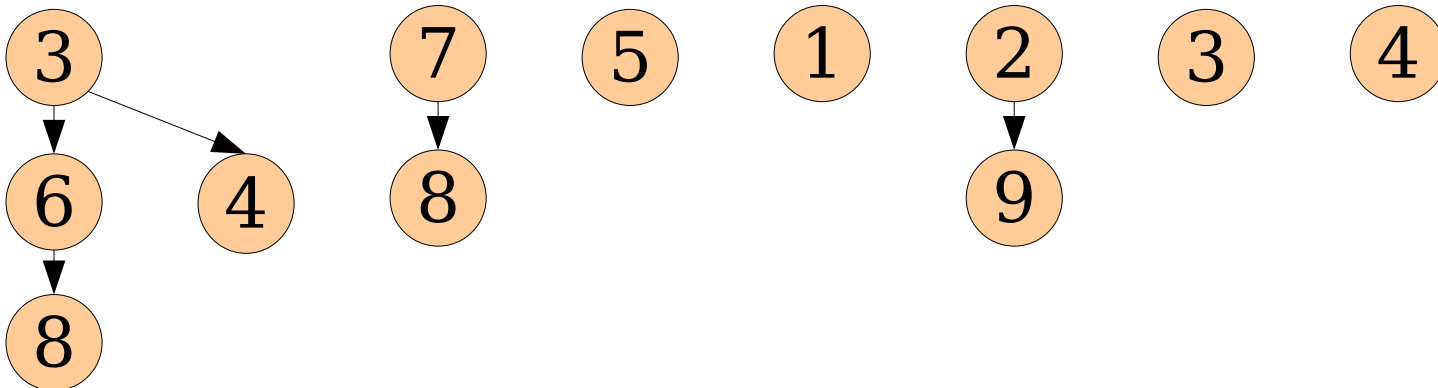
Analyzing a Meld

- Suppose that we *meld* two lazy binomial heaps B_1 and B_2 . Actual cost: $O(1)$.



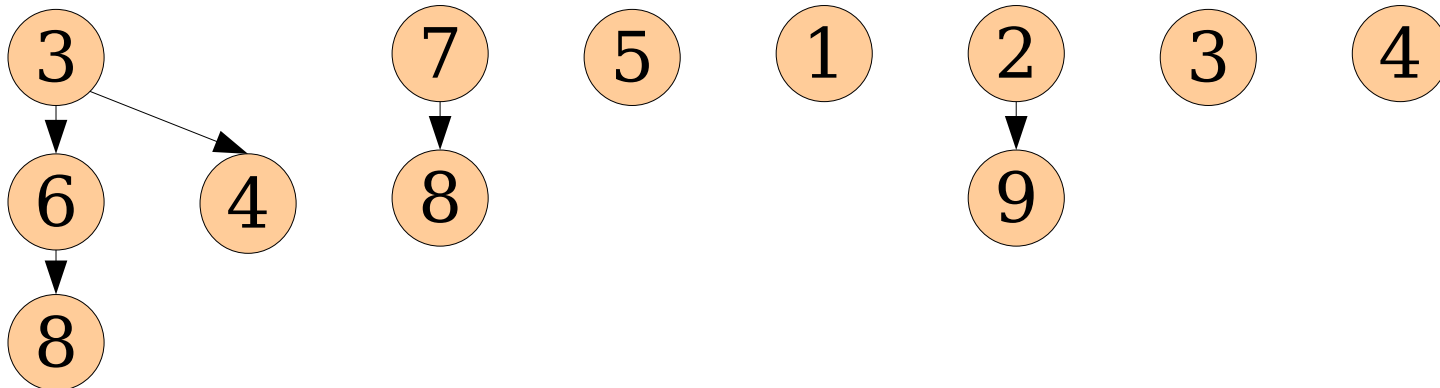
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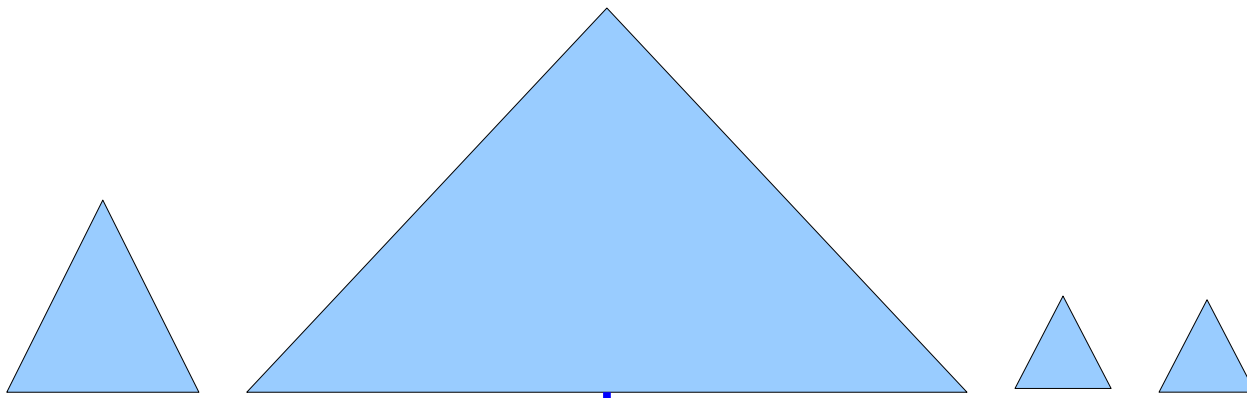


Analyzing a Meld

- Suppose that we *meld* two lazy binomial heaps B_1 and B_2 . Actual cost: $O(1)$.
- We have the same number of trees before and after we do this, so $\Delta\Phi = 0$.
- Amortized cost: **$O(1)$** .

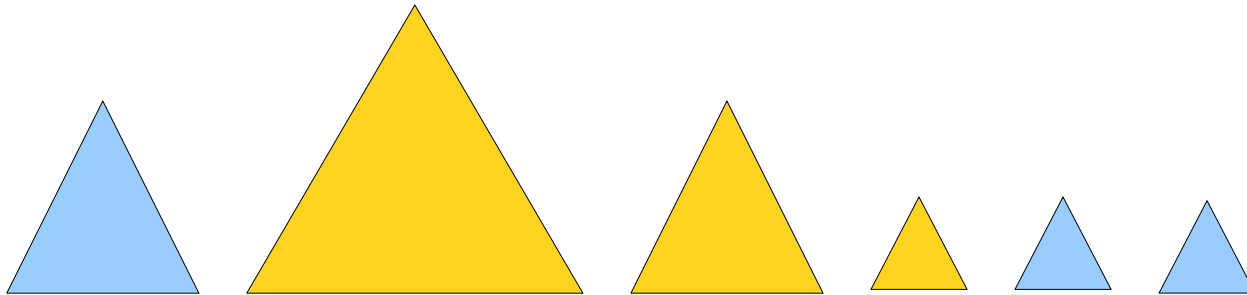


Analyzing *extract-min*



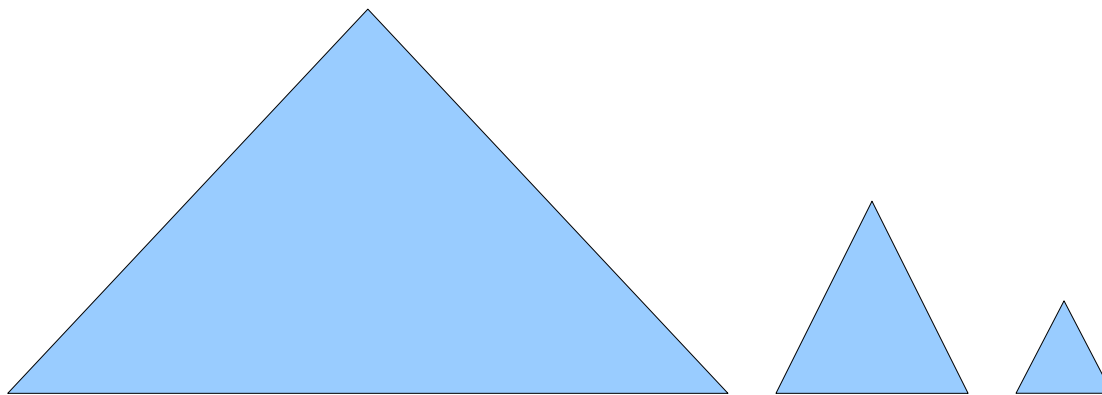
Find tree with minimum key.

Work: $O(t)$
 $\Phi = t$



*Remove min.
Add children to
list of trees.*

Work: $O(\log n)$

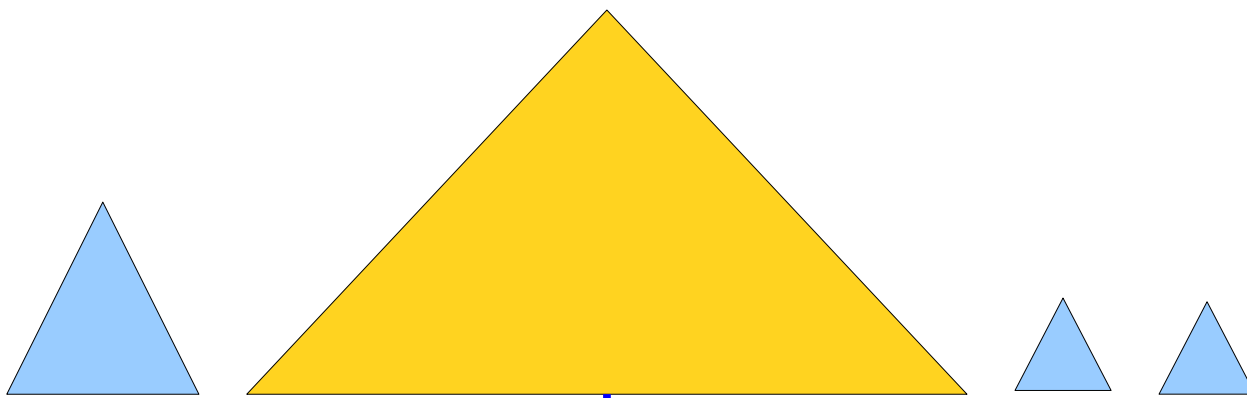


*Run the coalesce
algorithm.*

Work: $O(t + \log n)$
 $\Phi = O(\log n)$

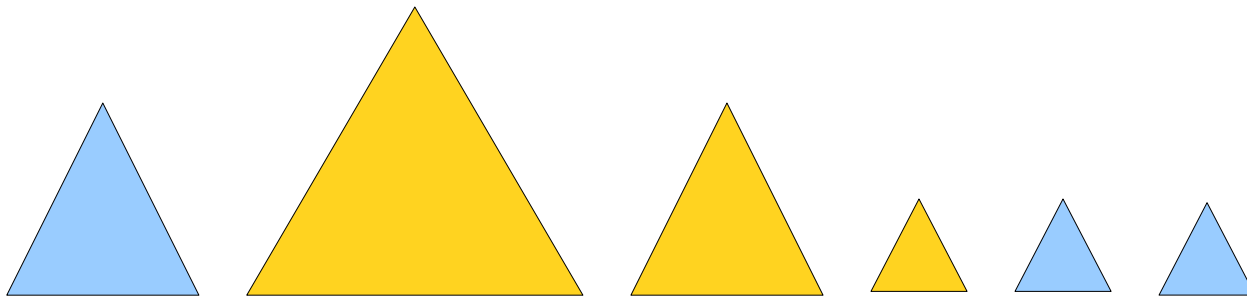
Work: $O(t + \log n)$

$\Delta\Phi: O(-t + \log n)$



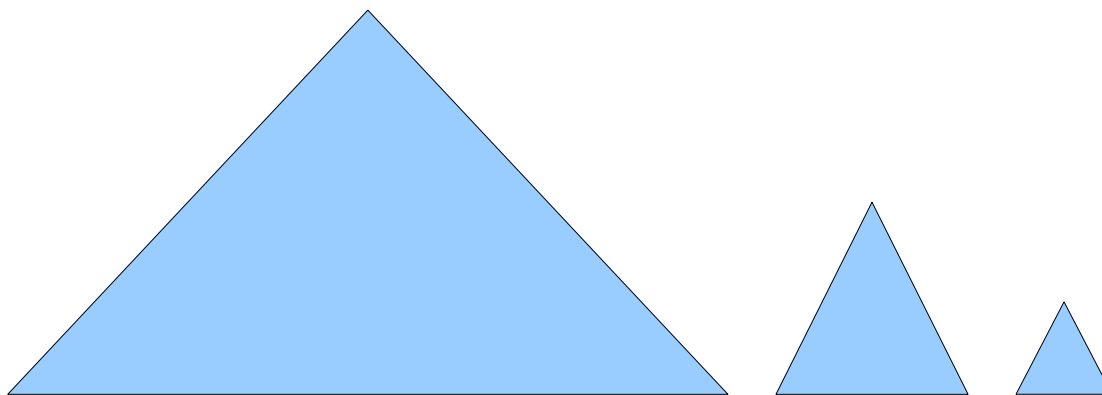
Find tree with minimum key.

Work: $O(t)$
 $\Phi = t$



*Remove min.
Add children to list of trees.*

Work: $O(\log n)$



Run the coalesce algorithm.

Work: $O(t + \log n)$
 $\Phi = O(\log n)$

Amortized cost: **$O(\log n)$** .

Analyzing Extract-Min

- Suppose we perform an **extract-min** on a binomial heap with t trees in it.
- Initially, we expose the children of the minimum element. This increases the number of trees to $t + O(\log n)$.
- The runtime for coalescing these trees is $O(t + \log n)$.
- When we're done merging, there will be $O(\log n)$ trees remaining, so $\Delta\Phi = -t + O(\log n)$.
- Amortized cost is

$$\begin{aligned} & O(t + \log n) + O(1) \cdot (-t + O(\log n)) \\ &= O(t) - O(1) \cdot t + O(1) \cdot O(\log n) \\ &= O(\log n). \end{aligned}$$

The Final Scorecard

- Here's the final scorecard for our lazy binomial heap.
- These are *great* runtimes! We can't improve upon this except by making ***extract-min*** worst-case efficient.
 - This is possible! Check out ***bootstrapped skew binomial heaps*** or ***strict Fibonacci heaps*** for details!

Lazy Binomial Heap

- ***Insert***: $O(1)$
- ***Find-Min***: $O(1)$
- ***Extract-Min***: $O(\log n)^*$
- ***Meld***: $O(1)$

* *amortized*

Major Ideas from Today

- Isometries are a *great* way to design data structures.
 - Here, binomial heaps come from binary arithmetic.
- Designing for amortized efficiency is about building up messes slowly and rapidly cleaning them up.
 - Each individual *enqueue* isn't too bad, and a single *extract-min* fixes all the prior problems.

Next Time

- ***The Need for decrease-key***
 - A powerful and versatile operation on priority queues.
- ***Fibonacci Heaps***
 - A variation on lazy binomial heaps with efficient decrease-key.
- ***Implementing Fibonacci Heaps***
 - ... is harder than it looks!