Euler Tour Trees

Outline for Today

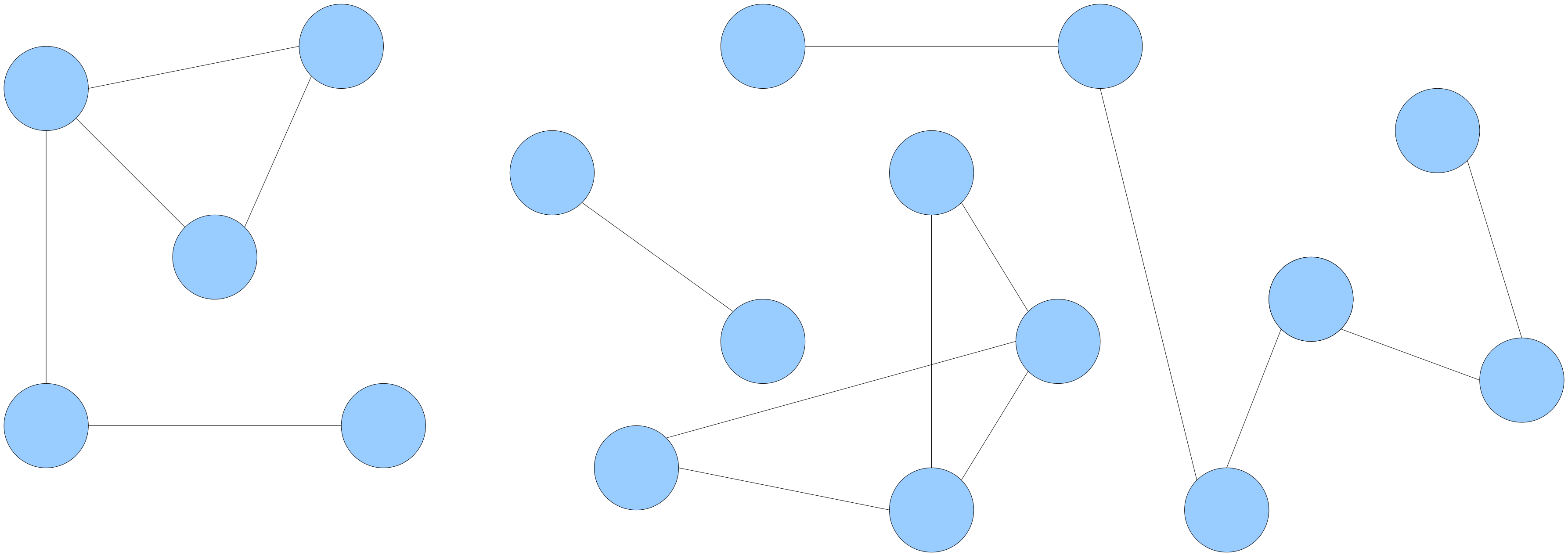
- ***Dynamic Connectivity***
  - Figuring out what's connected in a graph as the edges change.
- ***Euler Tour Representations***
  - An inspired and clever way to represent trees.
- ***Euler Tour Trees***
  - Encoding Euler tours in a creative way.
- ***Extending ETTs***
  - Extending our basic structure.

The Dynamic Connectivity Problem

The Connectivity Problem

- The ***graph connectivity problem*** is the following:

  Given an undirected graph *G*, preprocess the graph so we can answer queries of the form "are nodes *u* and *v* connected?"



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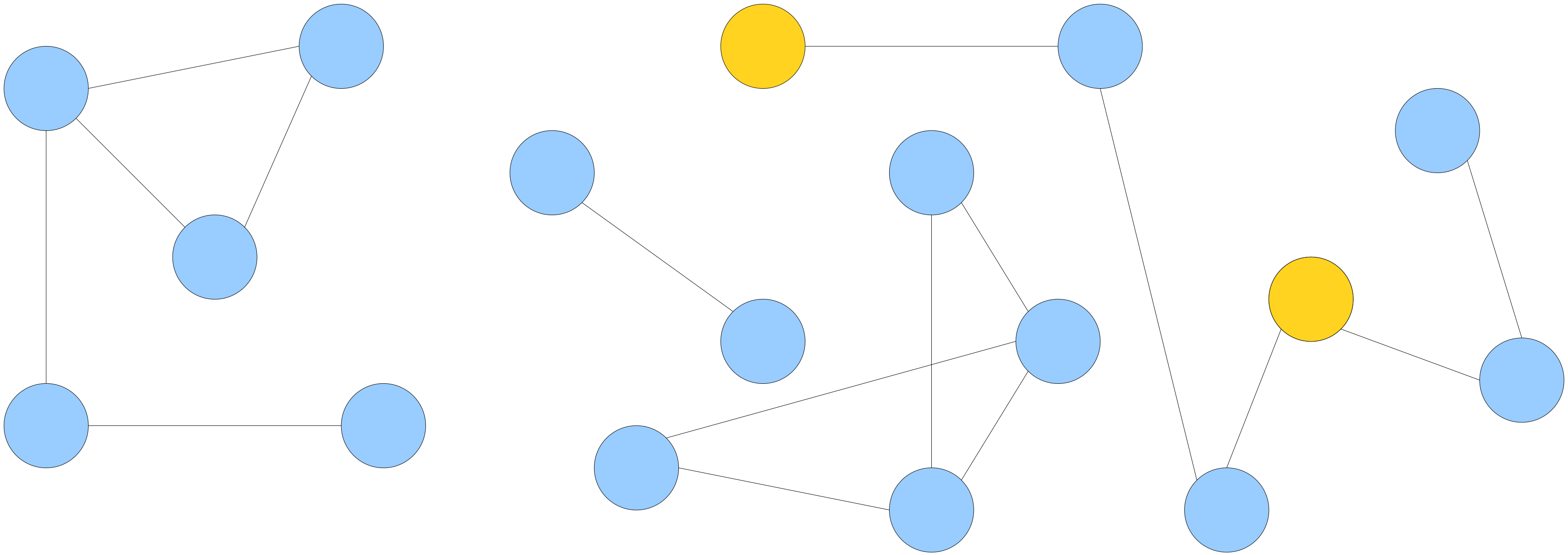

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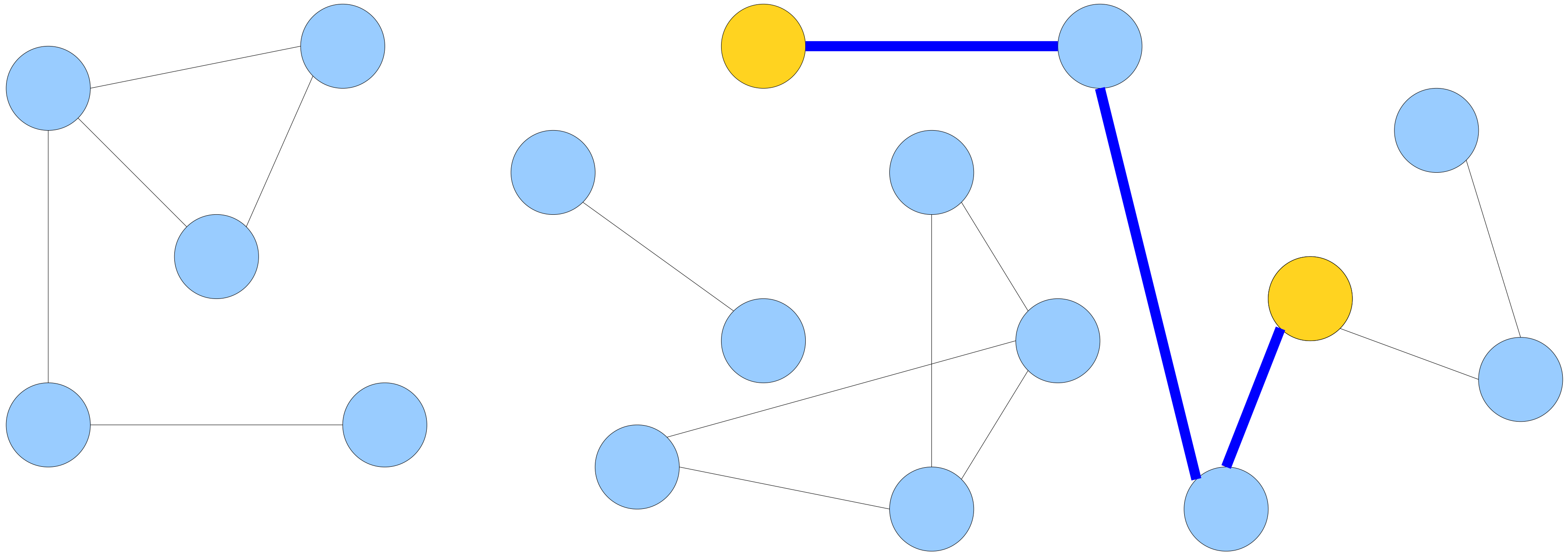

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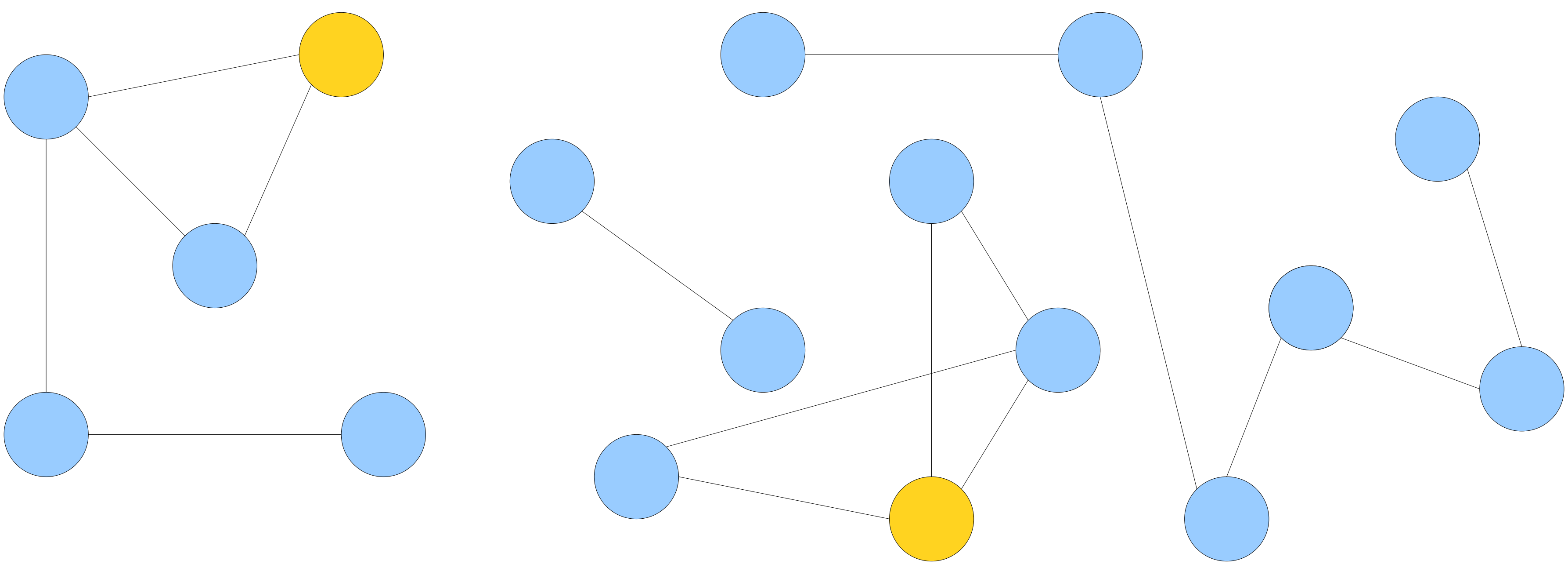

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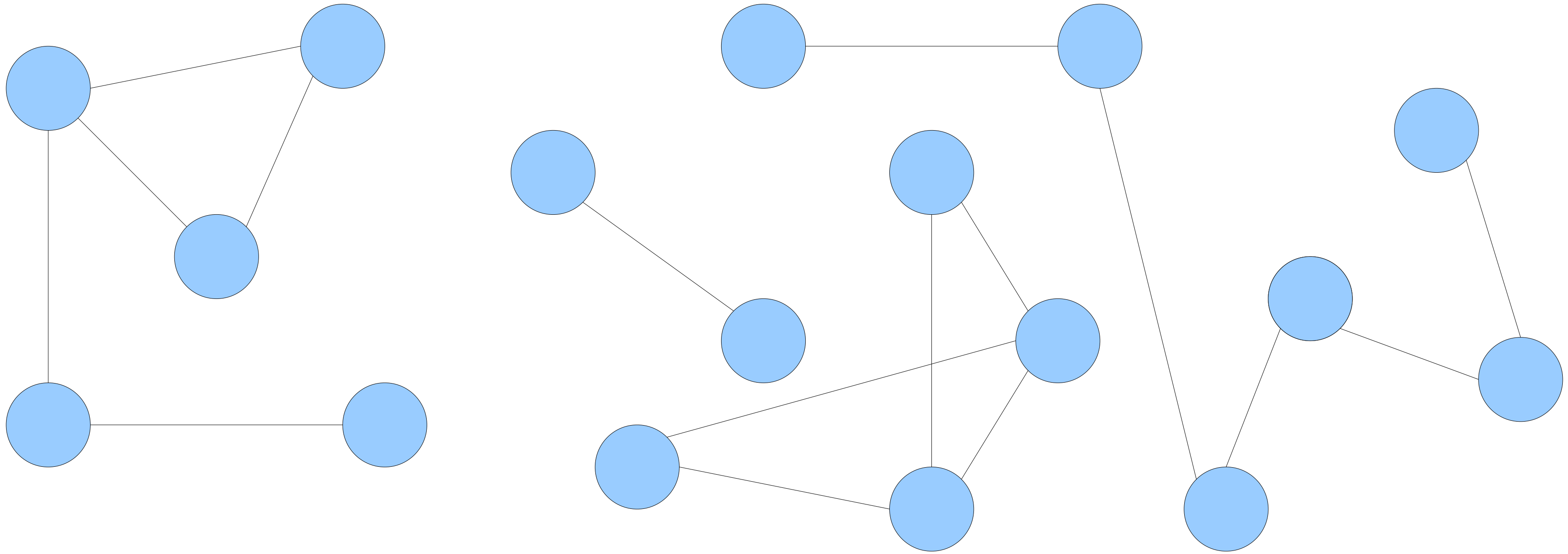
- Using $\Theta(m + n)$ preprocessing, can preprocess the graph to answer queries in time O(1).



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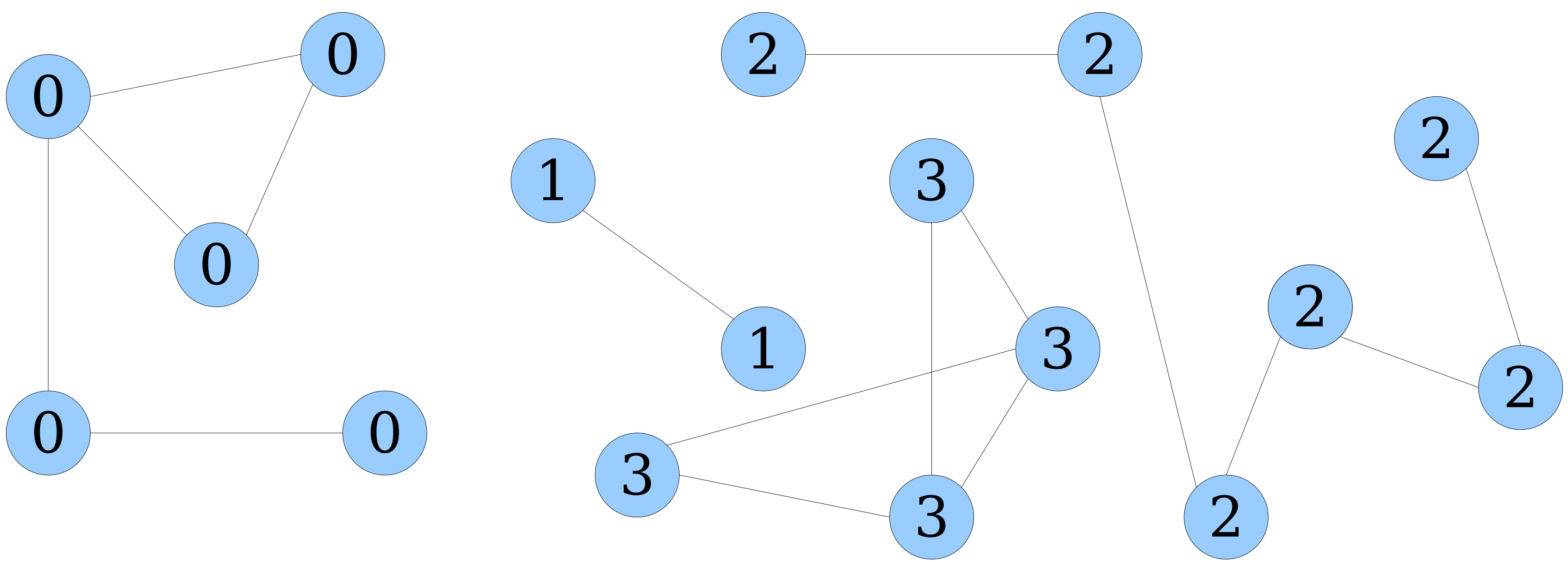

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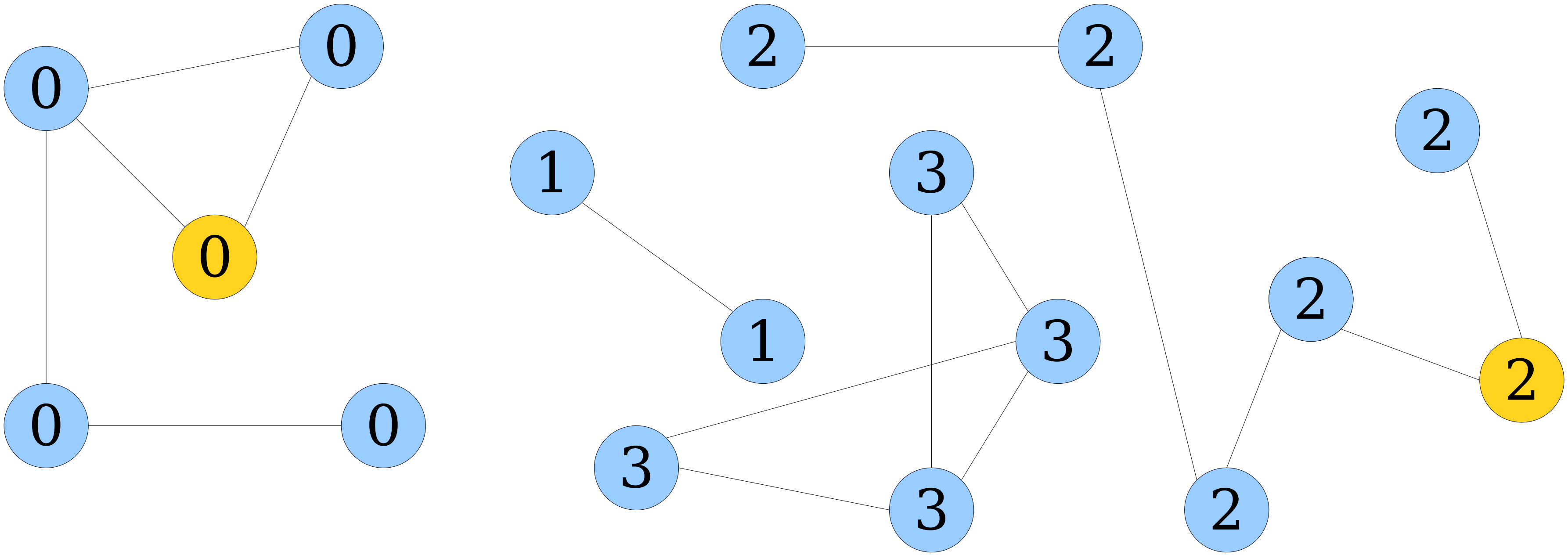

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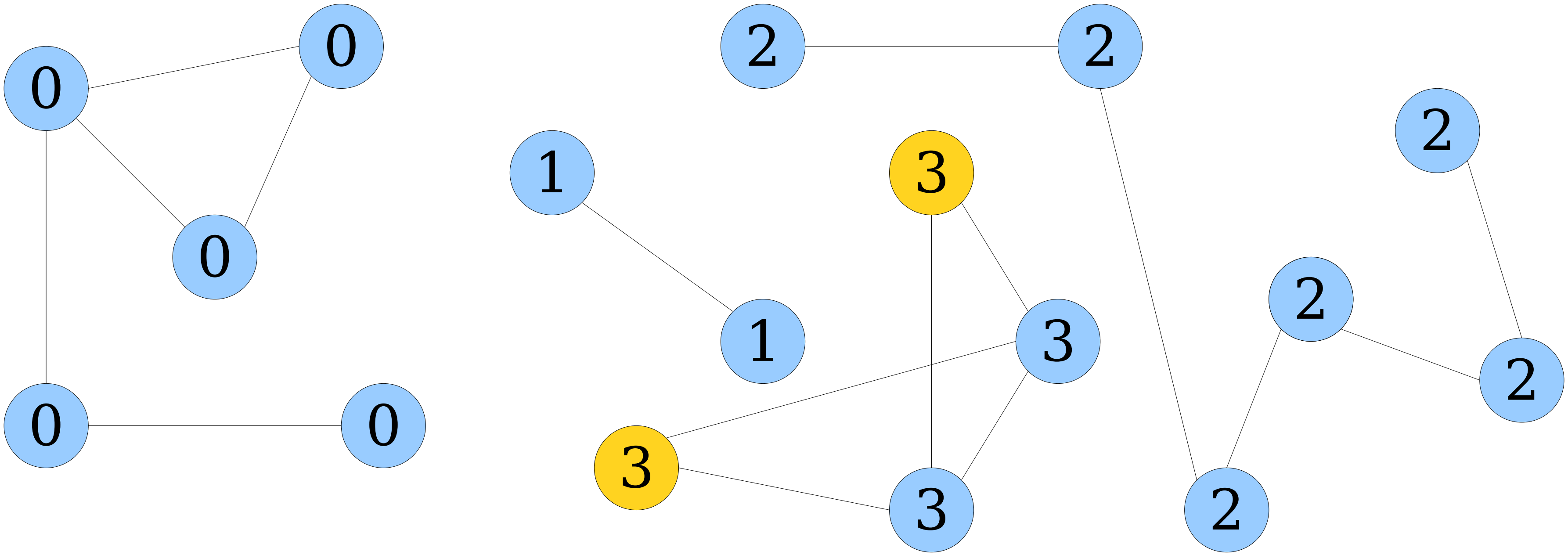

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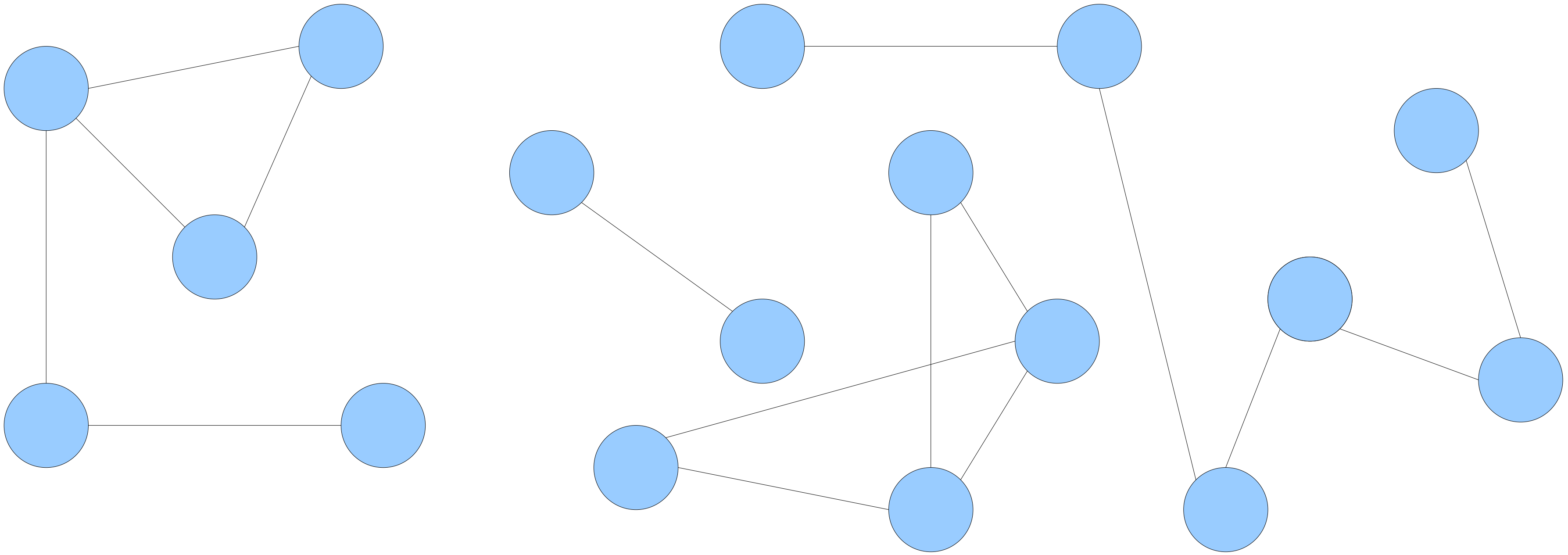

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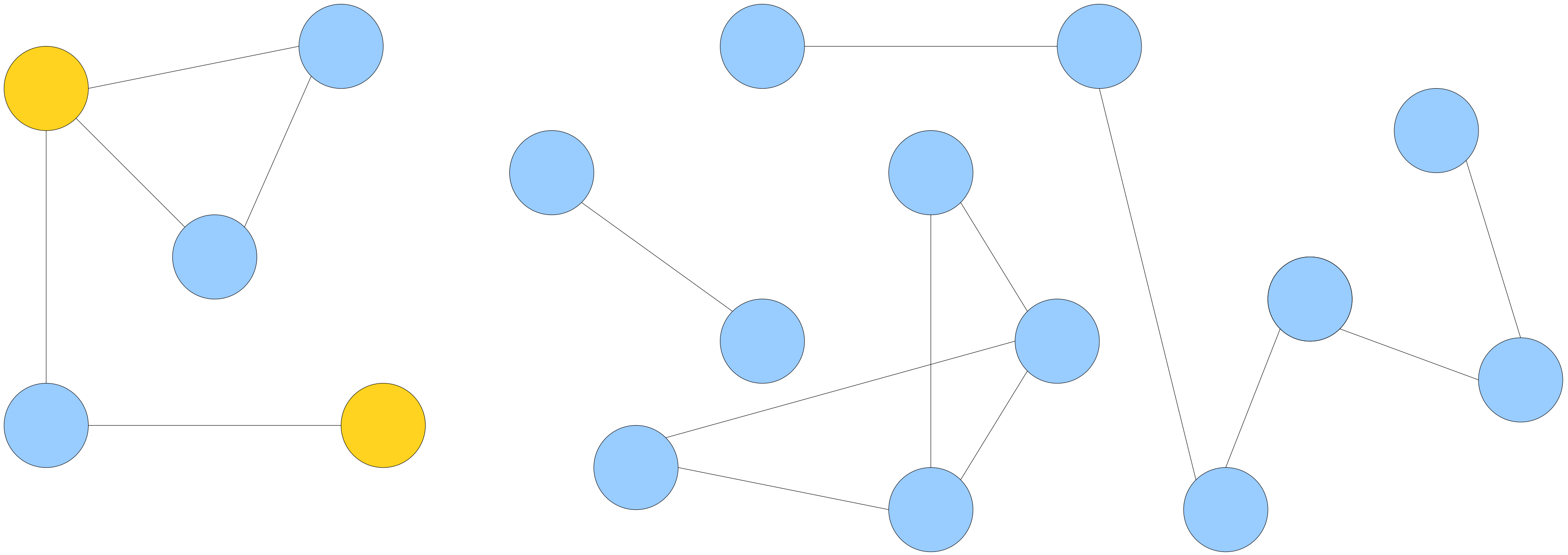

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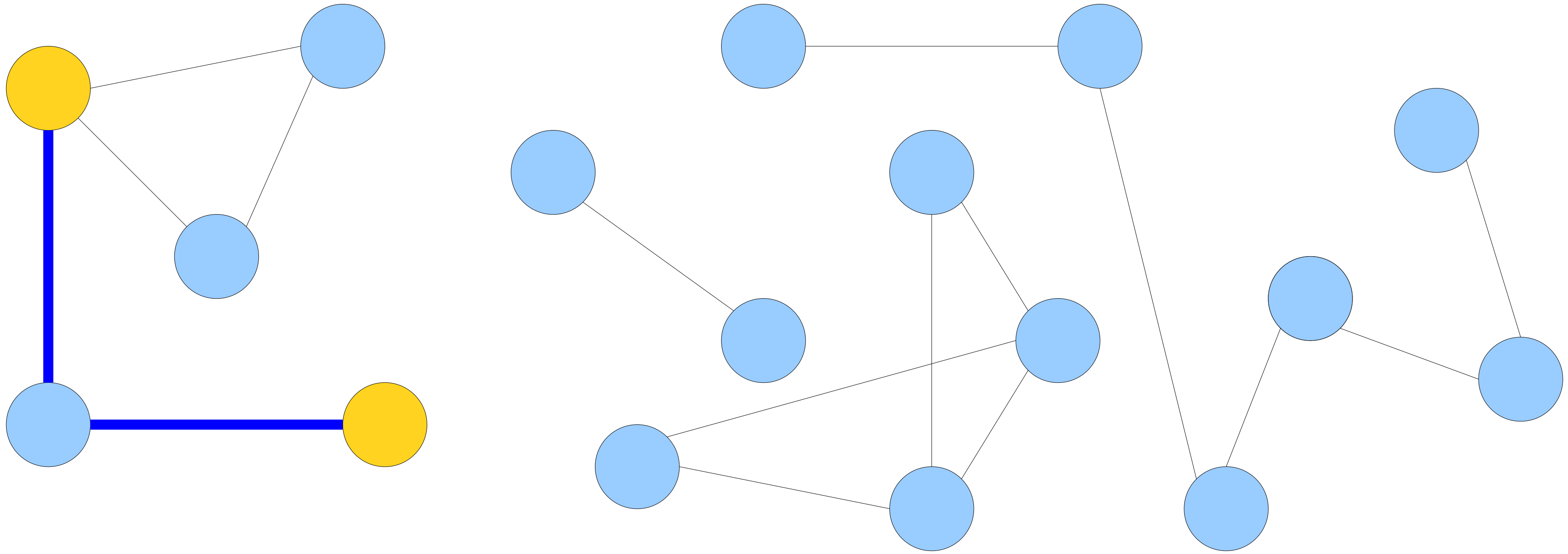

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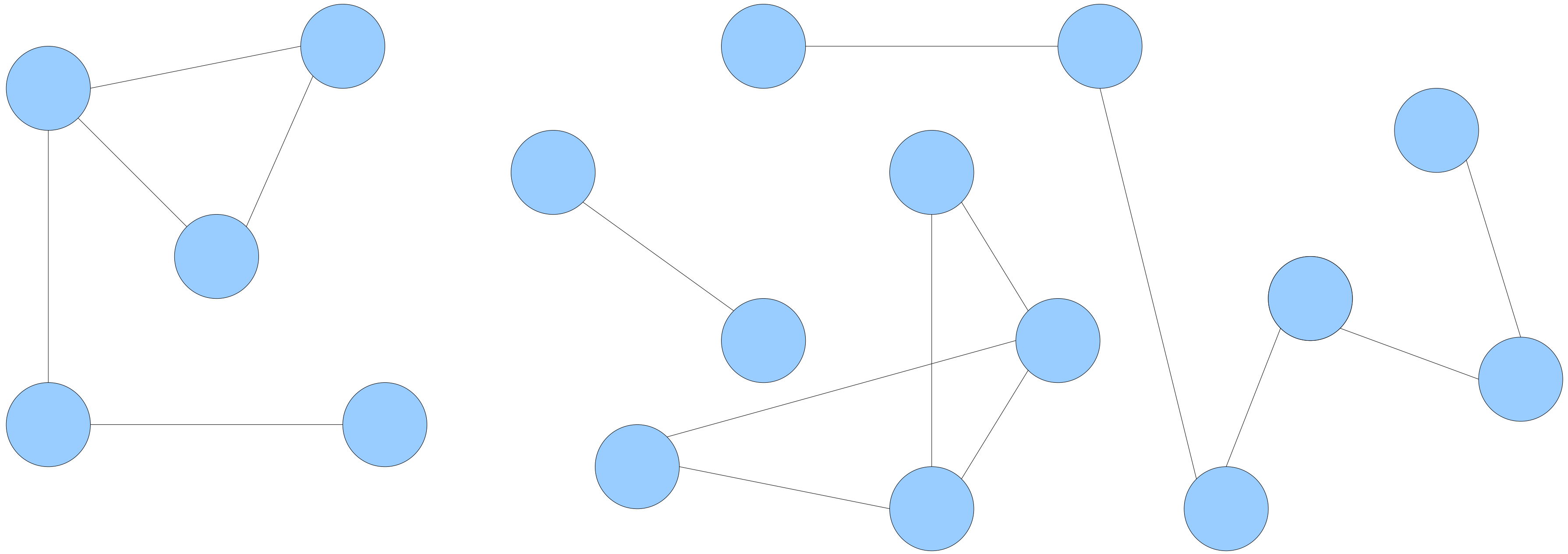

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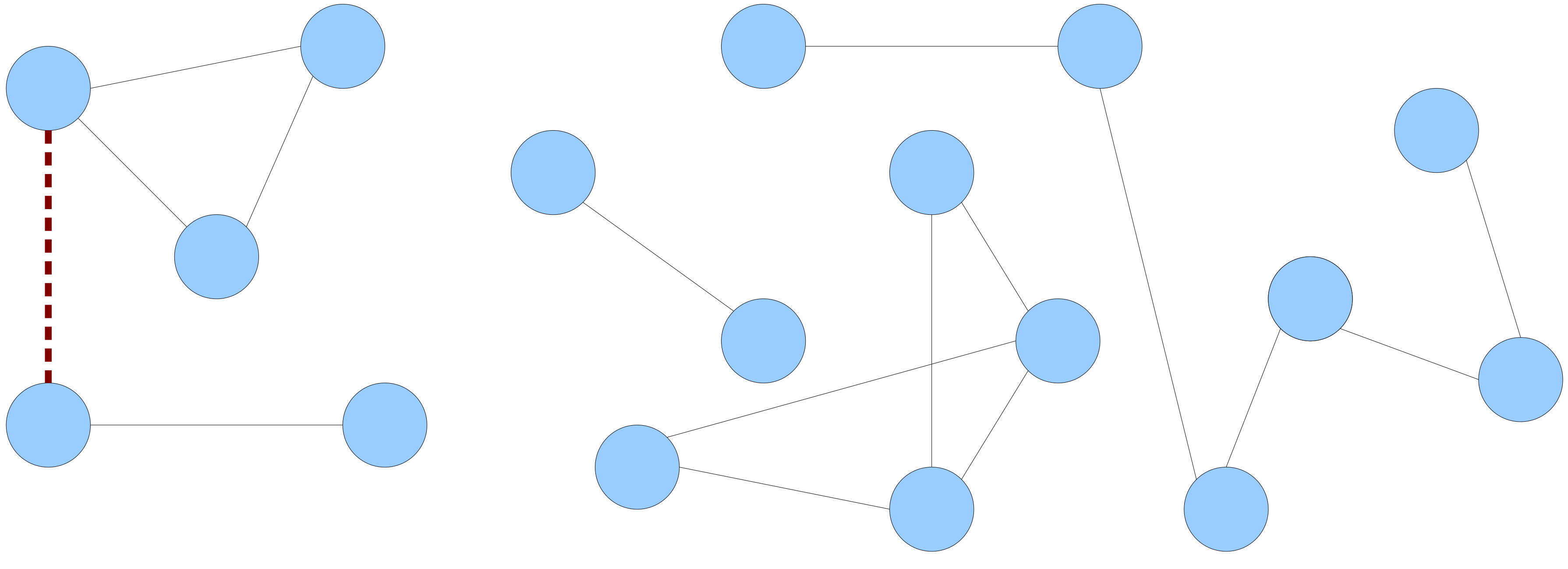

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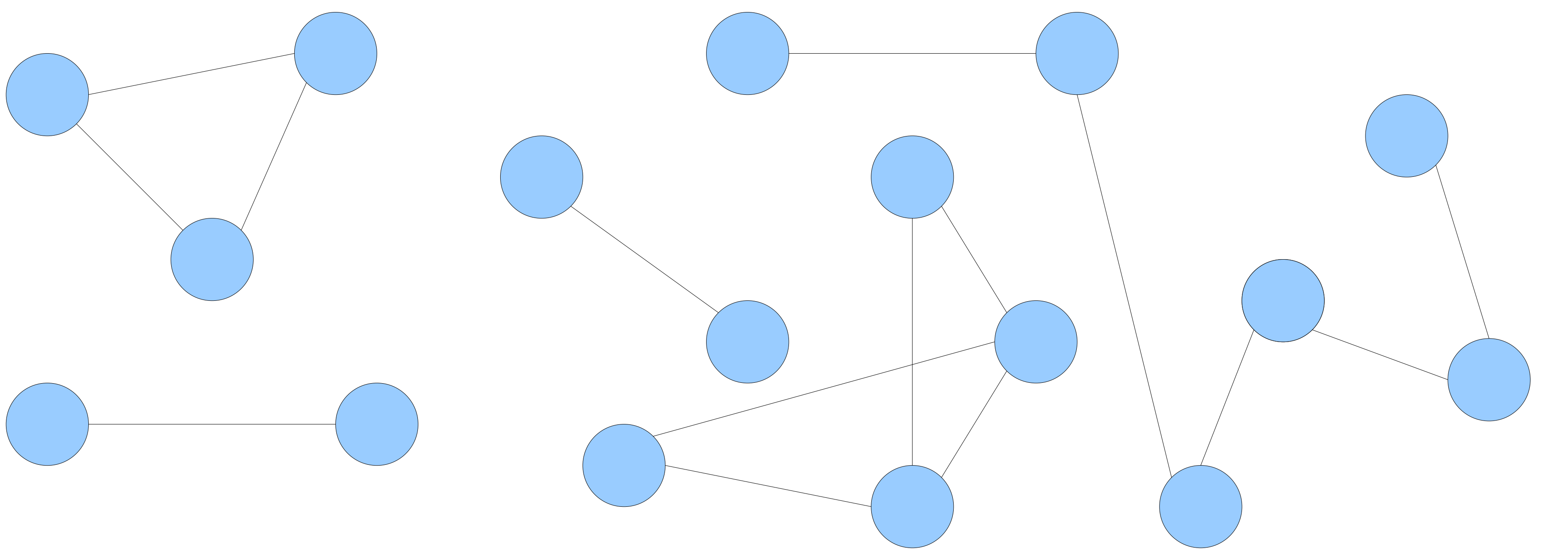

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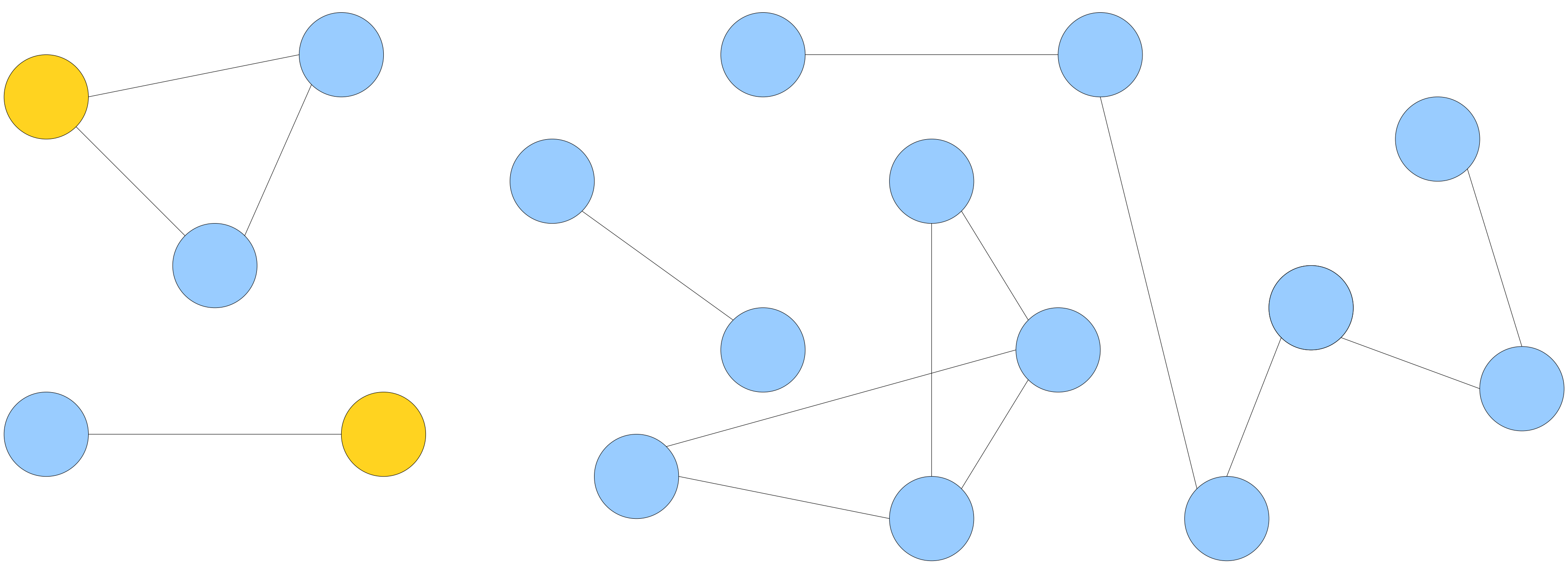

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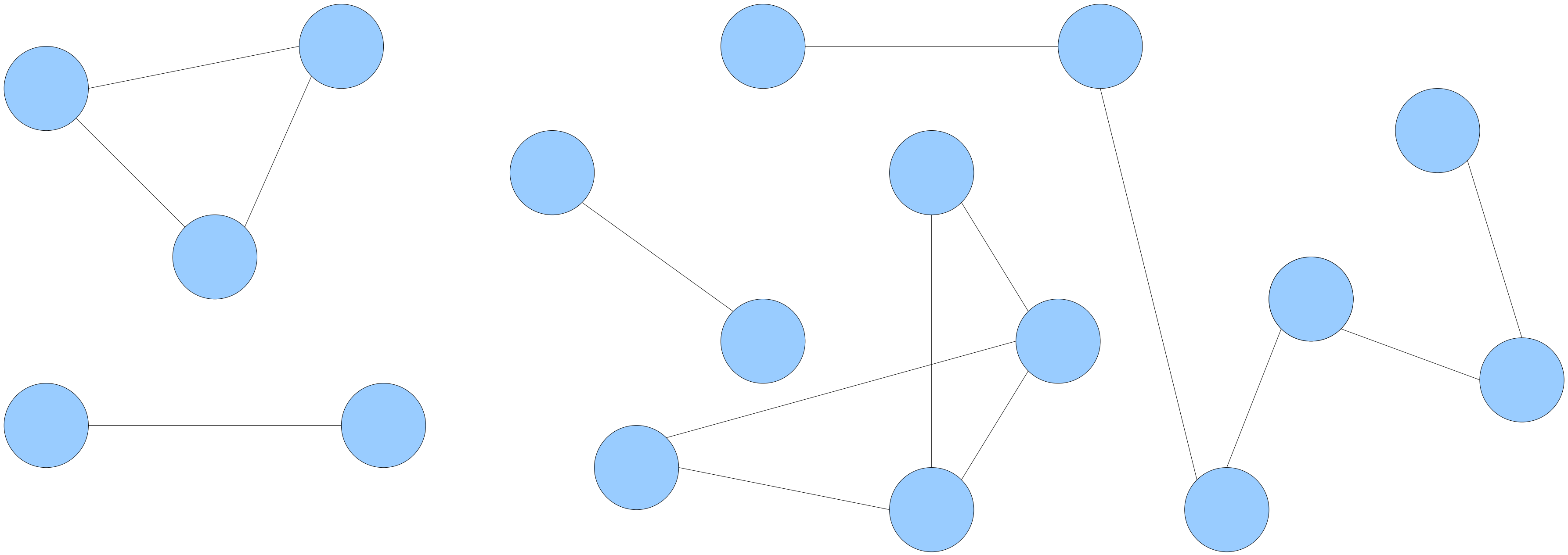

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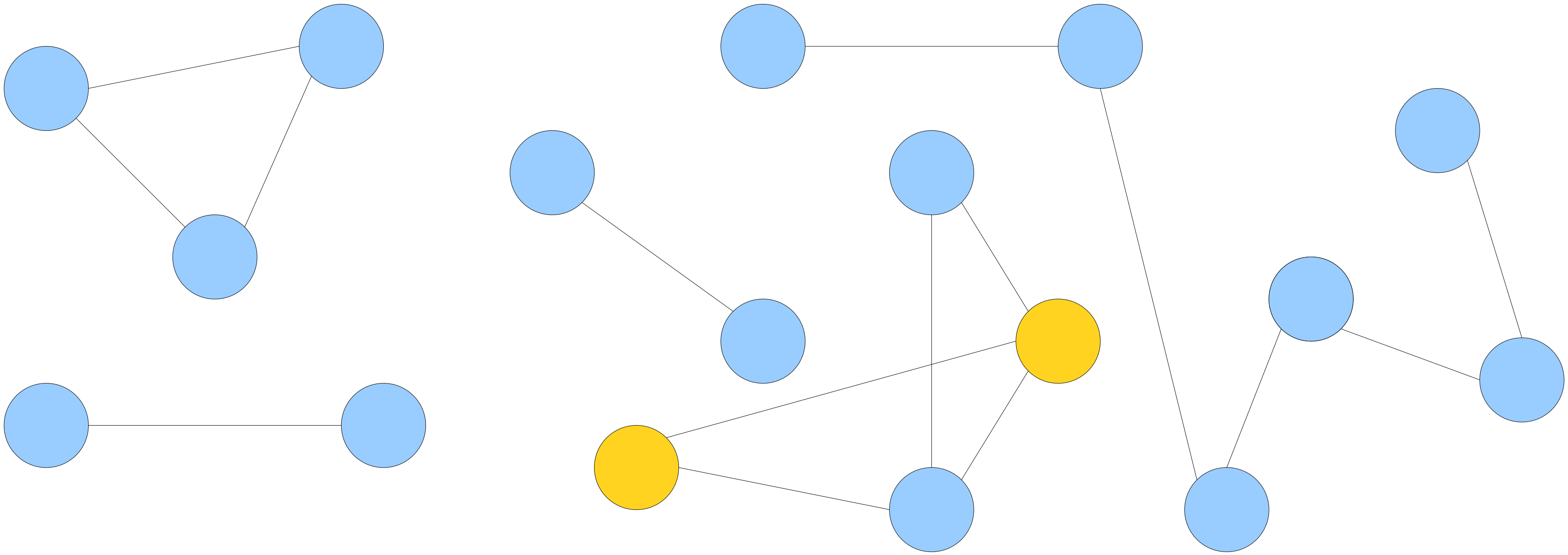

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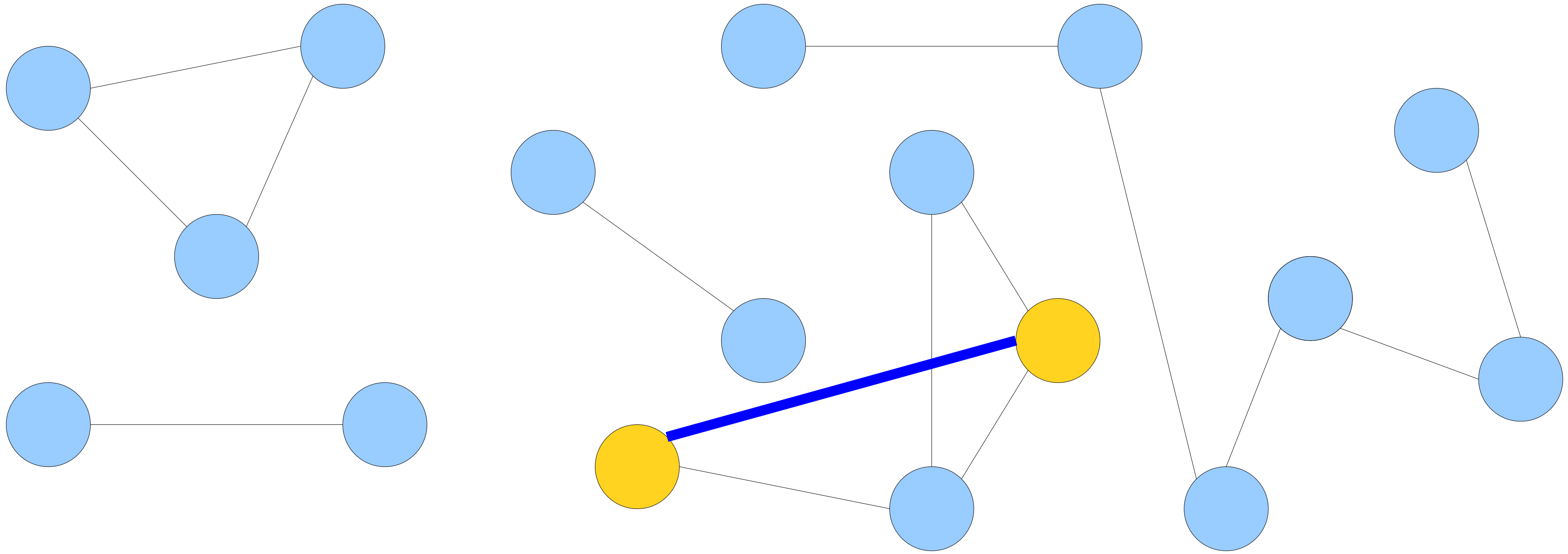

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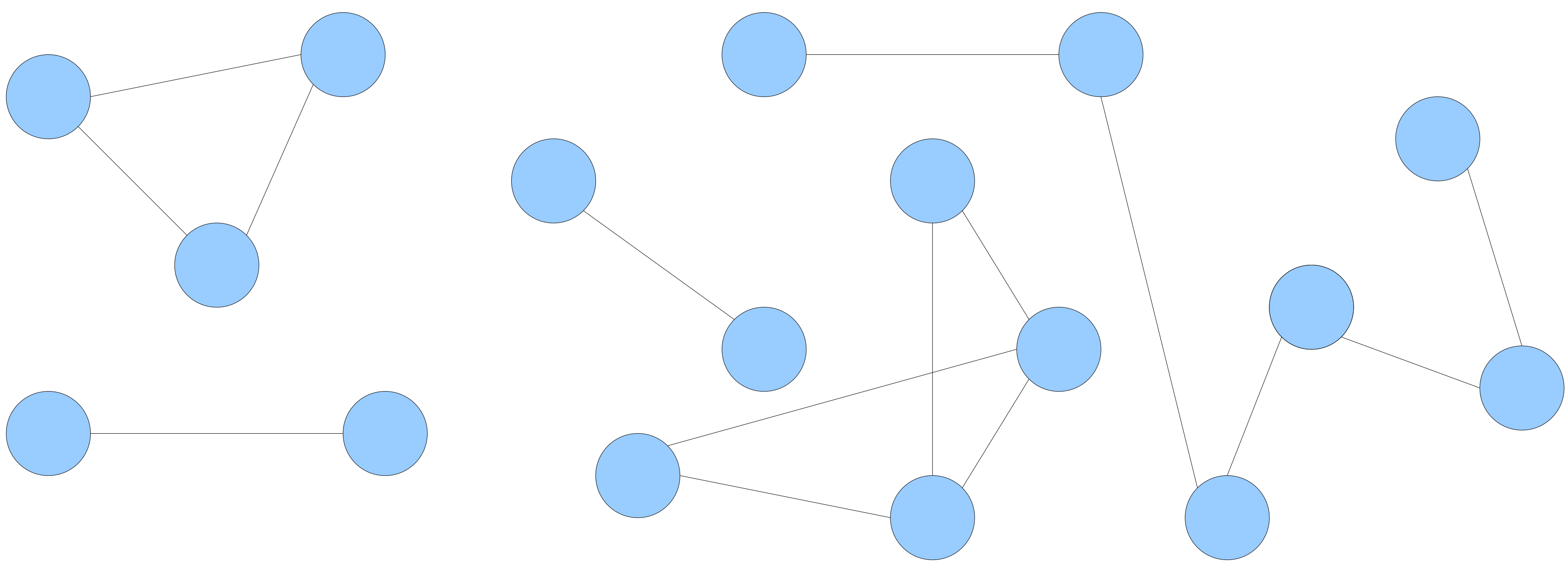

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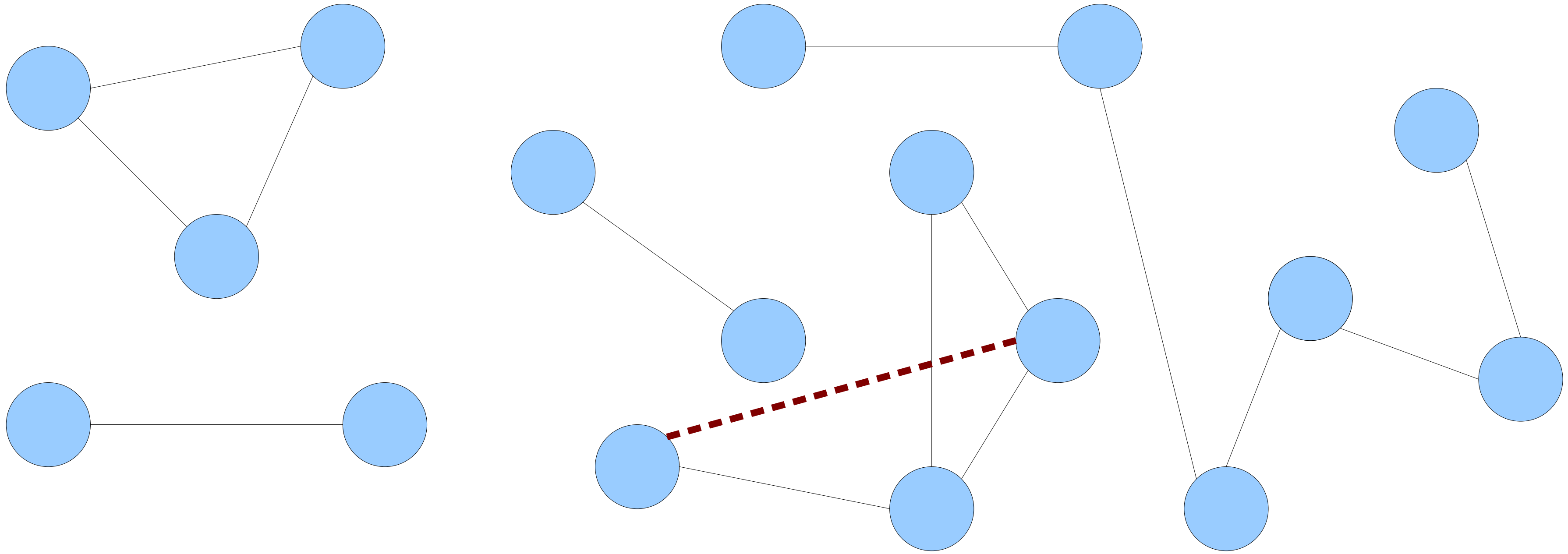

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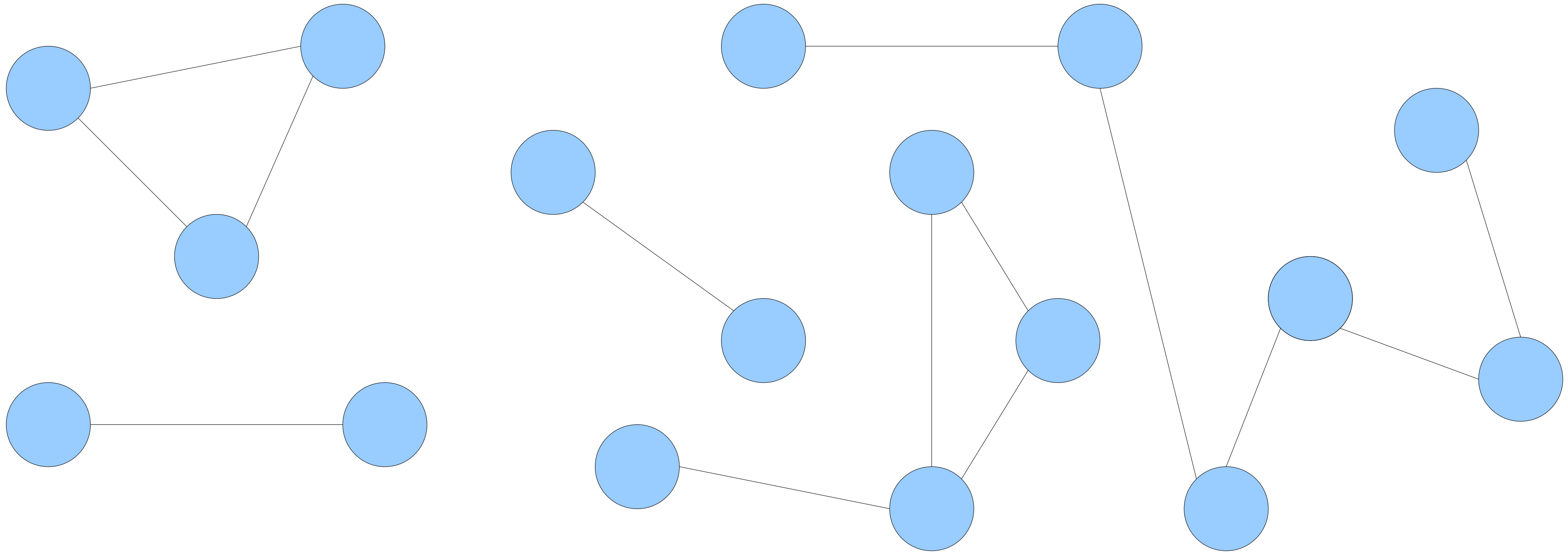

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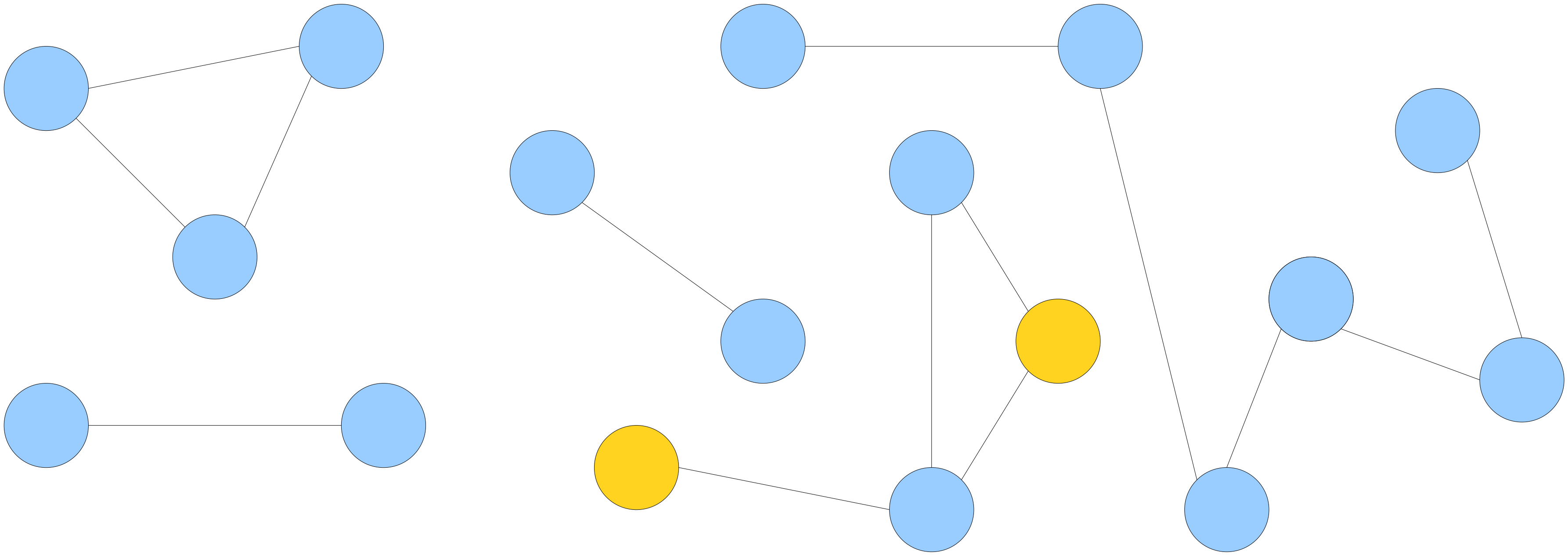

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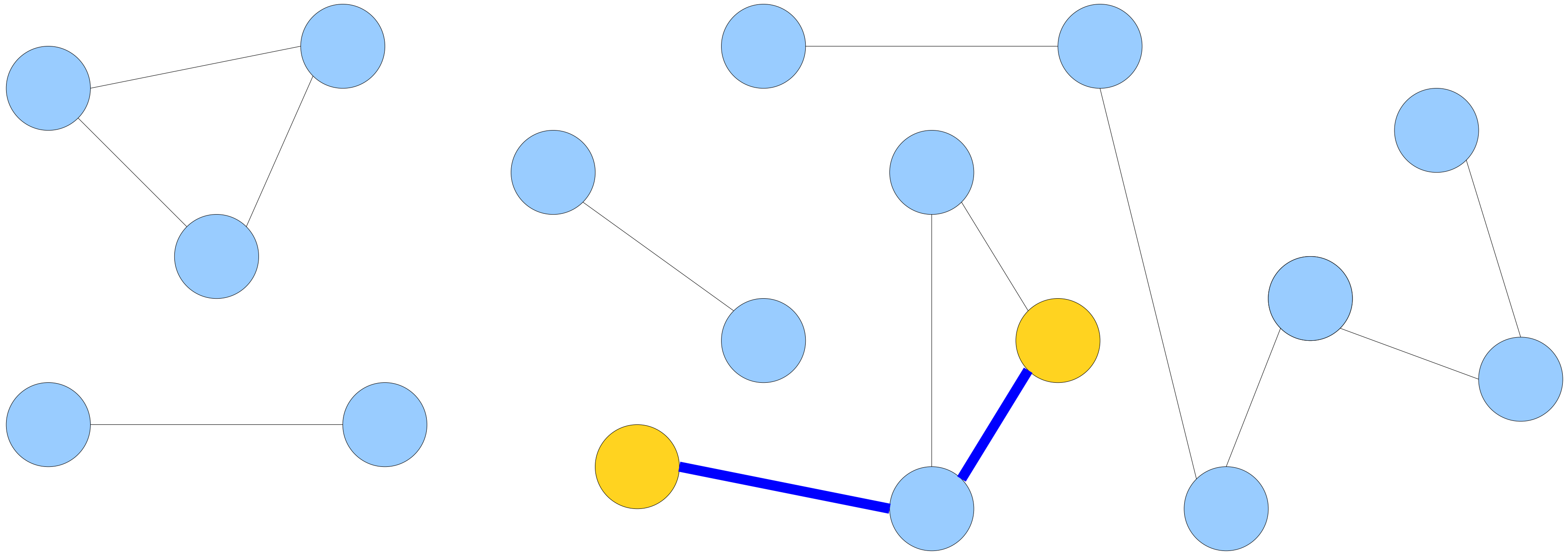

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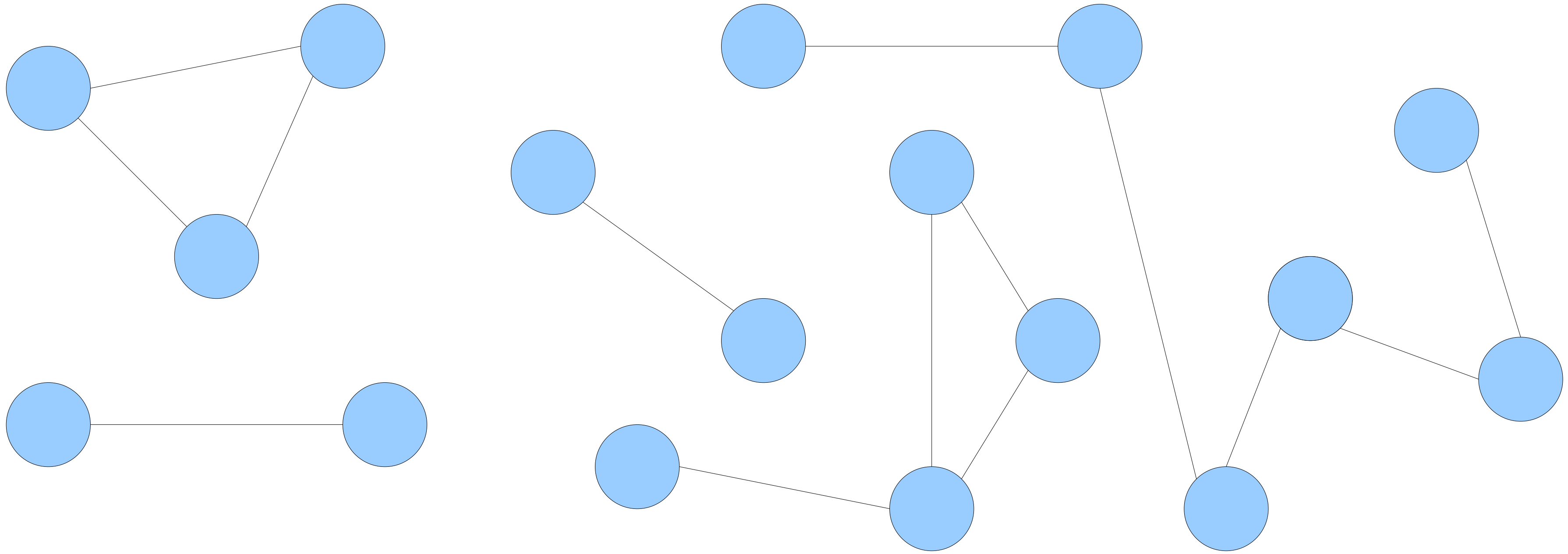

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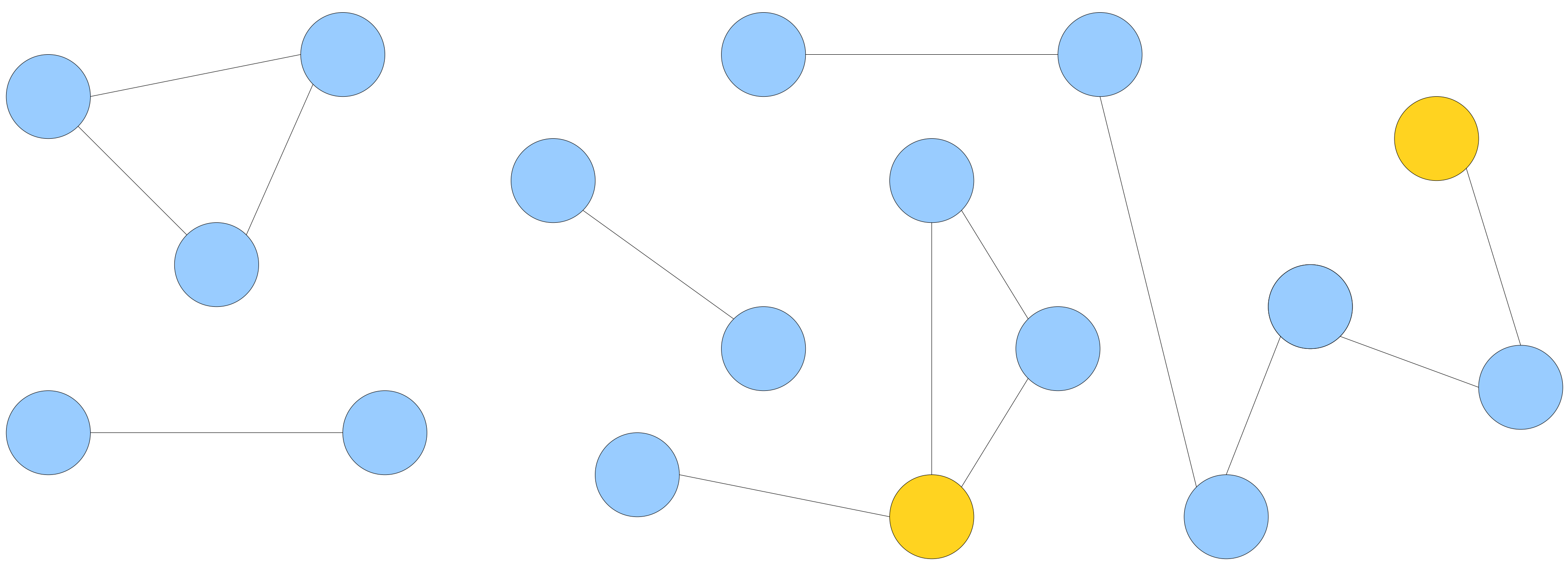

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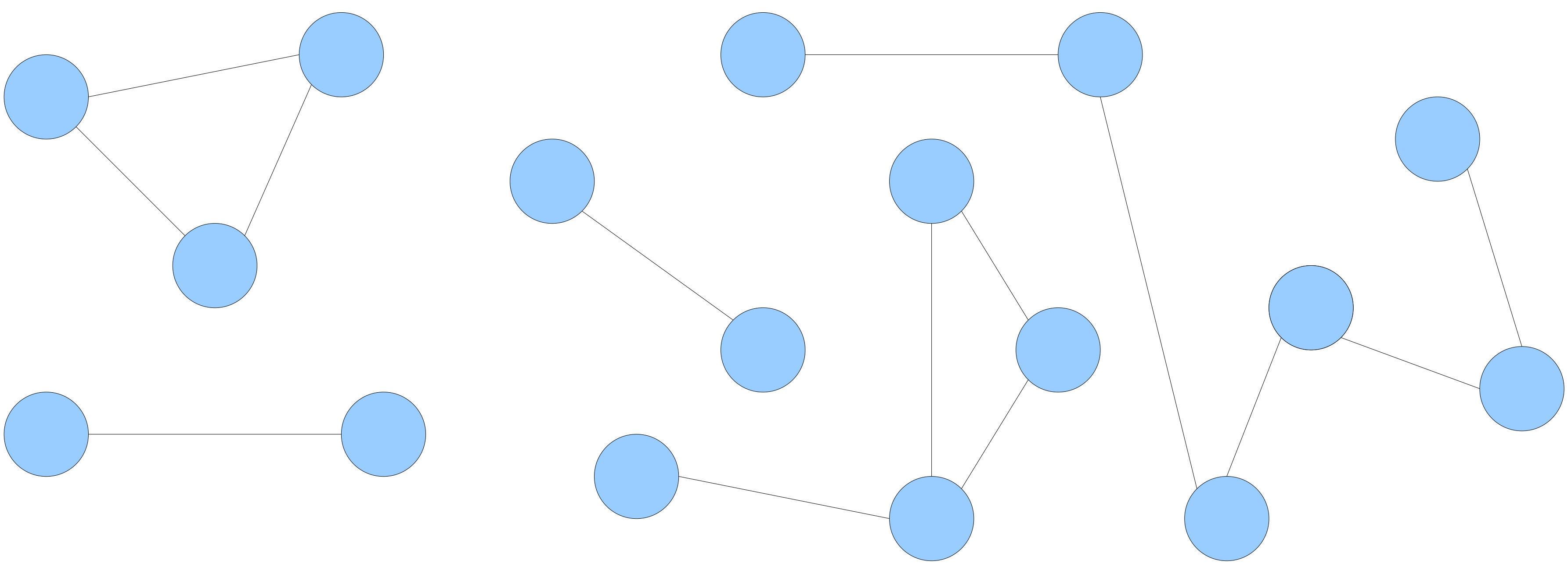

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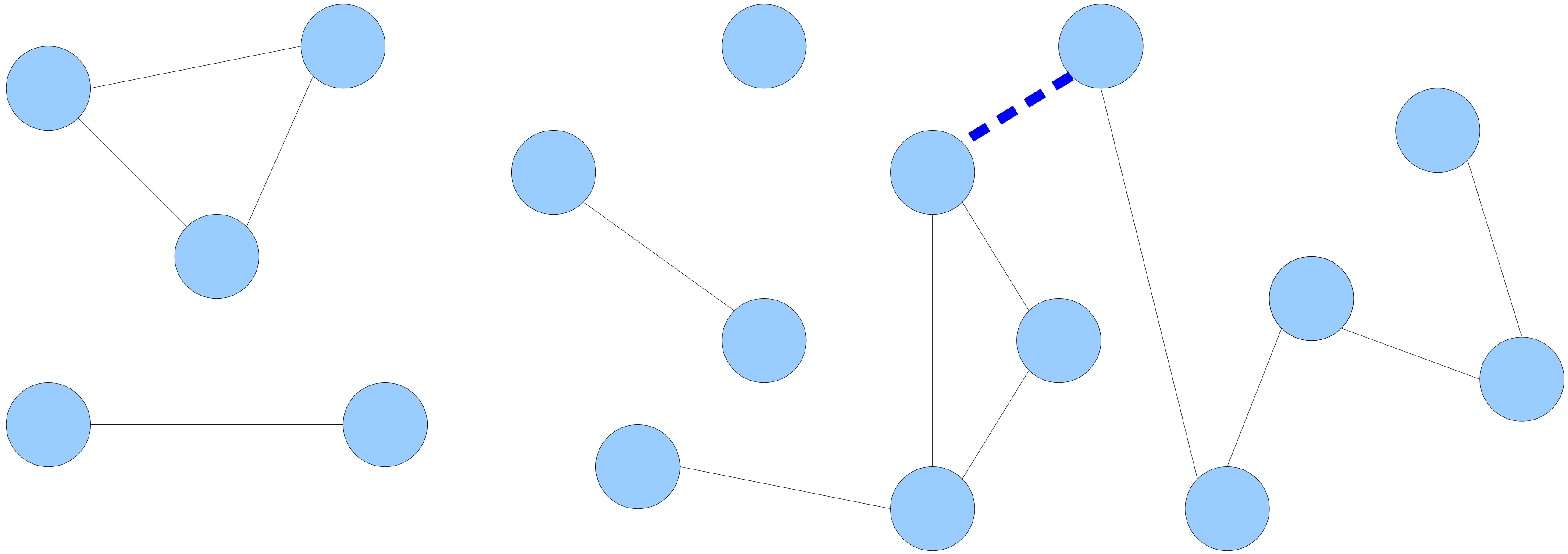

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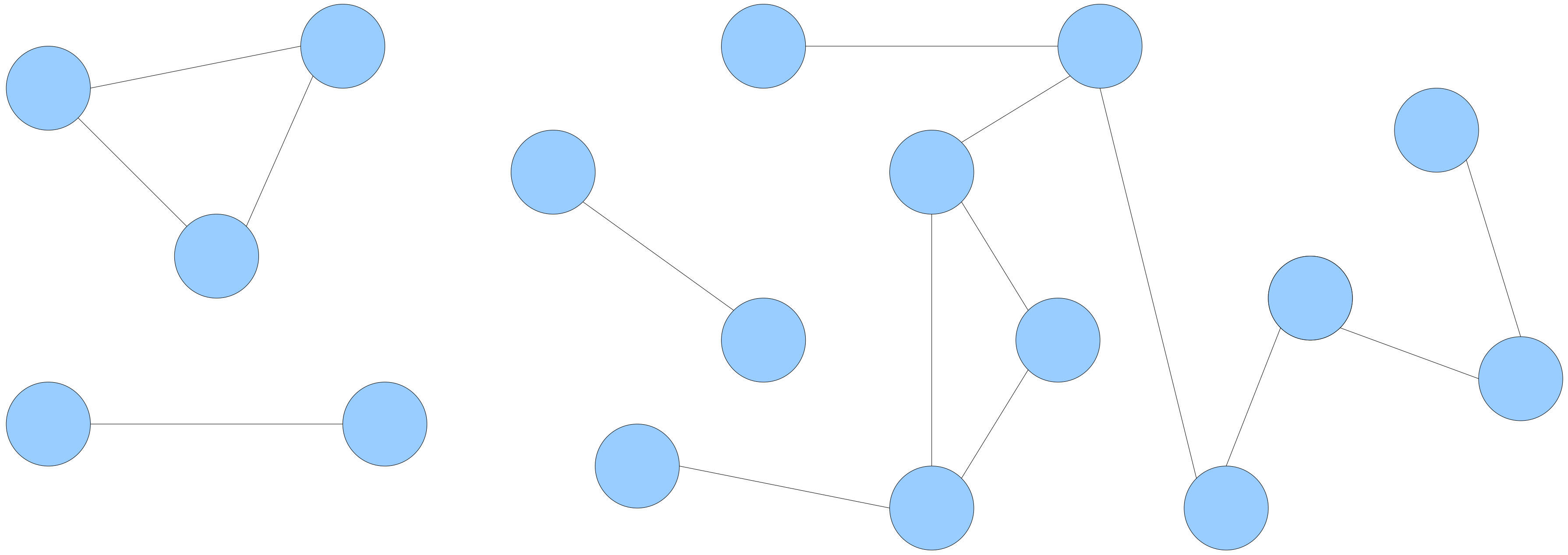

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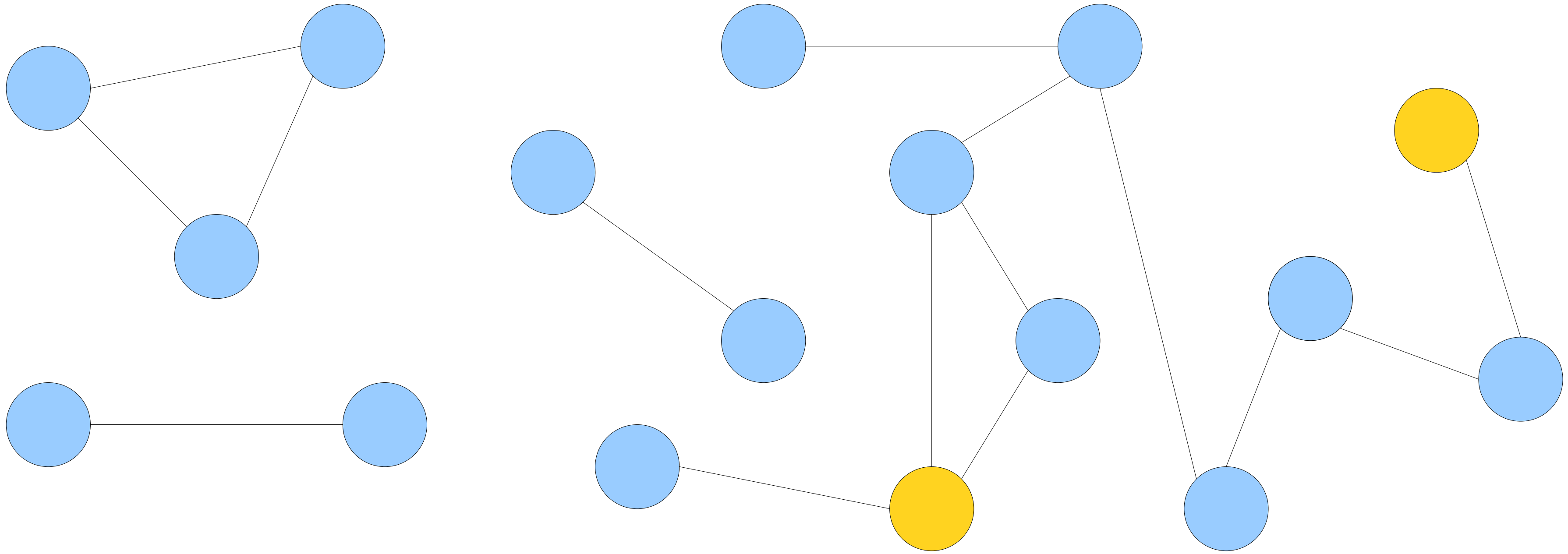

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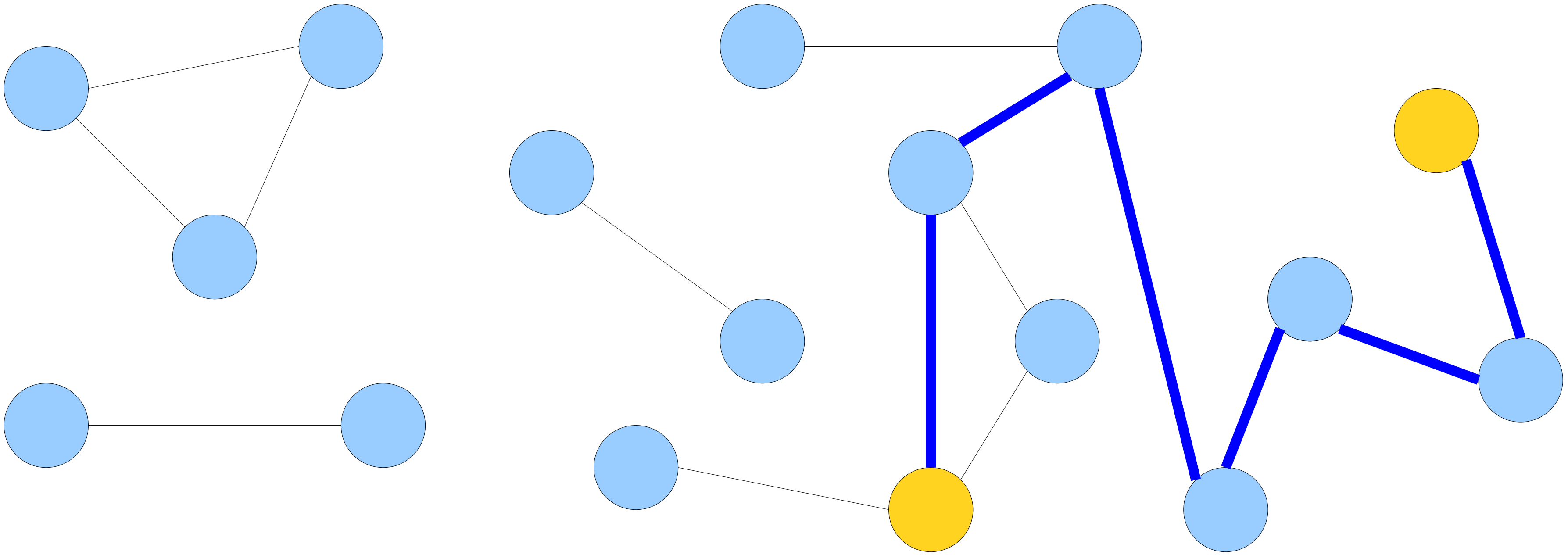

Special Cases

- Last time, we covered the ***incremental connectivity problem*** in which edges can only be added and not removed.
- Today, we'll cover ***dynamic connectivity in forests***, a special case in which the graph is known to be a forest.
- Next time, we'll cover ***fully-dynamic connectivity***, in which there are no restrictions on which edges can be added and removed.

Dynamic Connectivity in Forests

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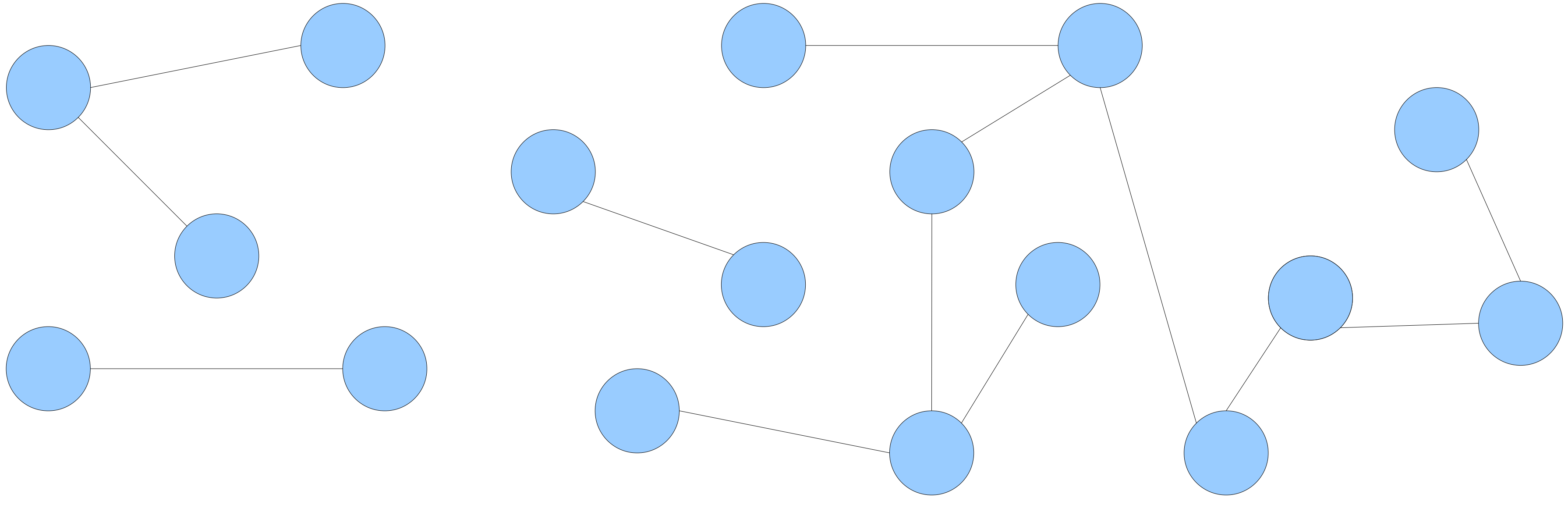
- Consider the following special-case of the dynamic connectivity problem:

  Maintain an undirected ***forest*** F so that edges may be inserted an deleted and connectivity queries may be answered efficiently.

- Each deleted edge splits a tree in two; each added edge joins two trees and never closes a cycle.



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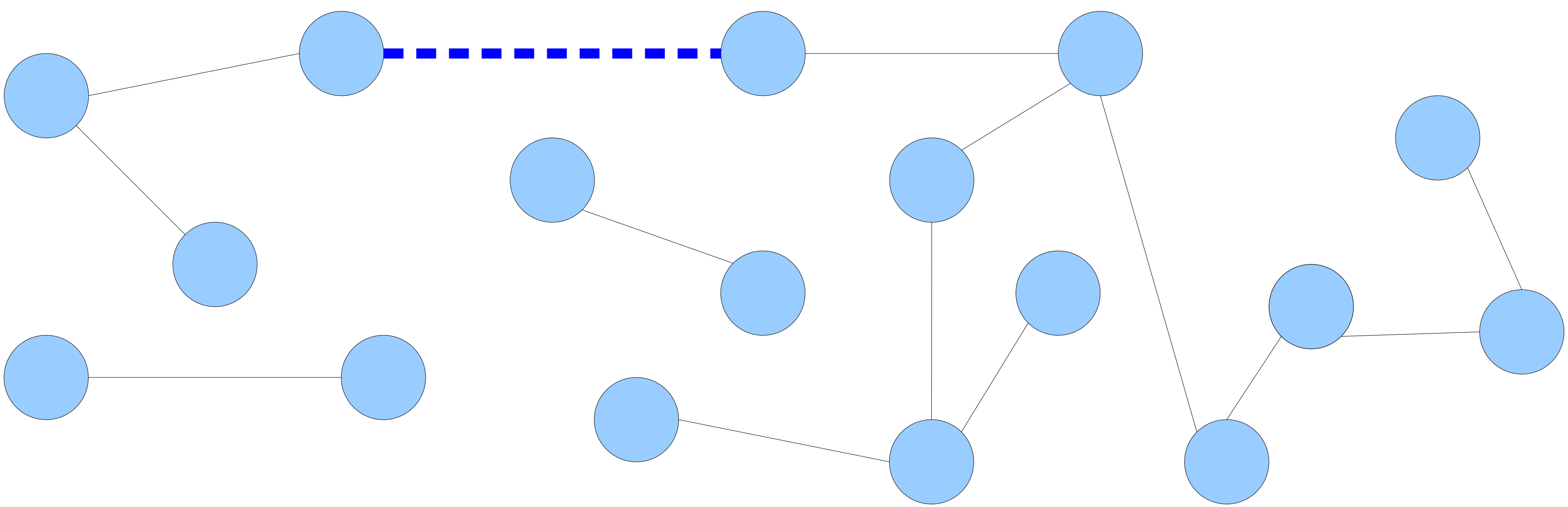

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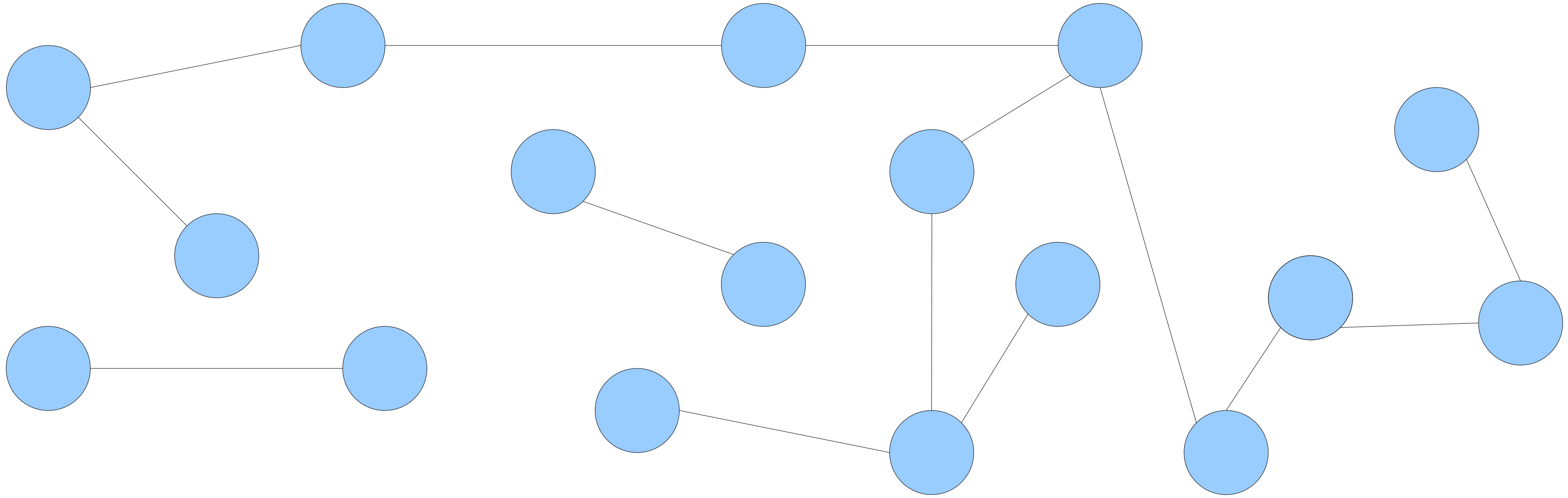

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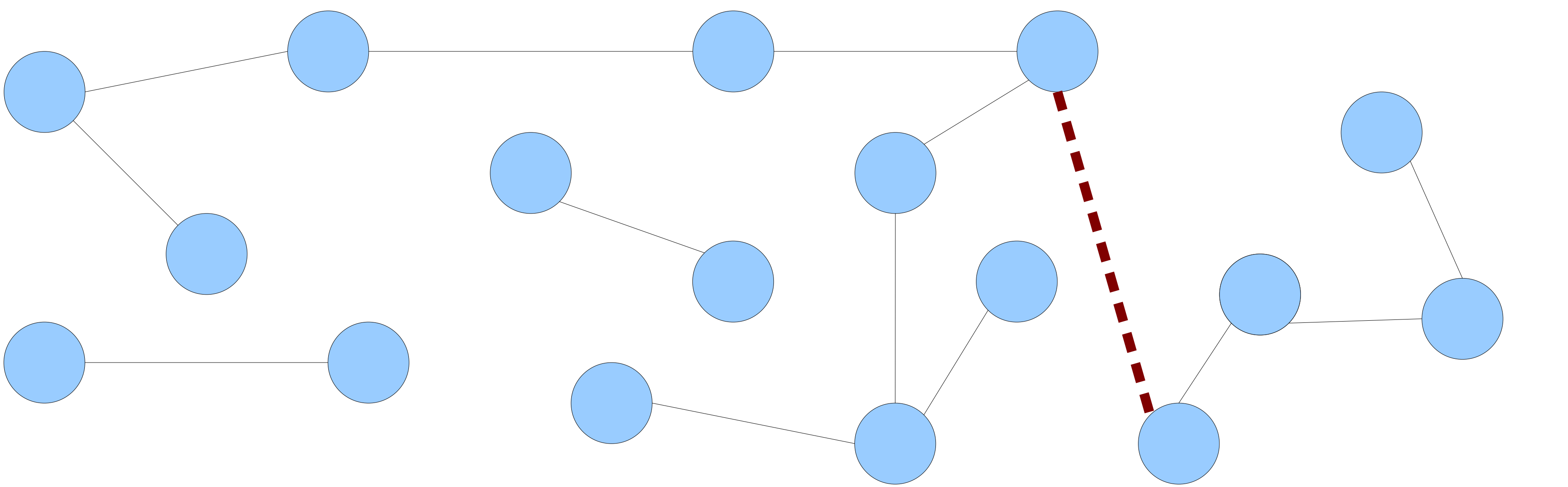

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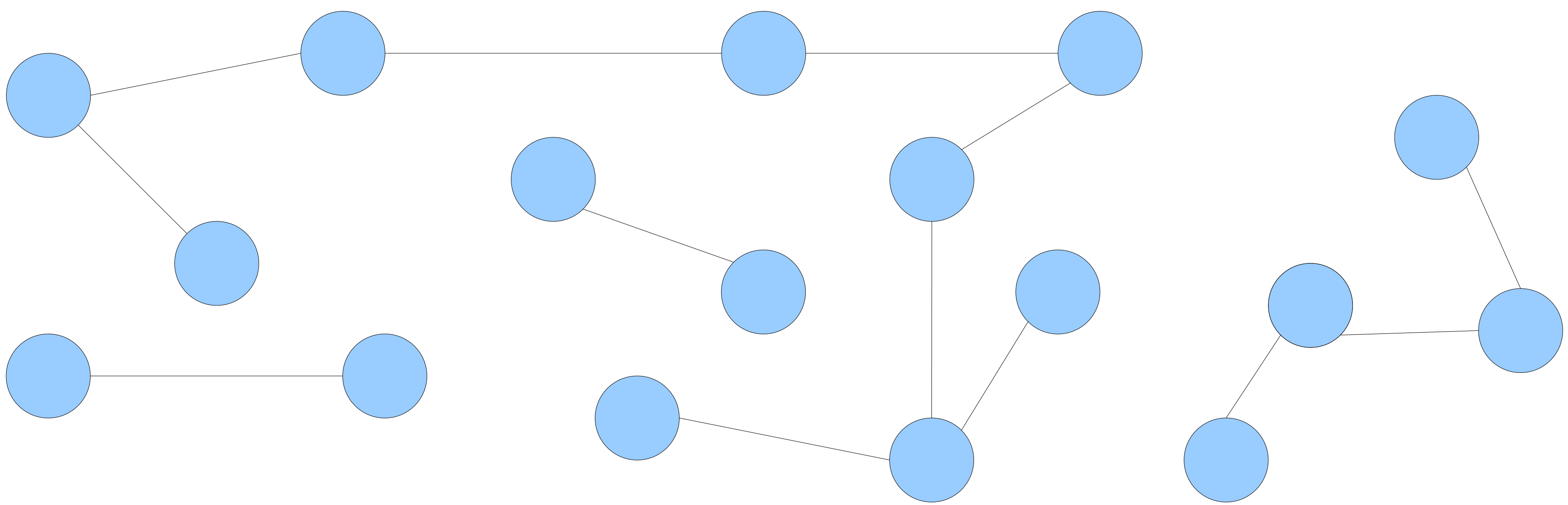

Dynamic Connectivity in Forests

- ***Goal***: Support these three operations:
  - ***link***(u, v): Add in edge uv. The assumption is that u and v are in separate trees.
  - ***cut***(u, v): Cut the edge uv. The assumption is that the edge exists in the forest.
  - ***are-connected***(u, v): Return whether u and v are connected.
- The data structure we'll develop can perform these operations time **O(log *n*)** each.

Euler Tours

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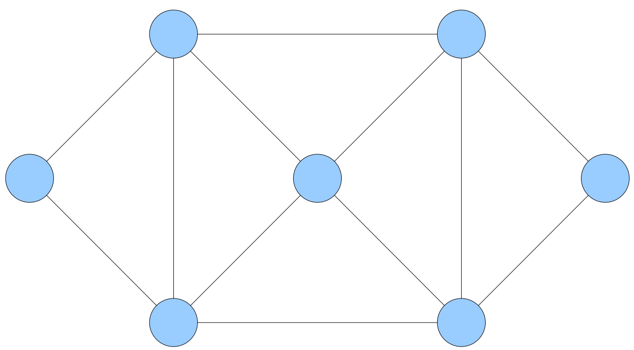
- An ***Euler tour*** is a path through a graph G that visits every edge exactly once.
- It mathematically formalizes the “trace this figure without picking up your pencil or redrawing any lines” puzzles.



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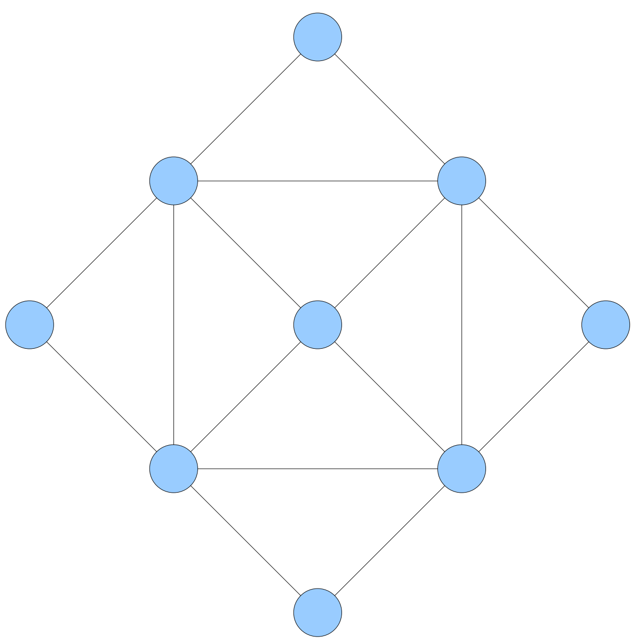

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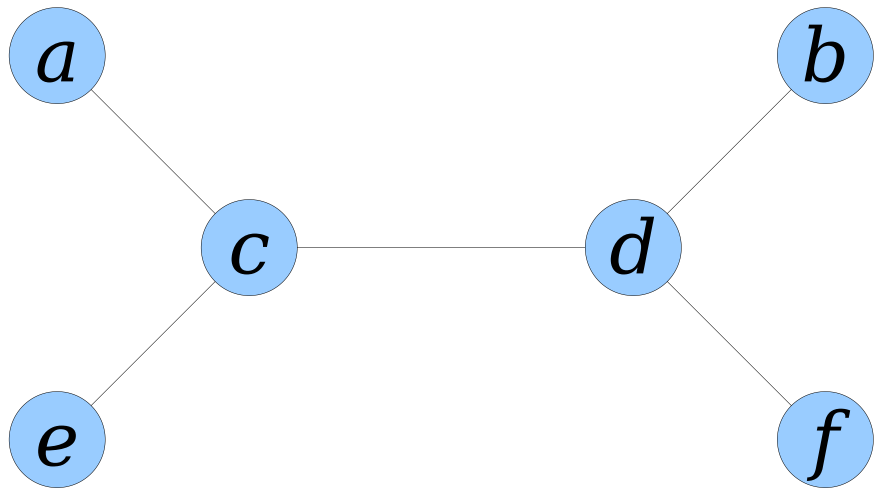
- ***Classic Theorem 1:*** A graph G has a closed Euler tour if and only if G is connected and every node in G has even degree.
- ***Classic Theorem 2:*** A directed graph G has a closed Euler tour if and only if G is strongly connected and every node's indegree equals its outdegree.

Euler Tours on Trees

- Trees do not have Euler tours.



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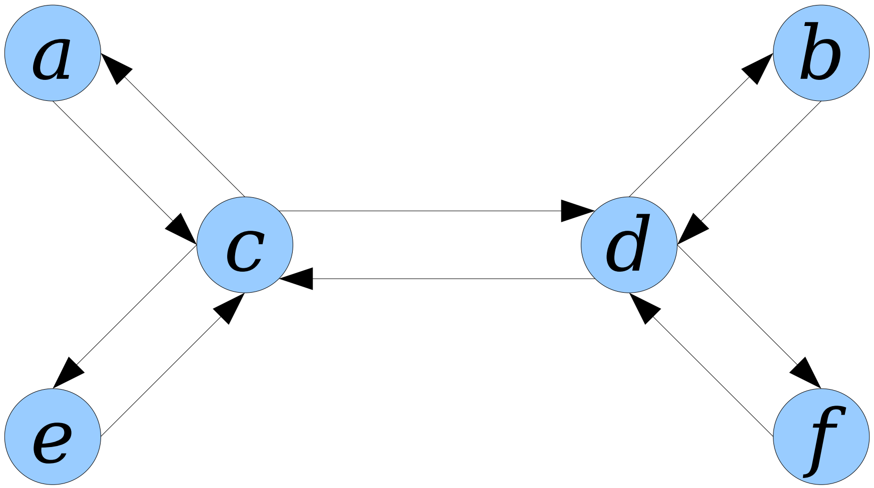

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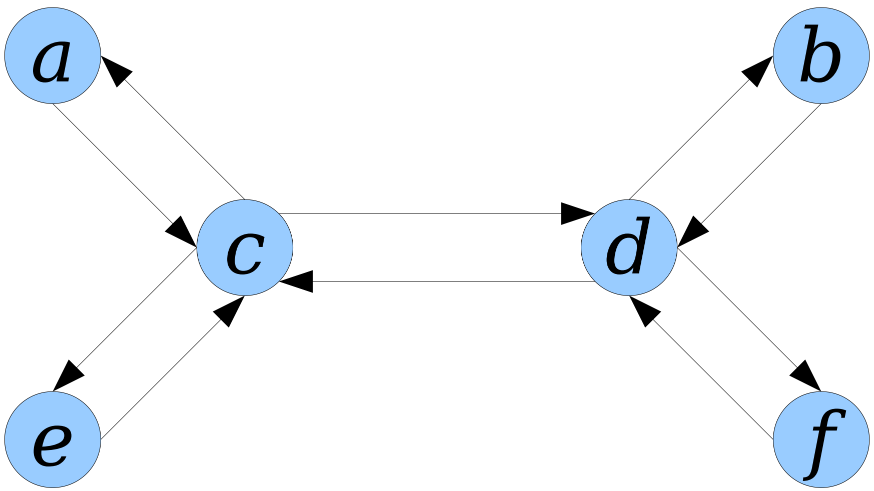

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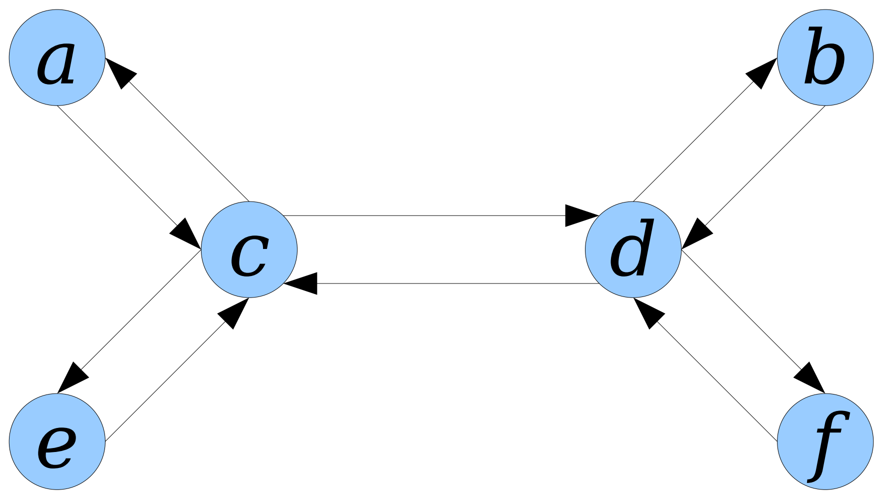

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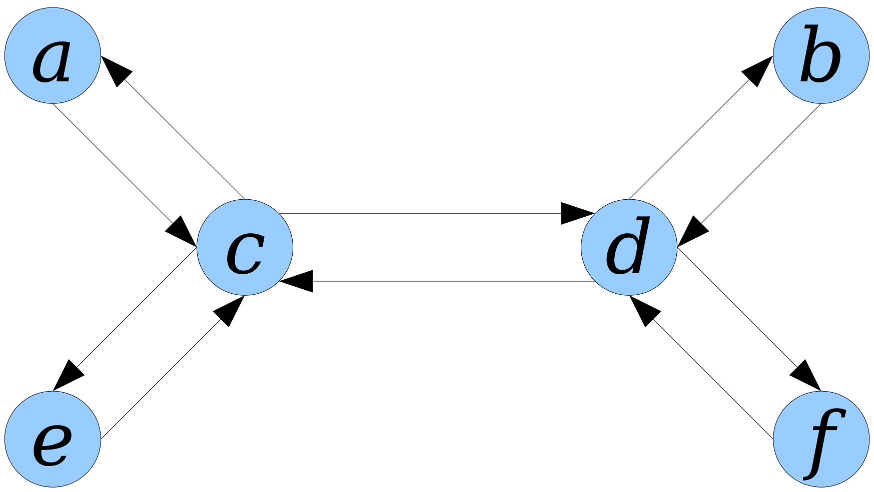

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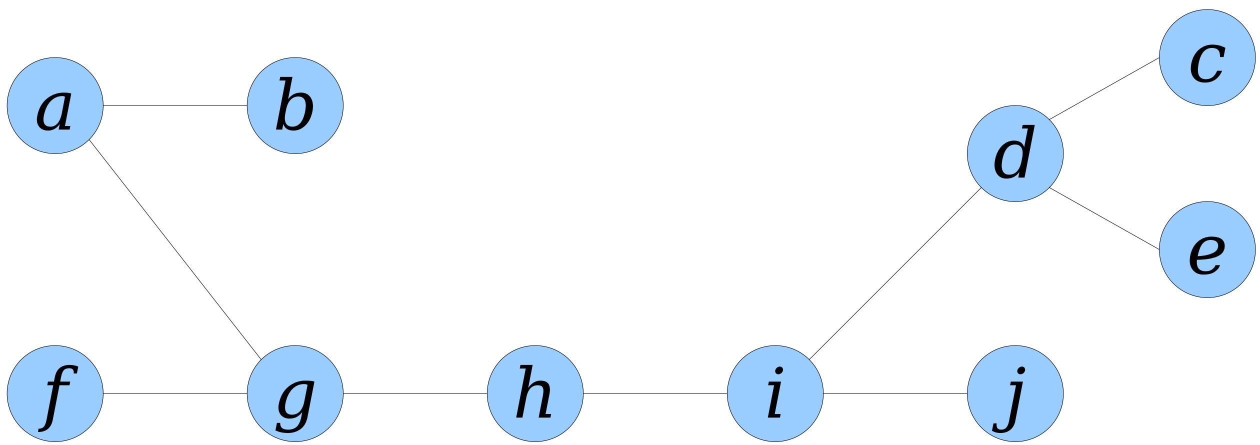
Properties of Euler Tours

- ***Fact:*** Any cyclic shift of an Euler tour of a tree is also an Euler tour.



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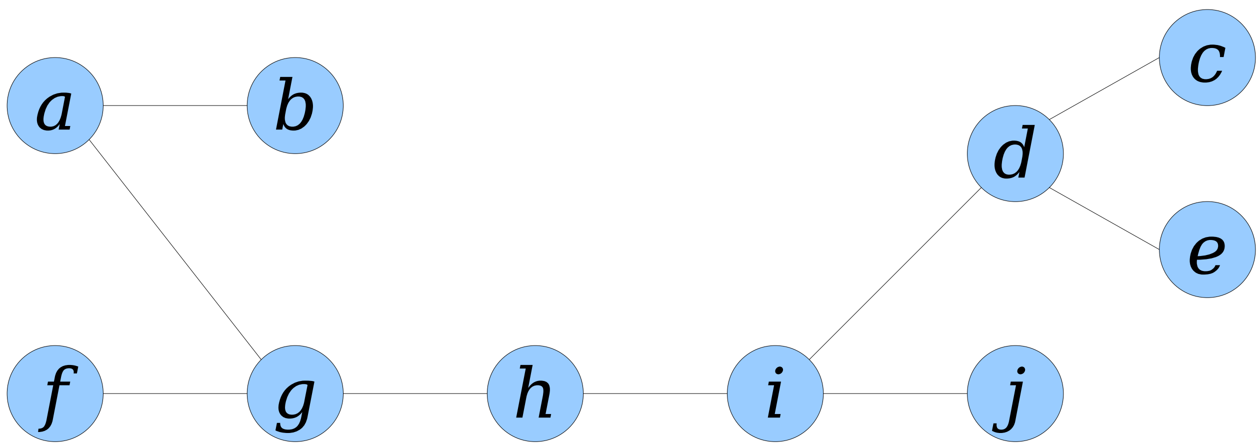

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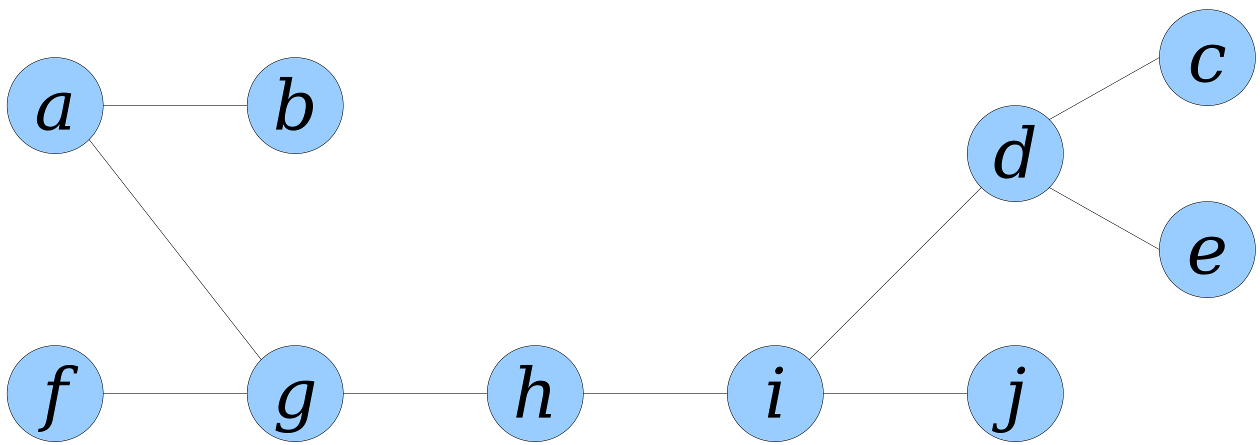

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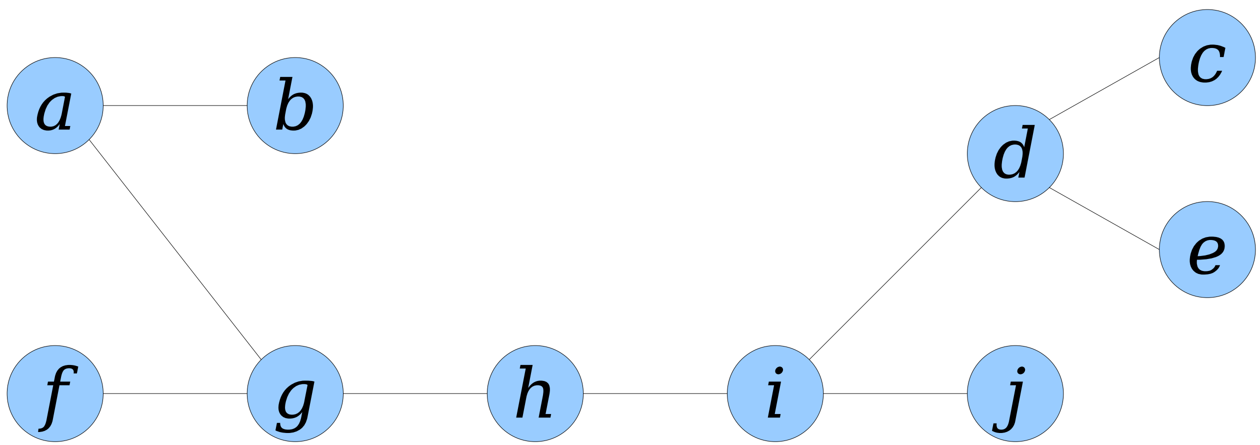

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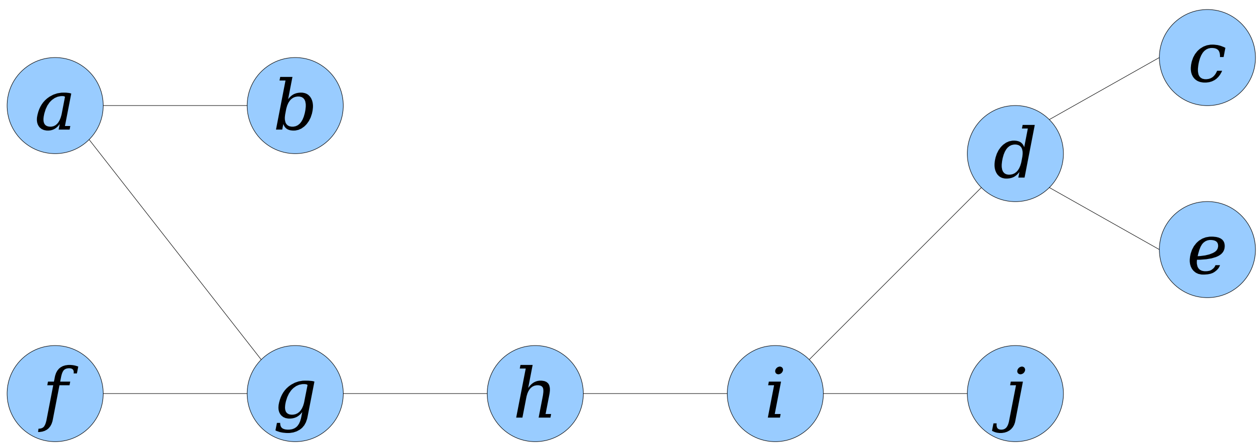

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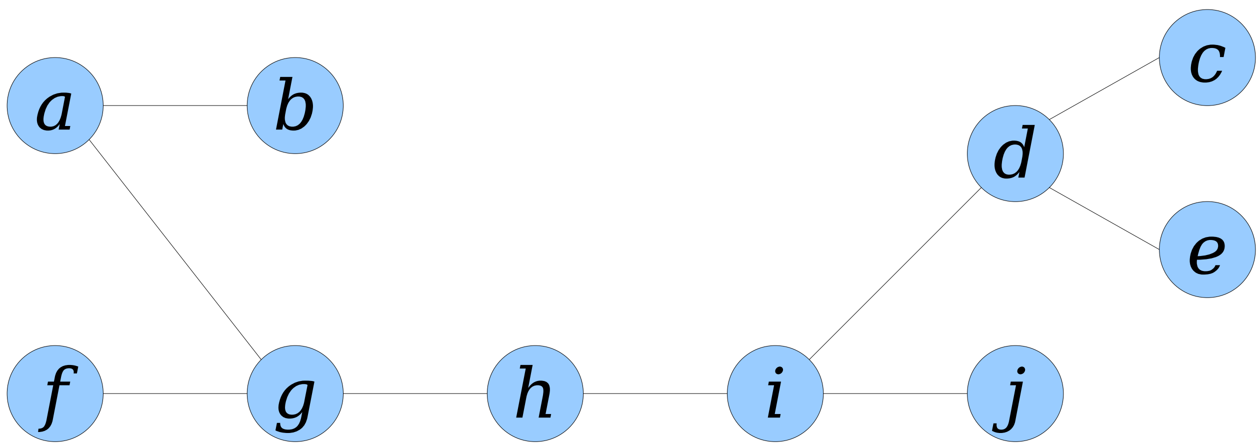

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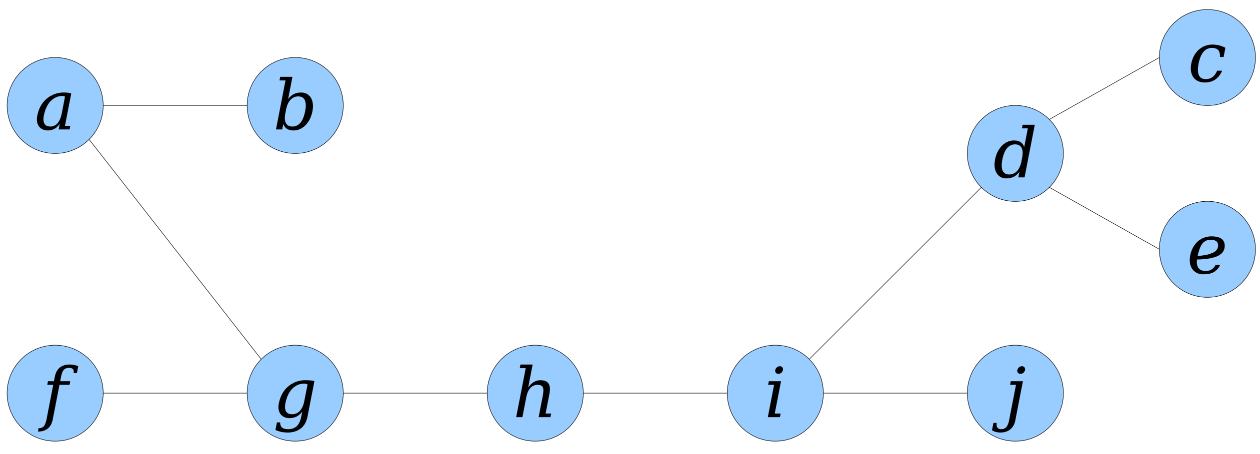

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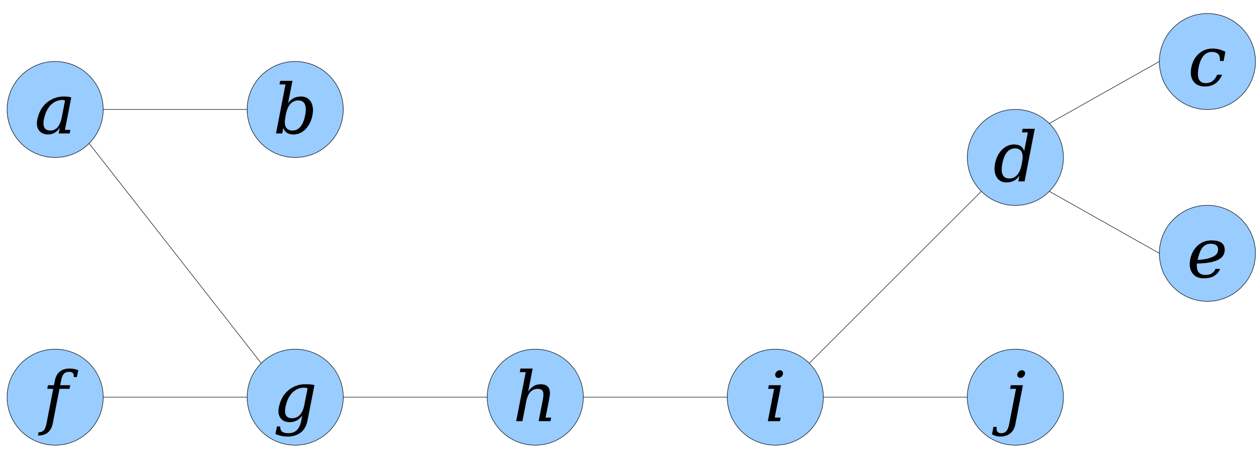

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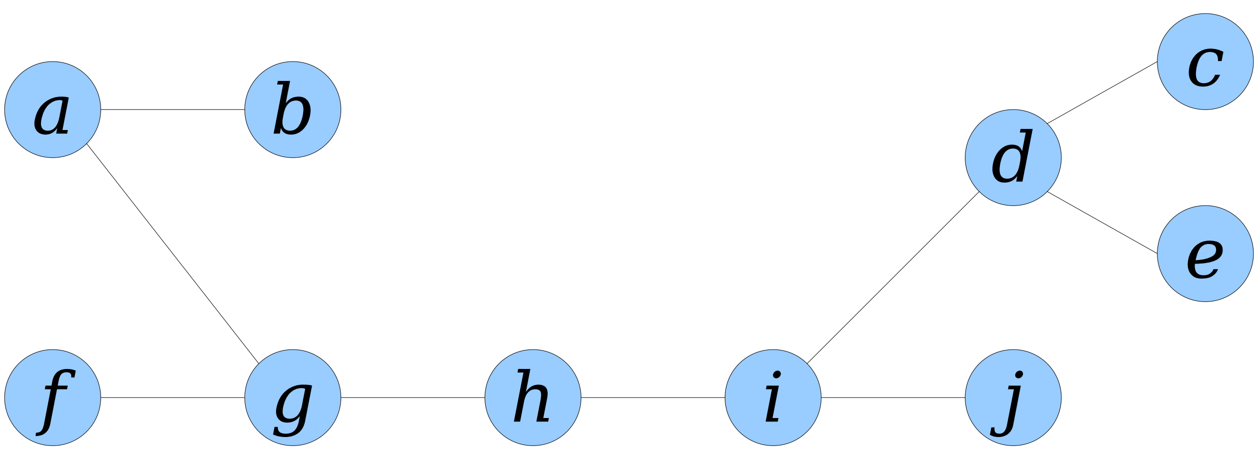

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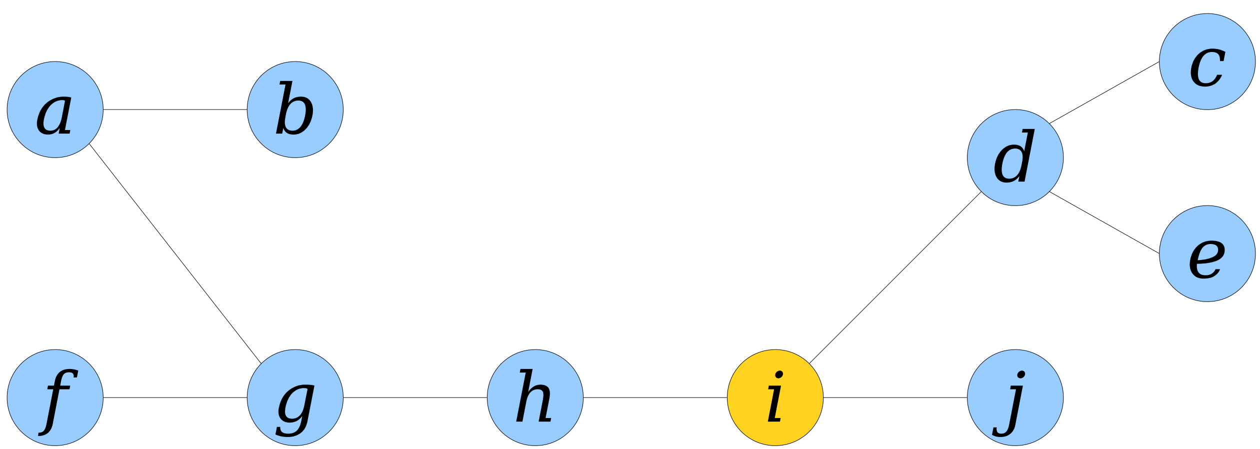

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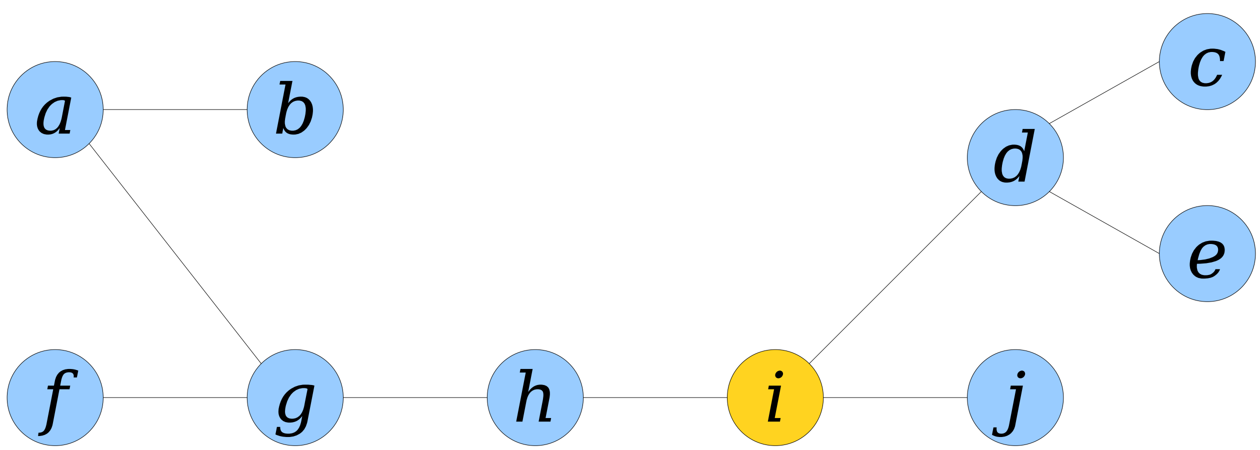

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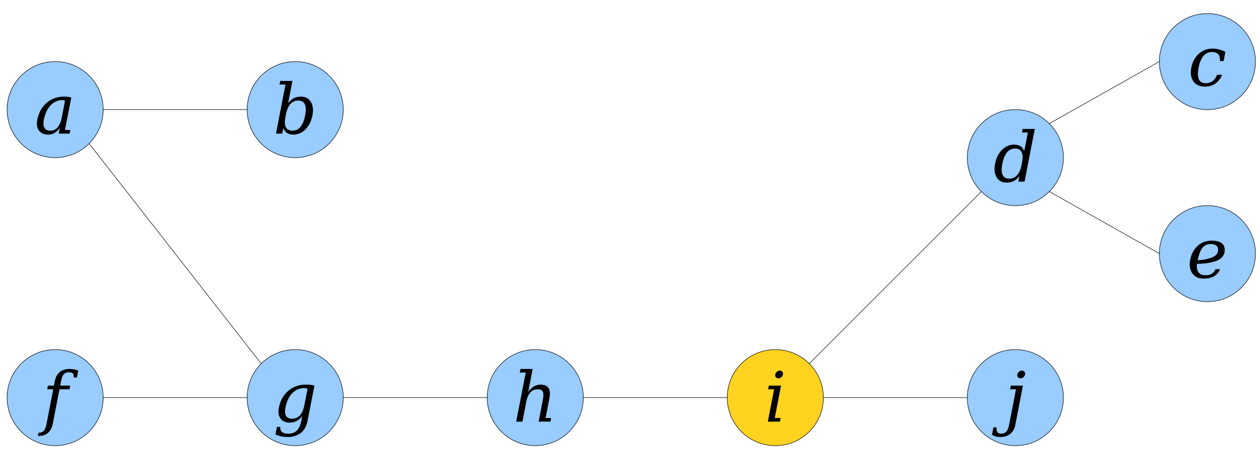

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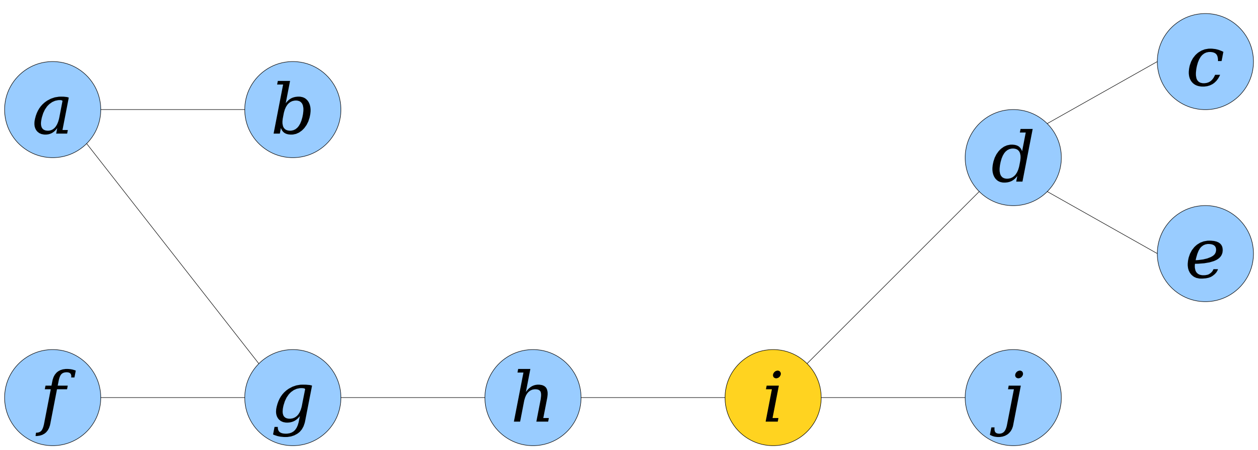

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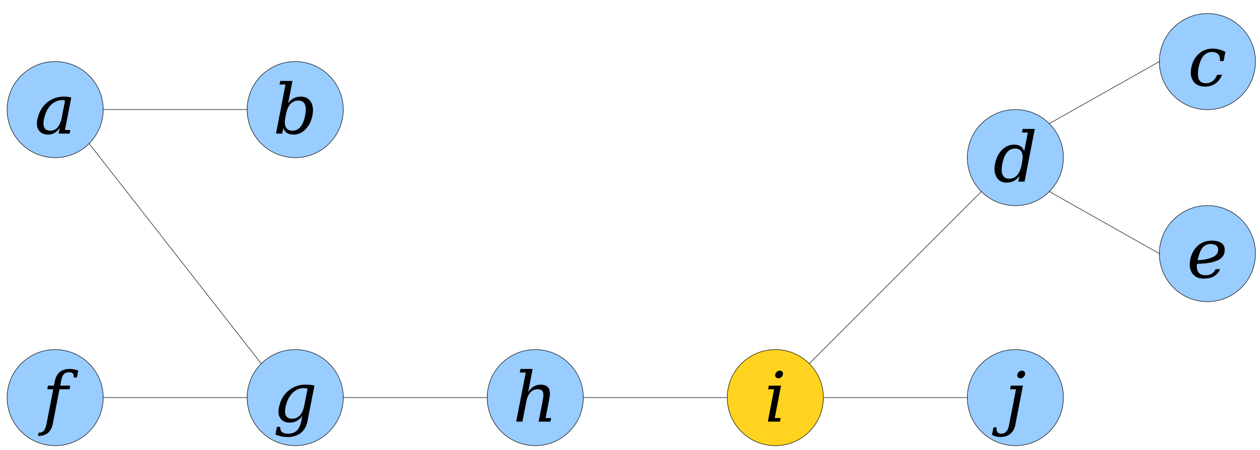

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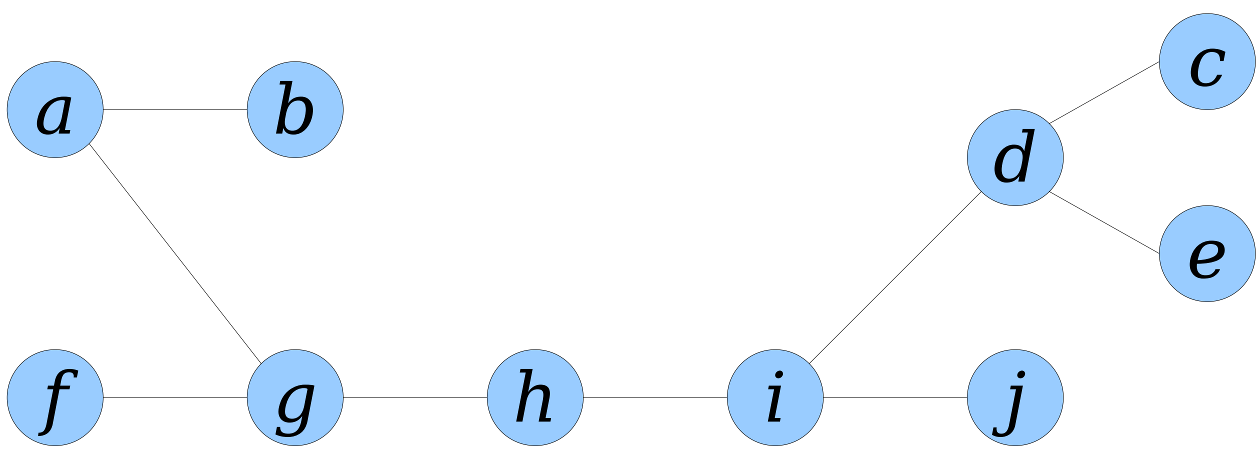

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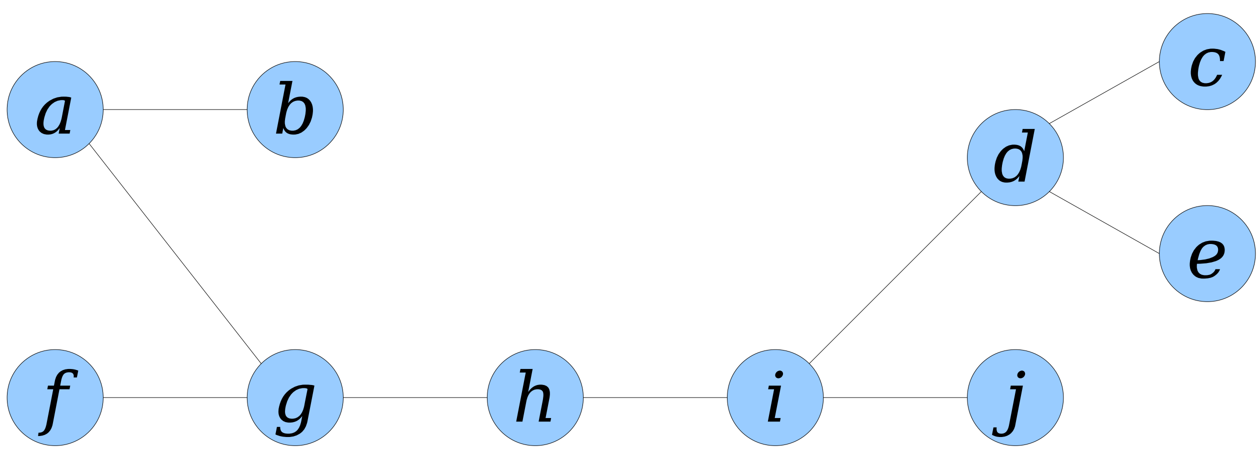

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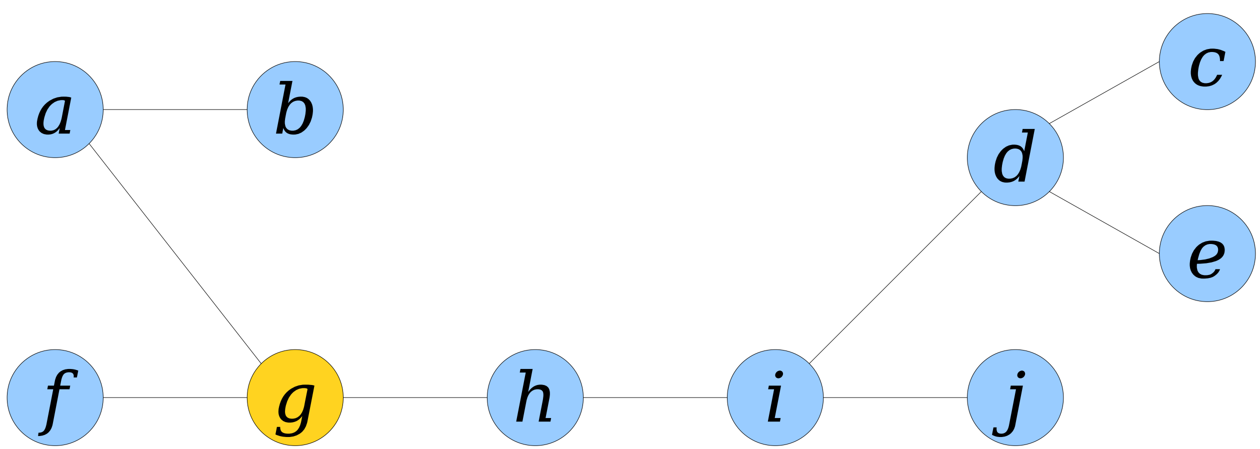

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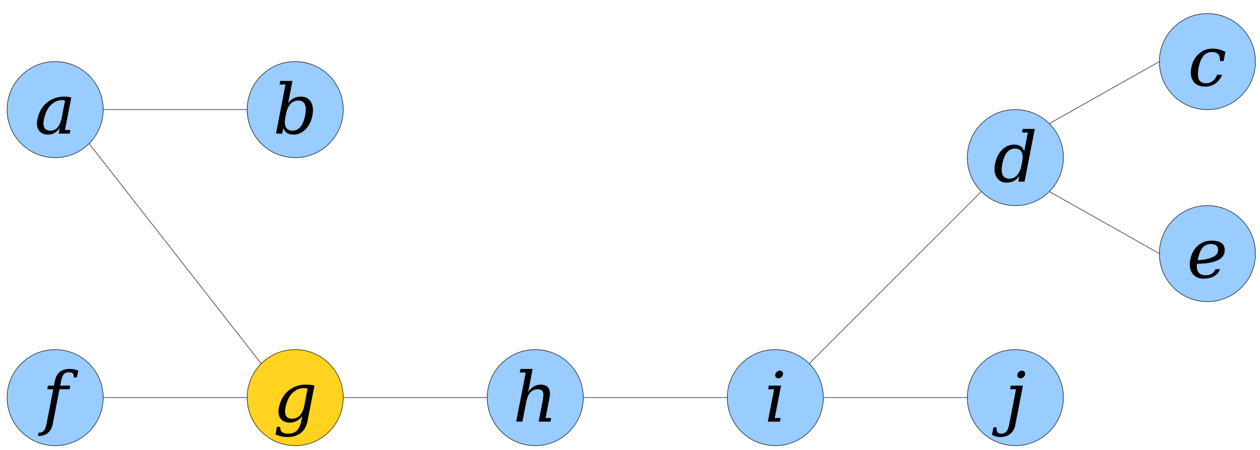

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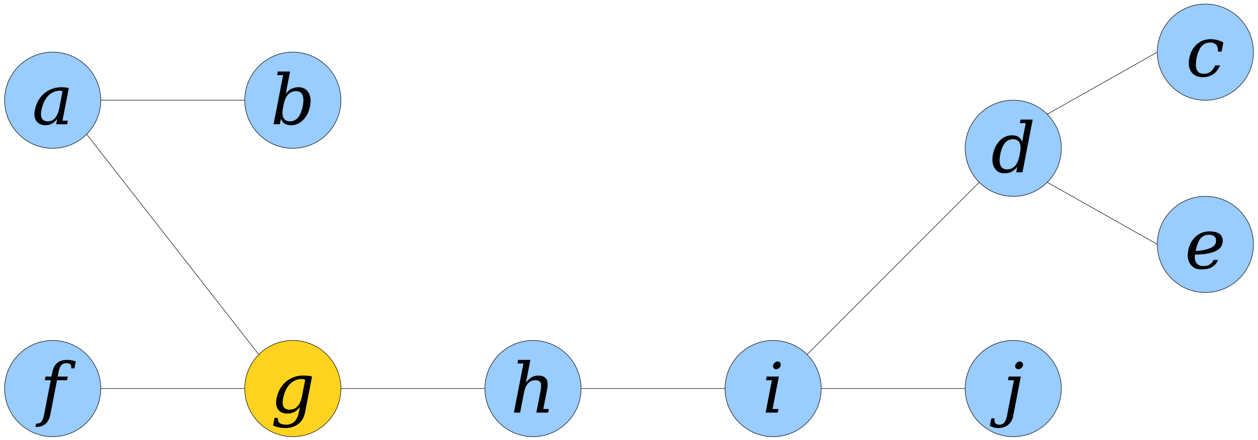

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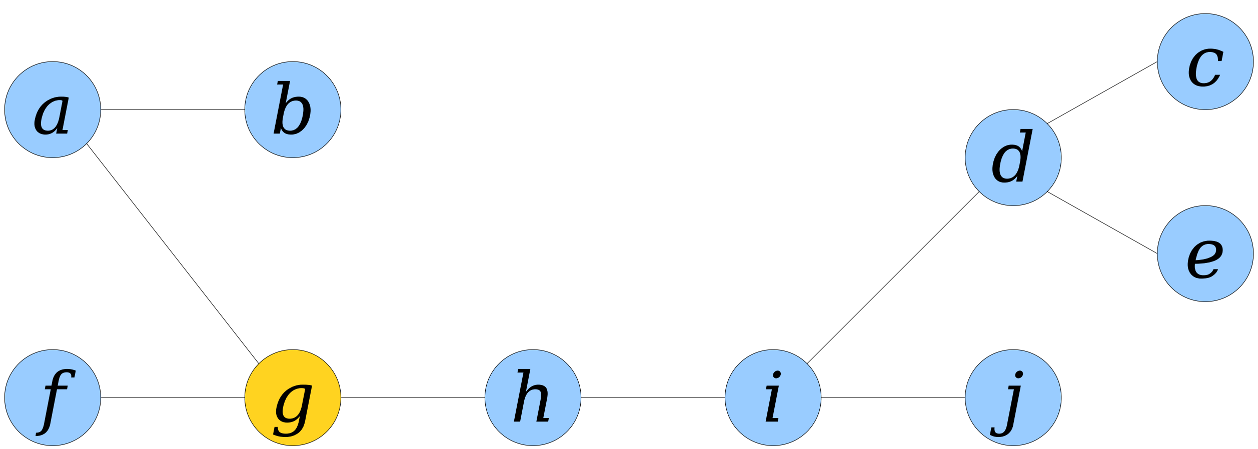

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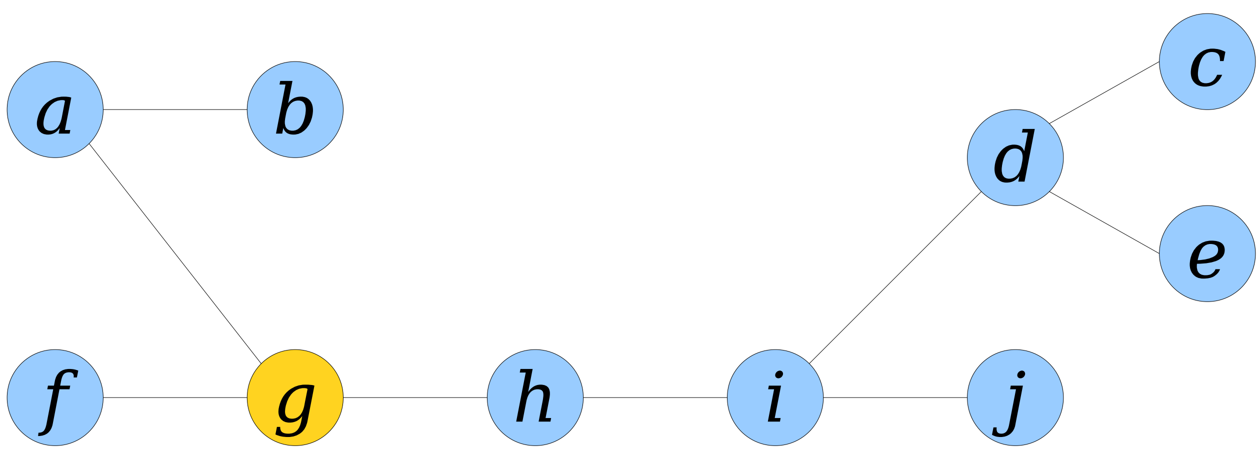

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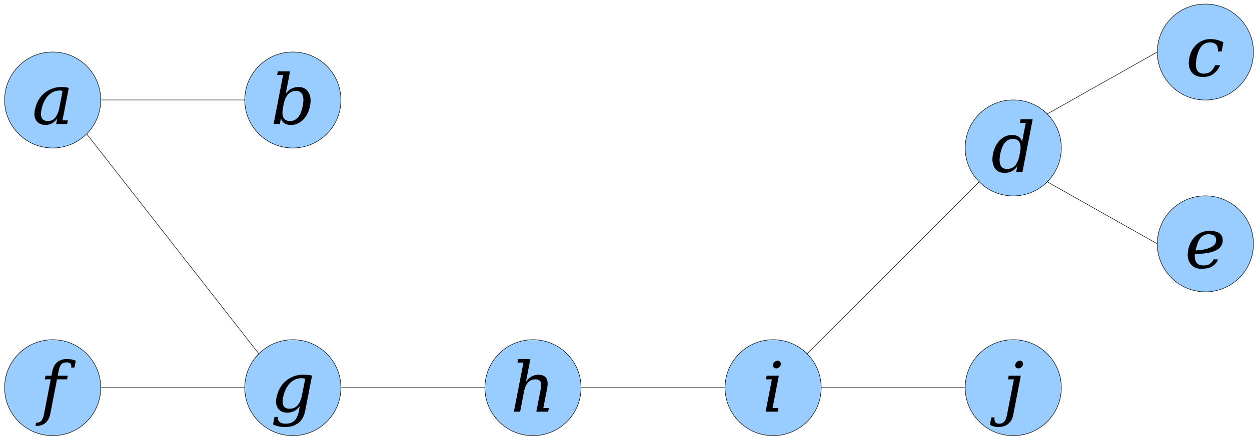

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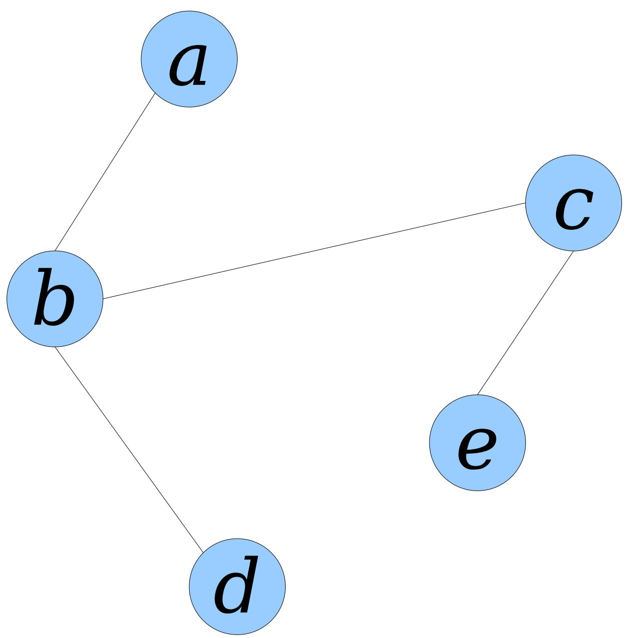

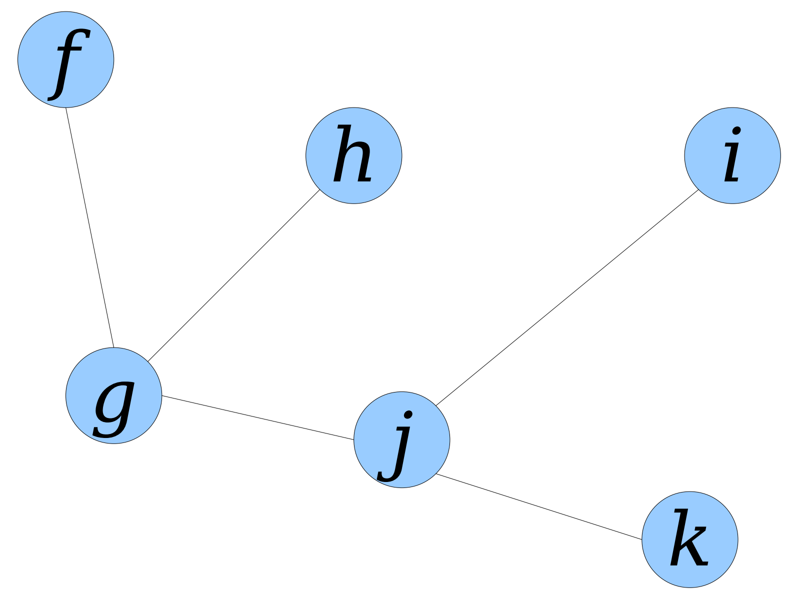

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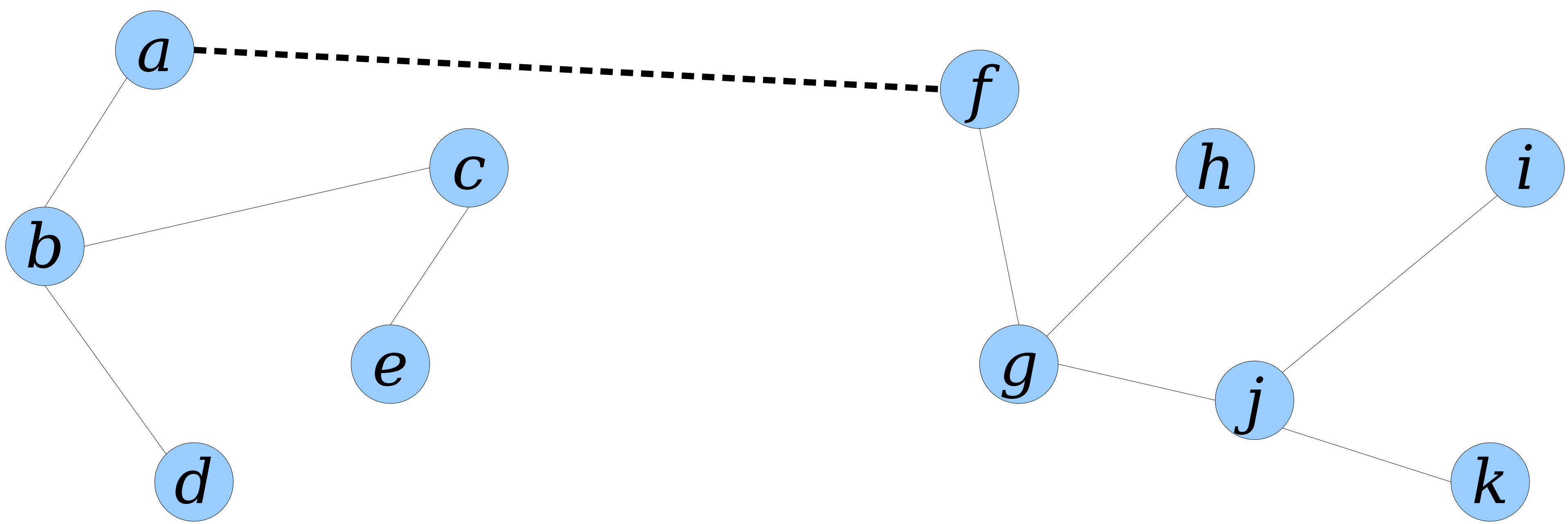

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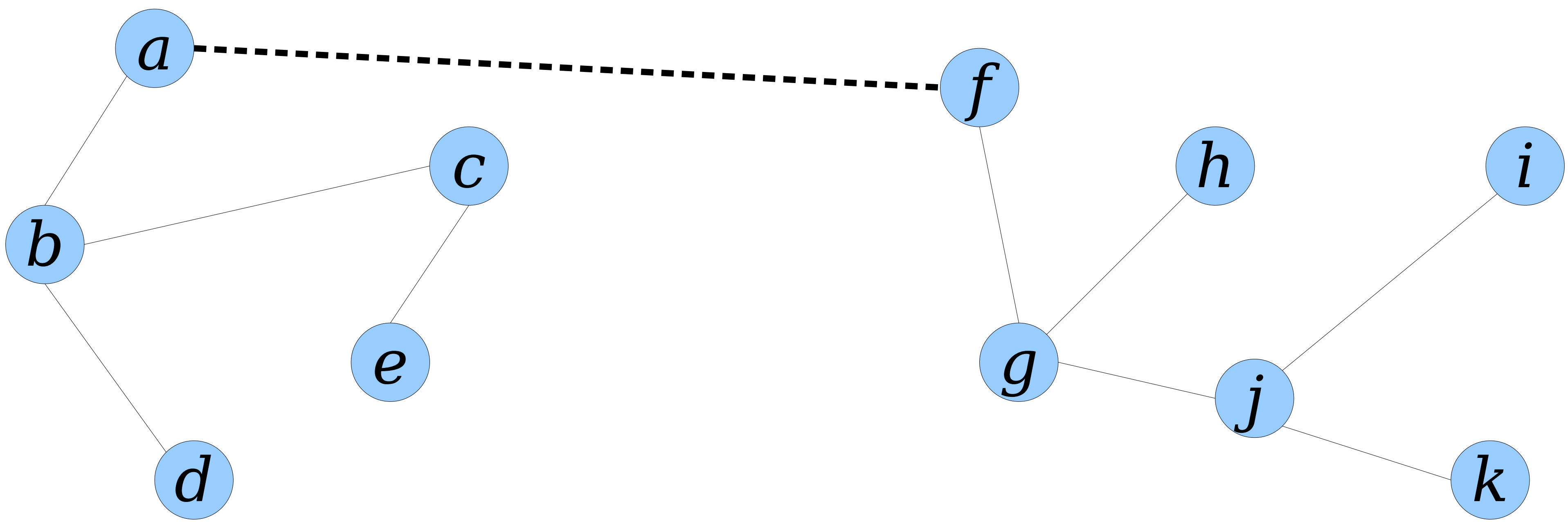

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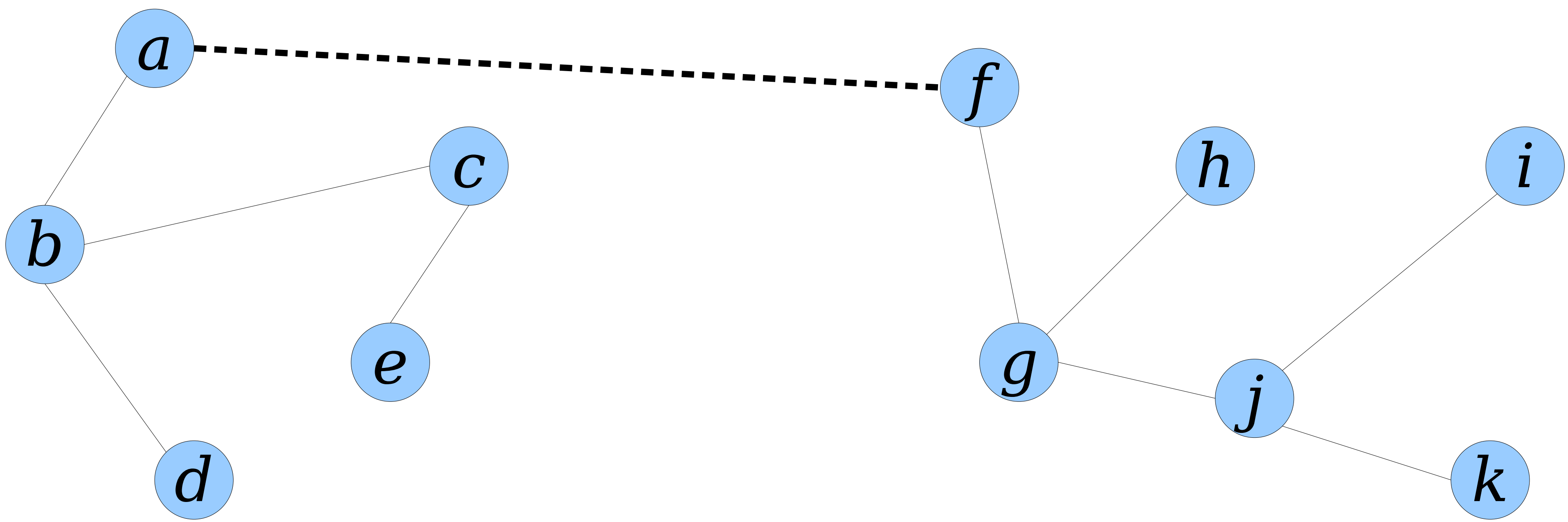

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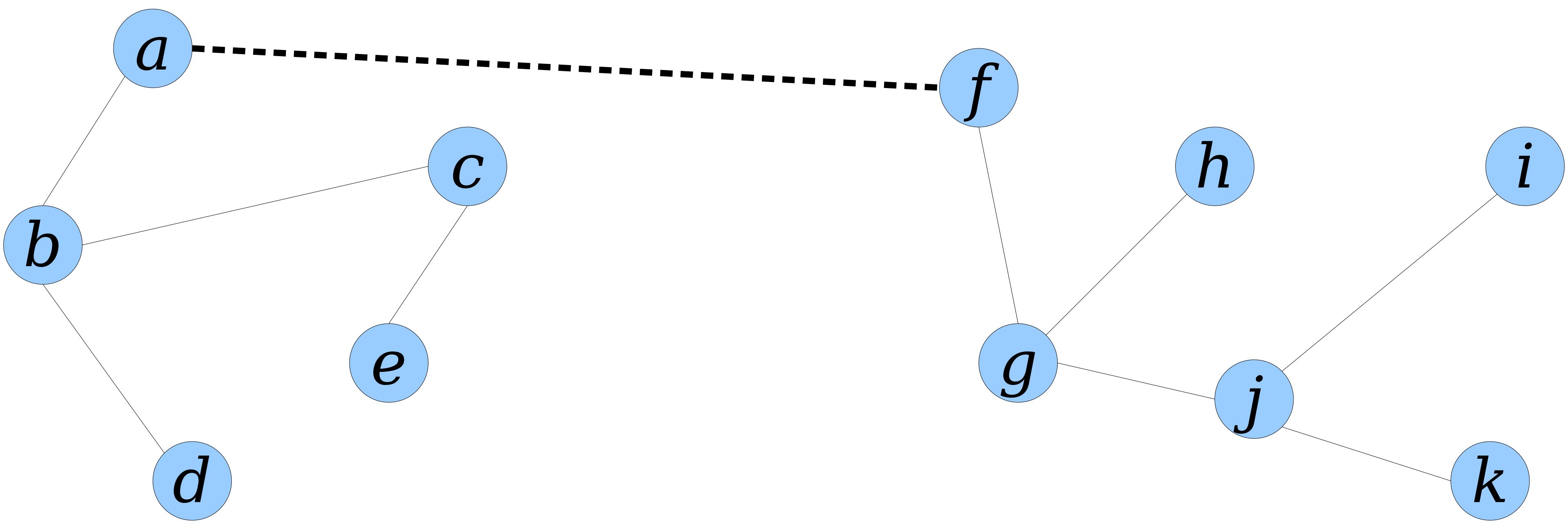

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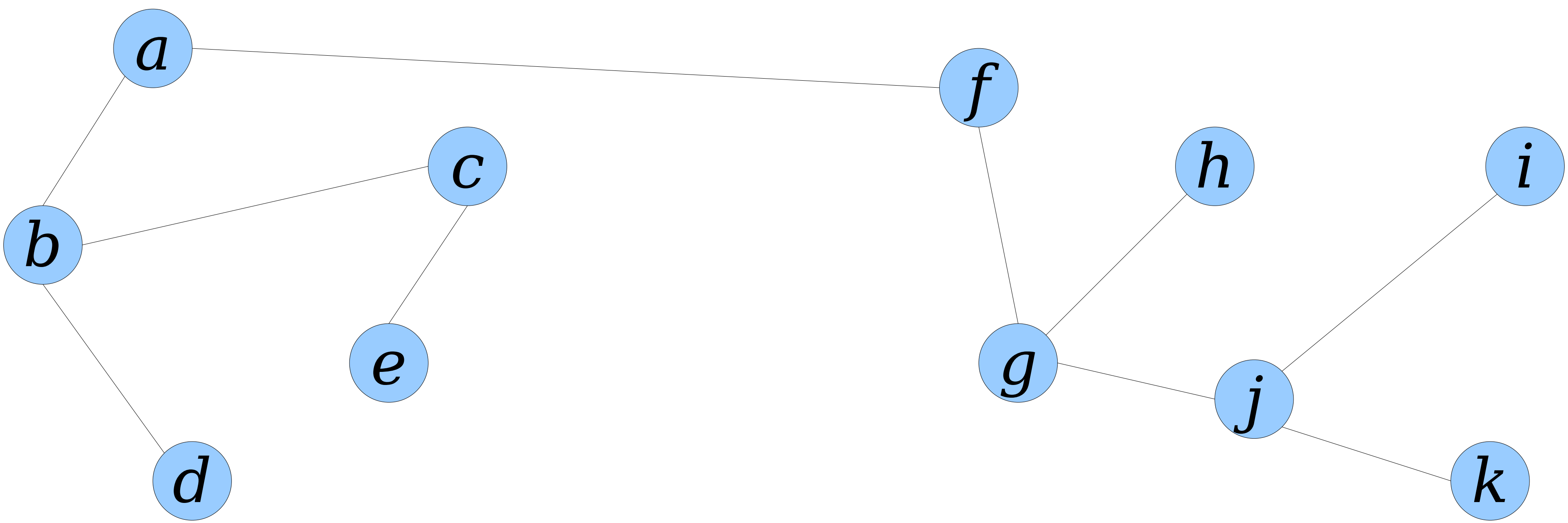

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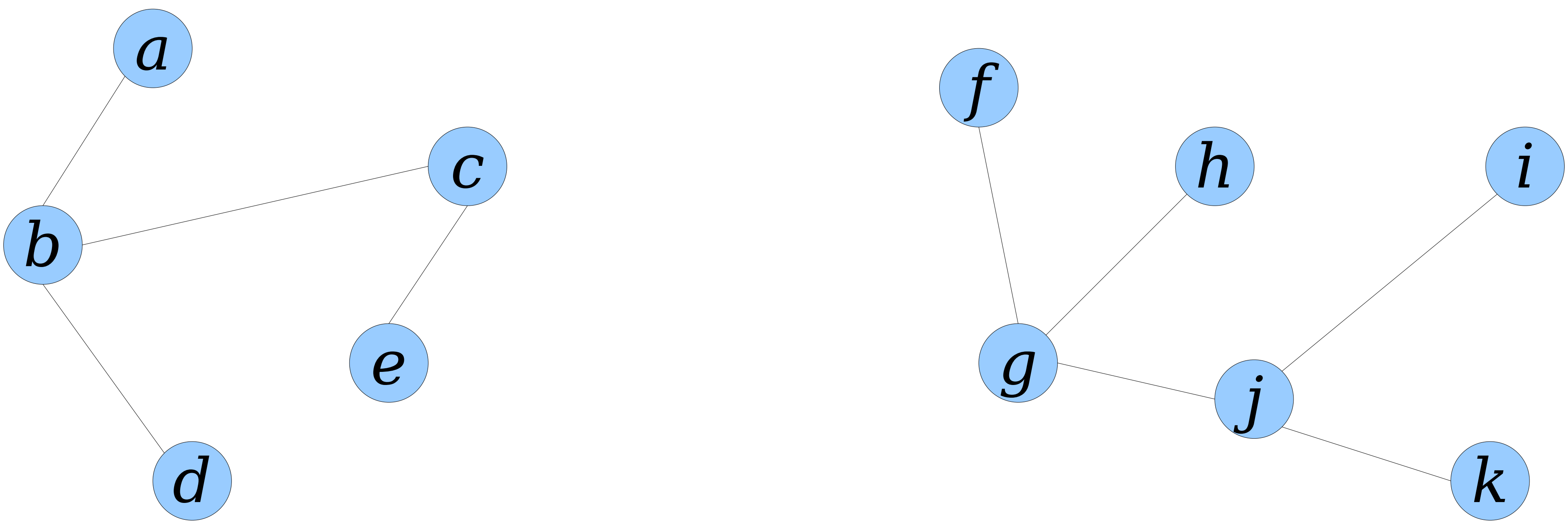

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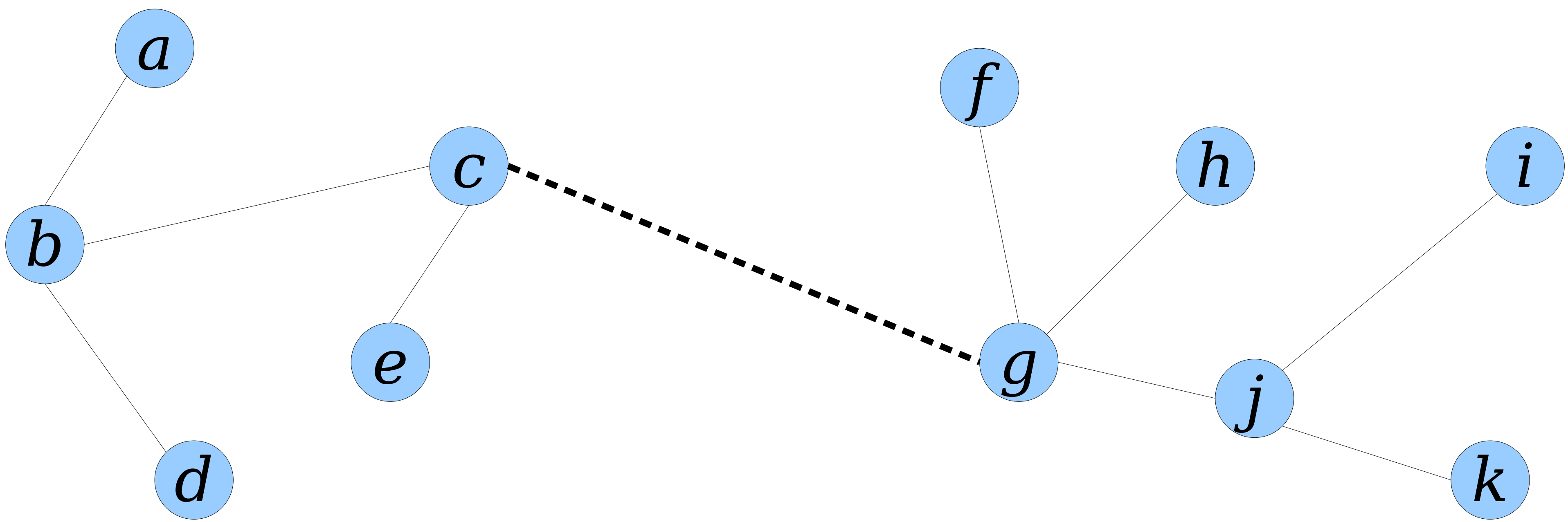

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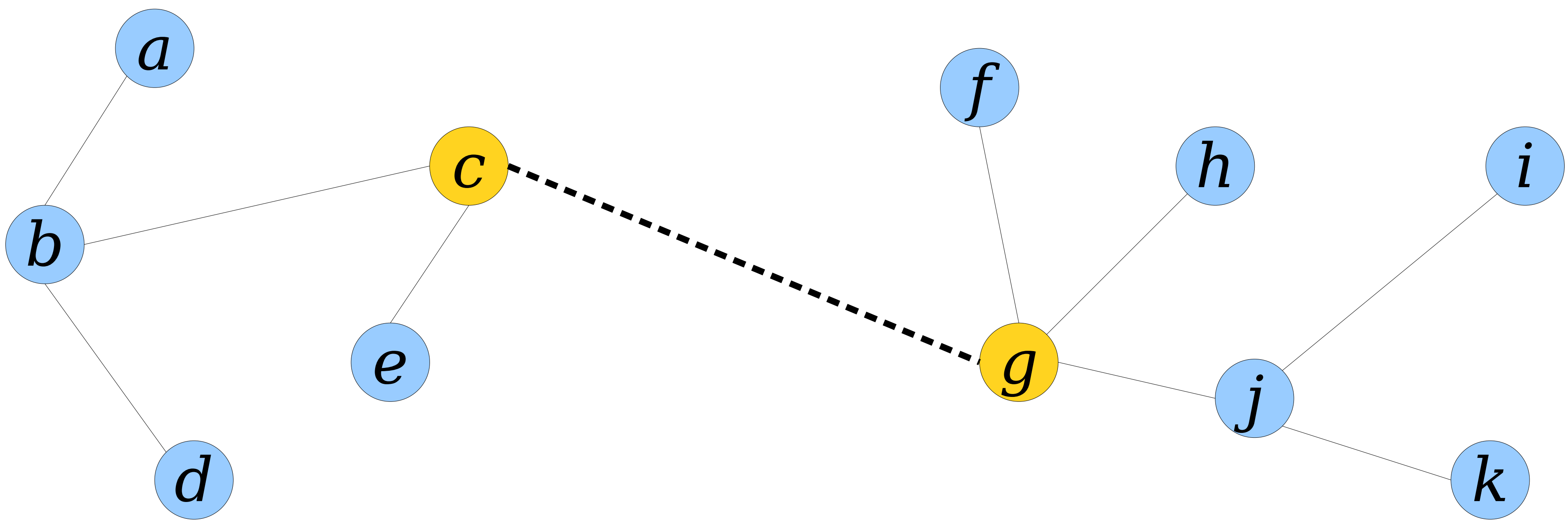

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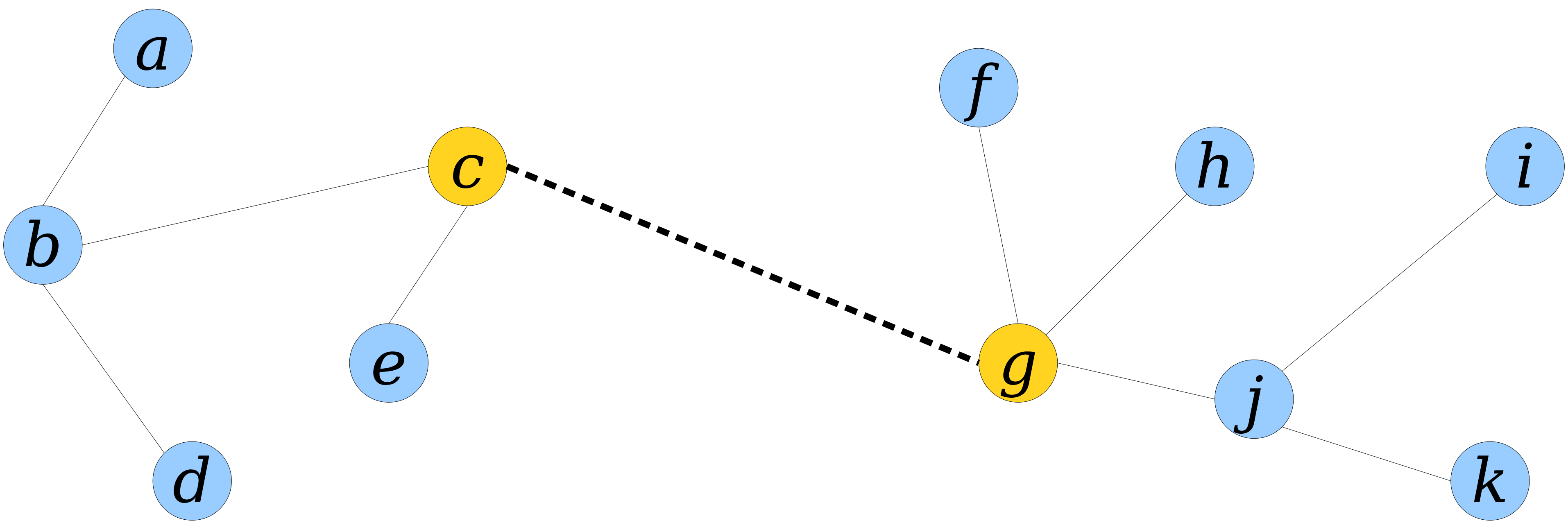

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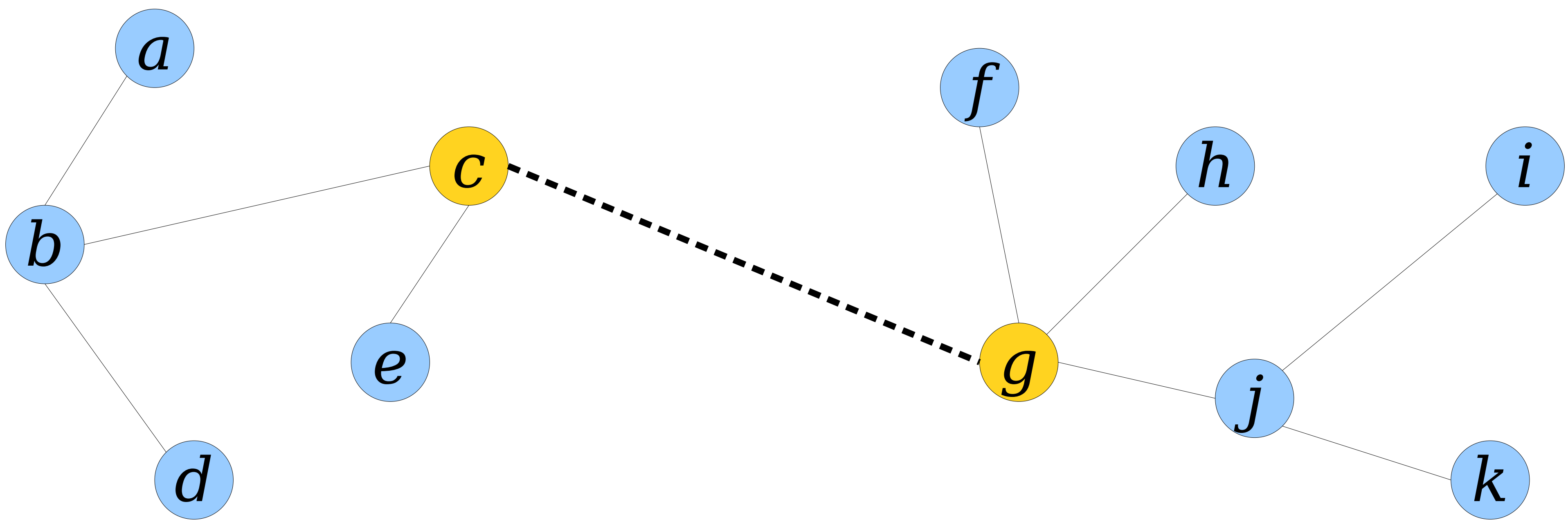

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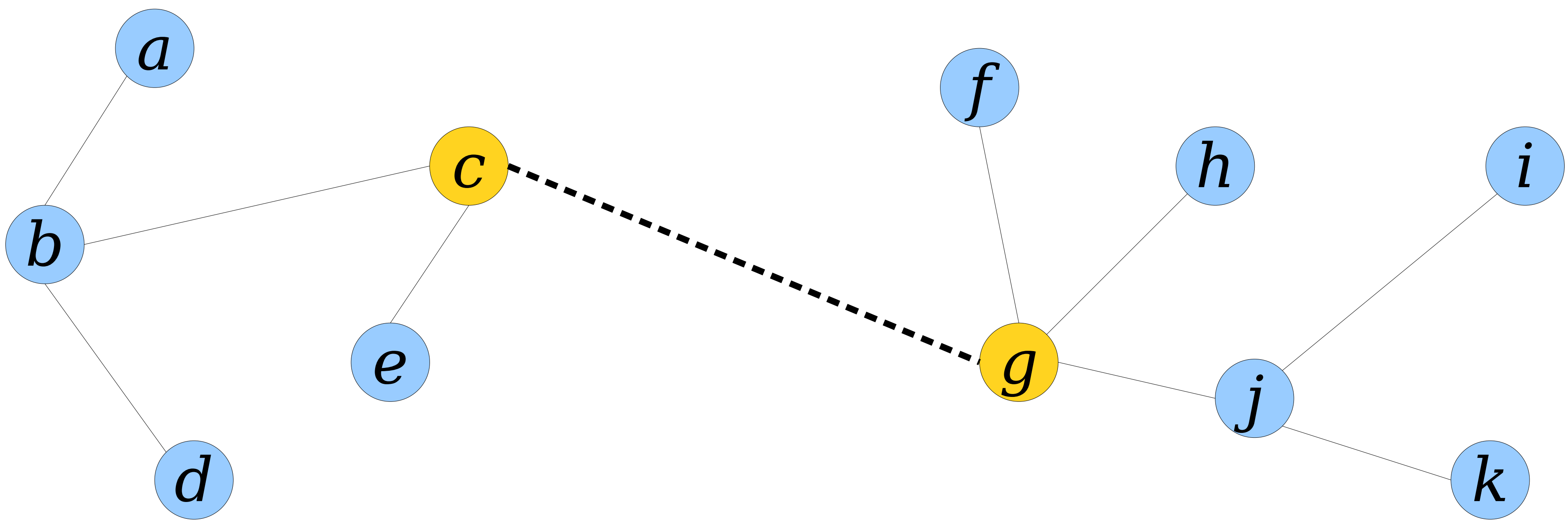

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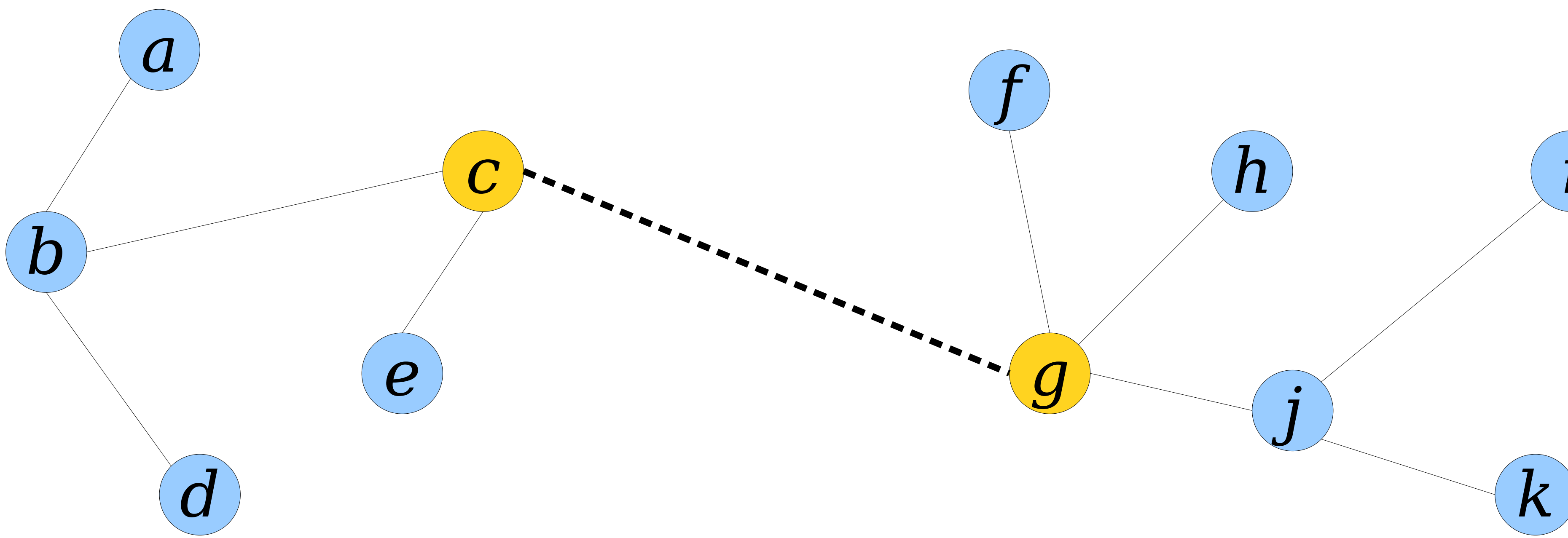

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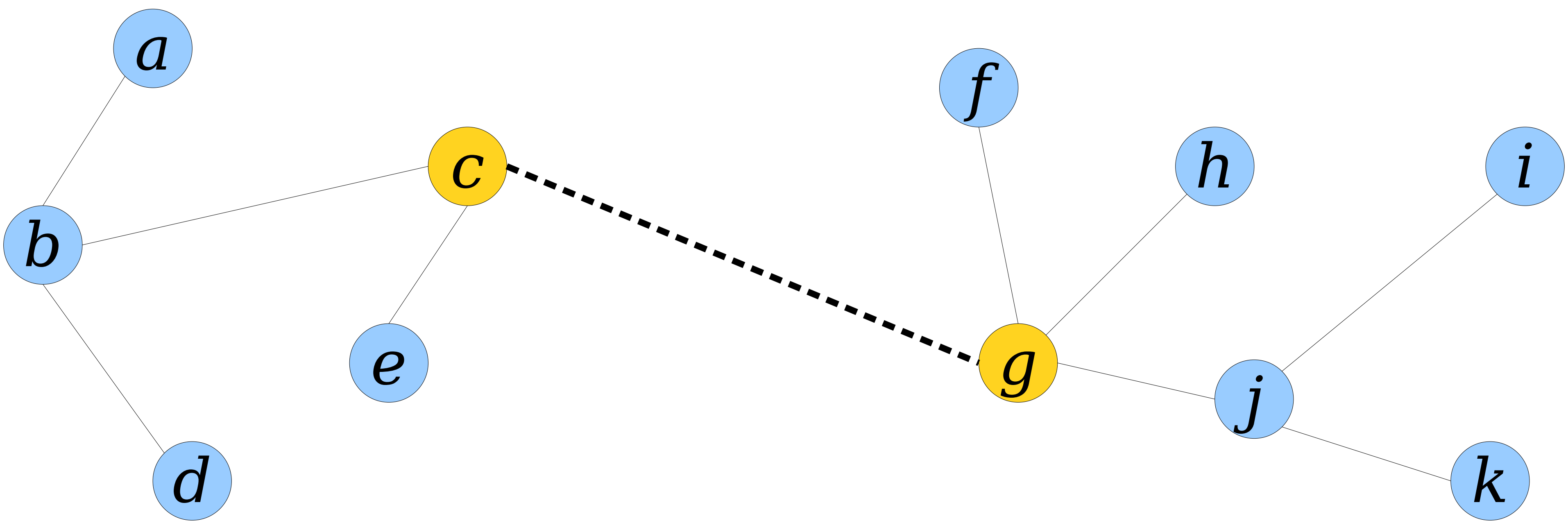

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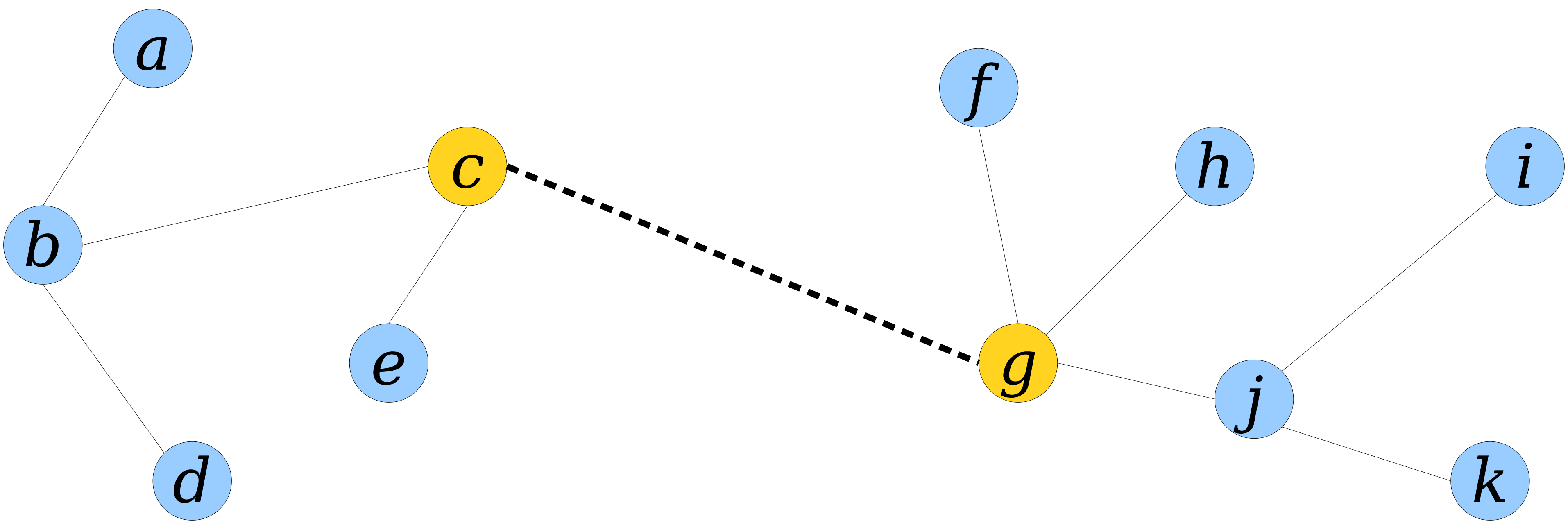

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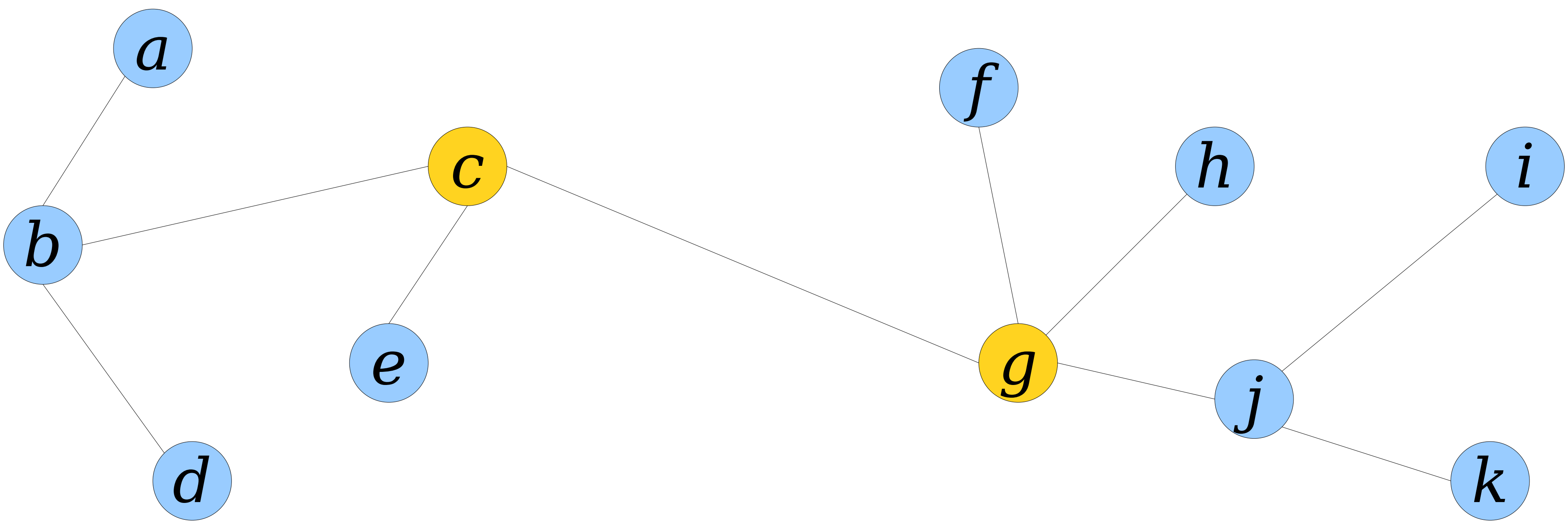

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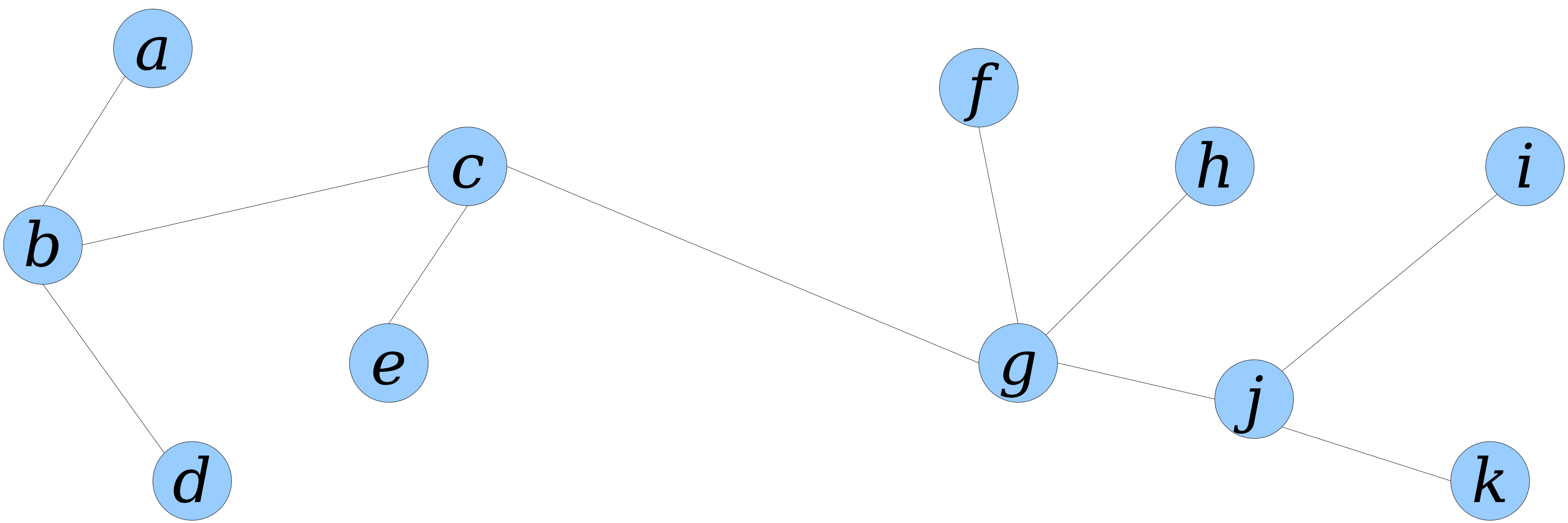

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- Given two trees T_1 and T_2, where $u \in T_1$ and $v \in T_2$, executing ***link***(u, v) links the trees together by adding edge uv.
- To ***link***(u, v):
  - Let E_1 and E_2 be Euler tours of T_1 and T_2, respectively.
  - ***reroot***(u).
  - ***reroot***(v).
  - Concatenate E_1 uv E_2 vu.

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- Given a tree T, executing ***cut***(u, v) cuts the edge uv from the tree (assuming it exists).
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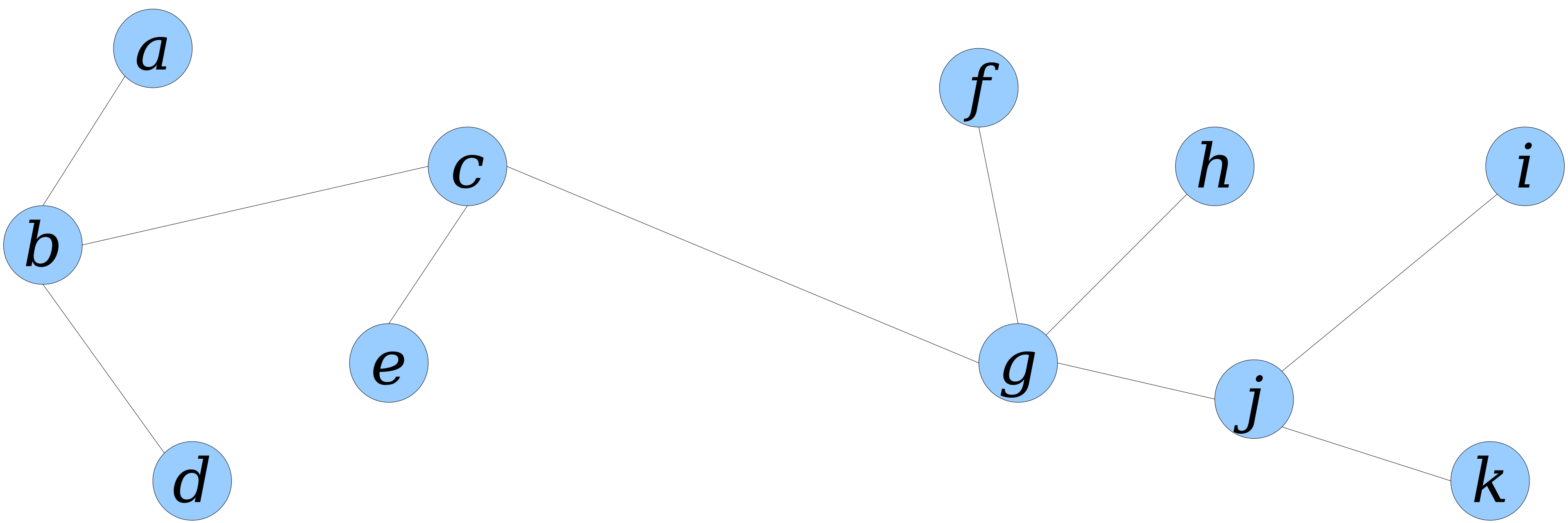

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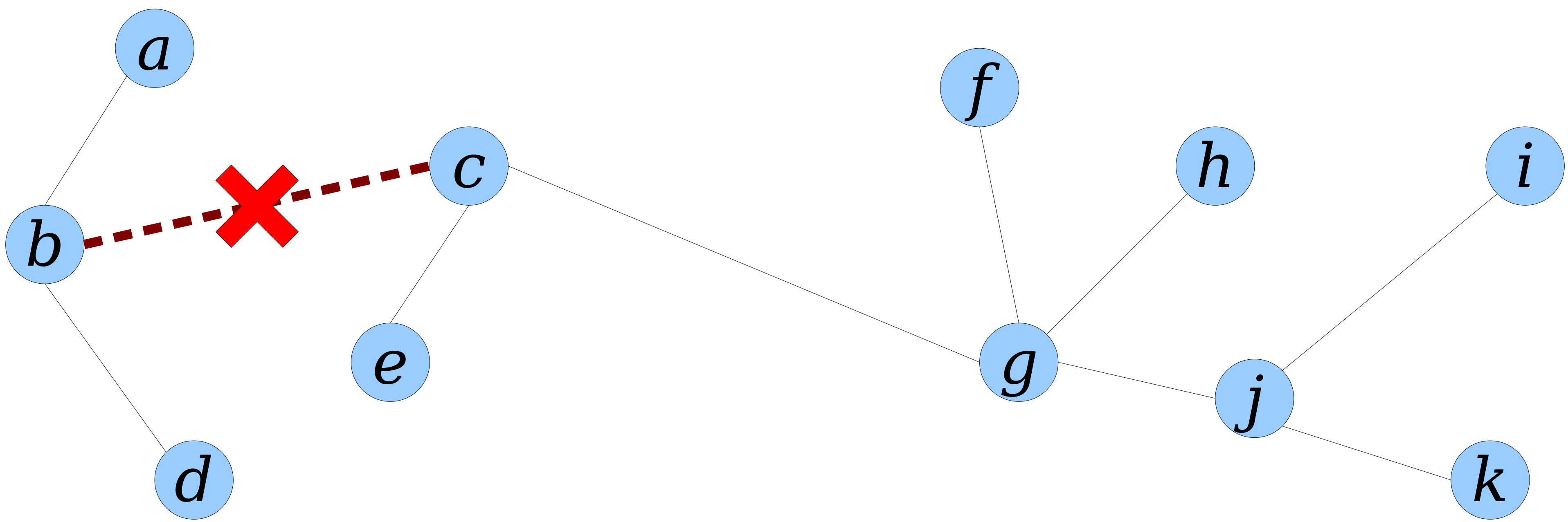

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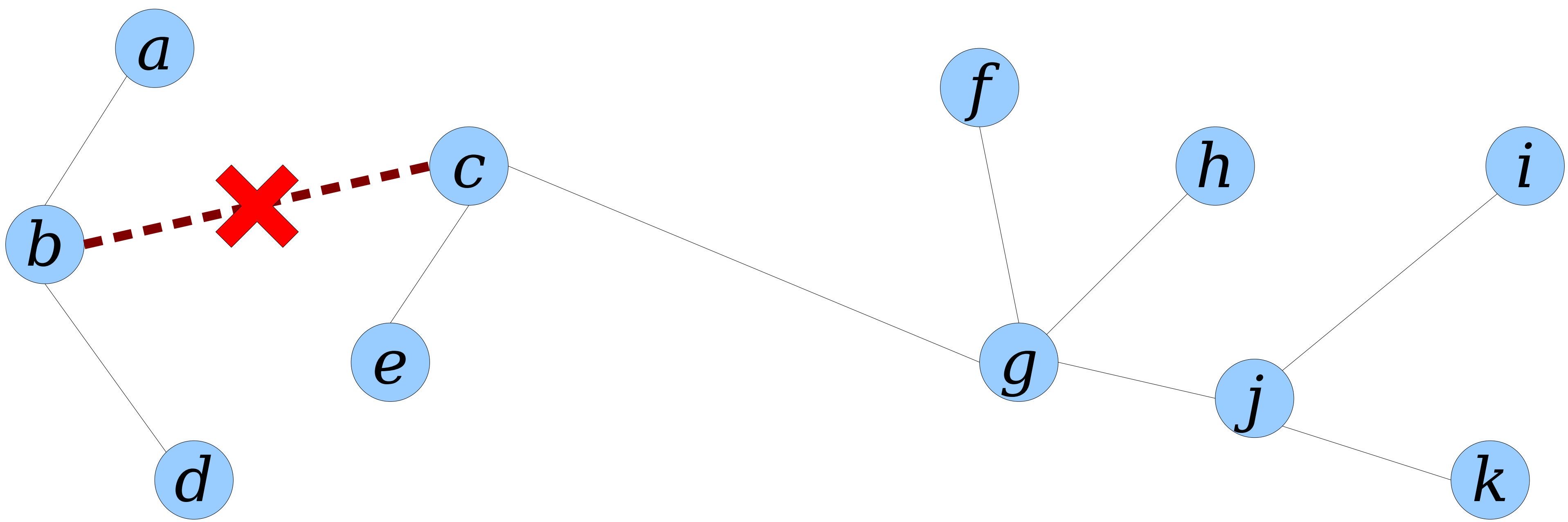

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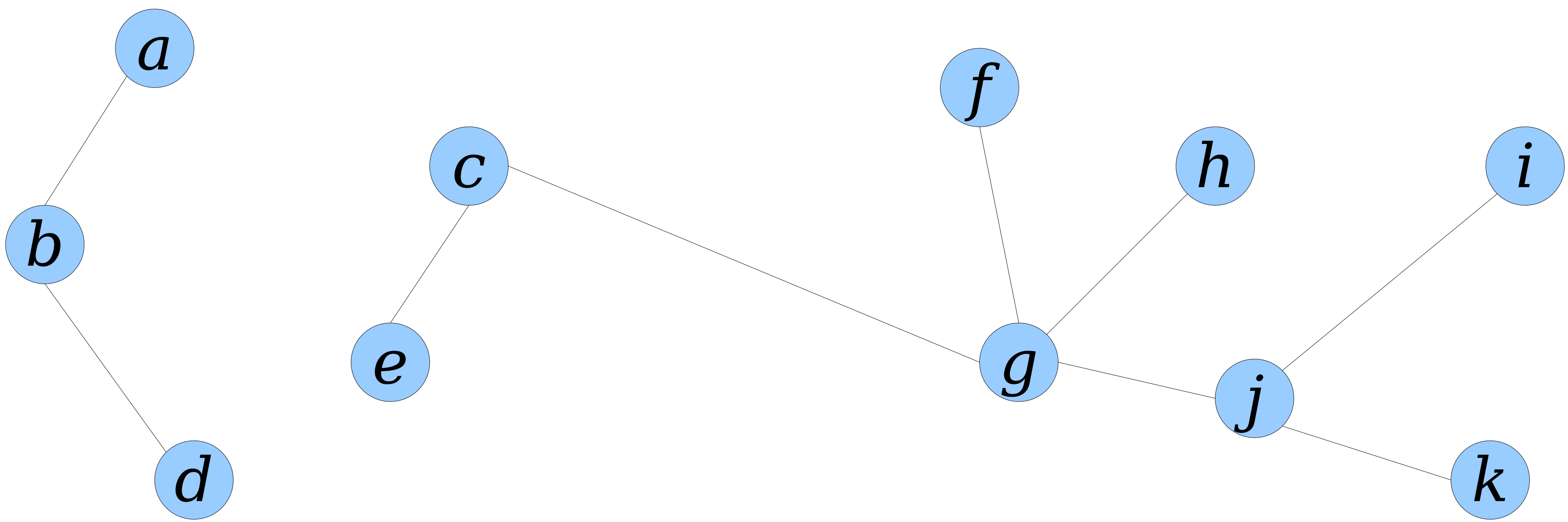

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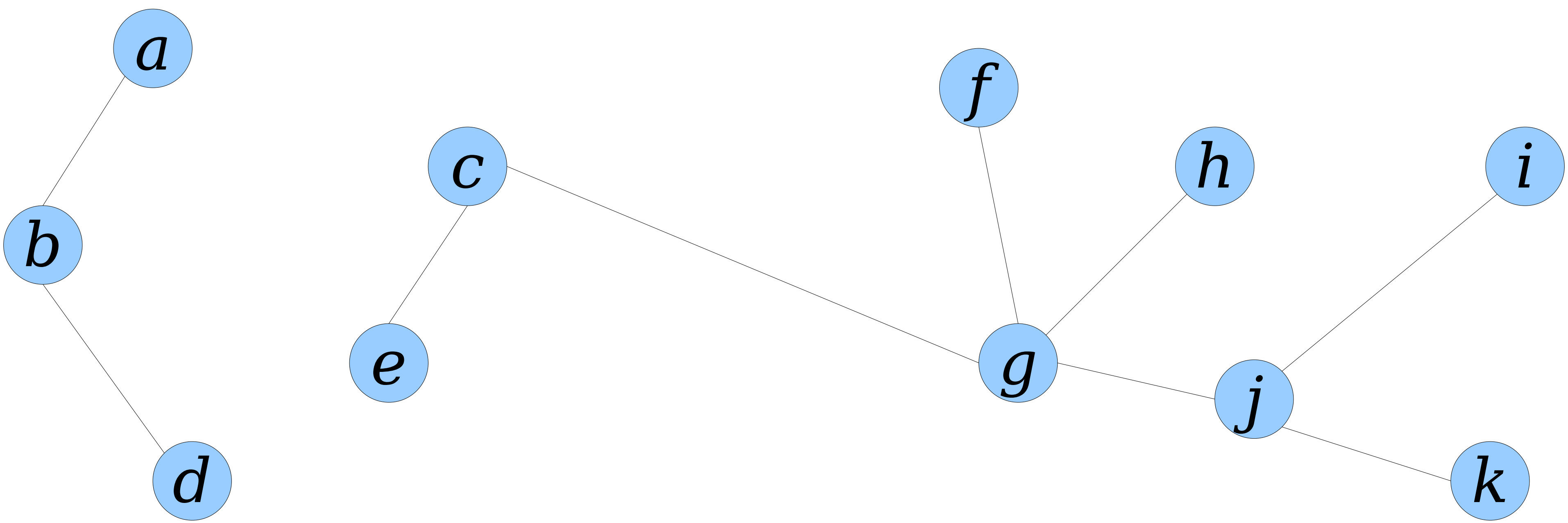

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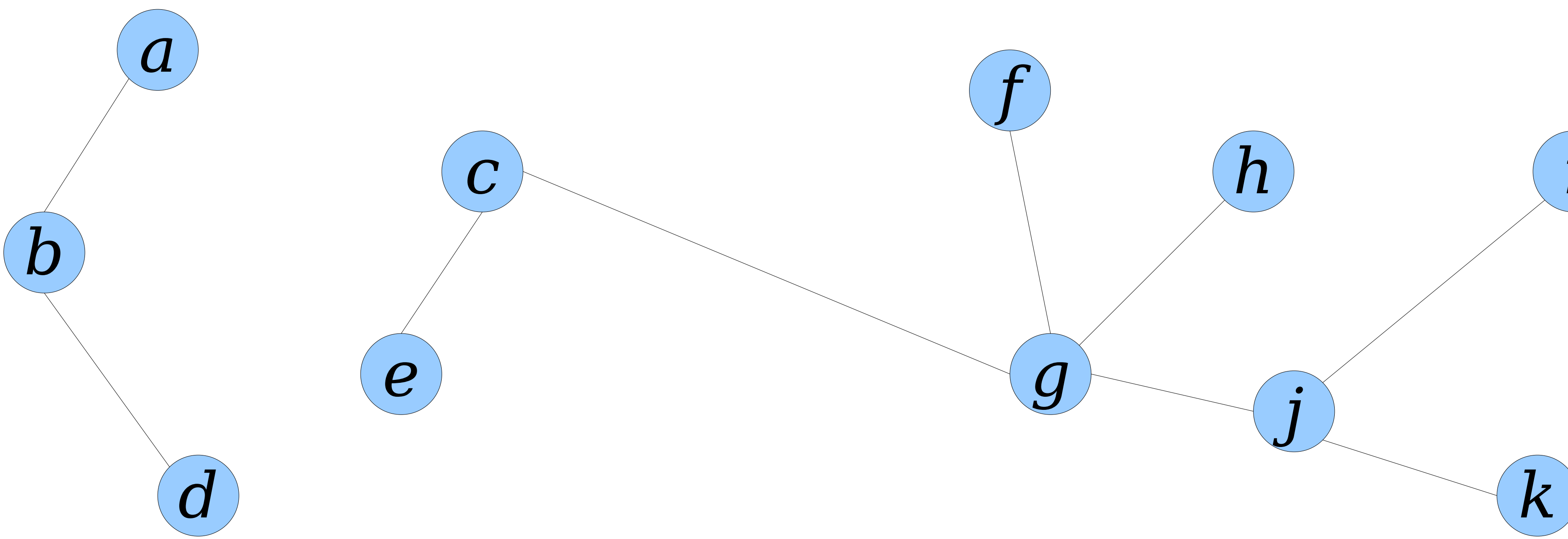

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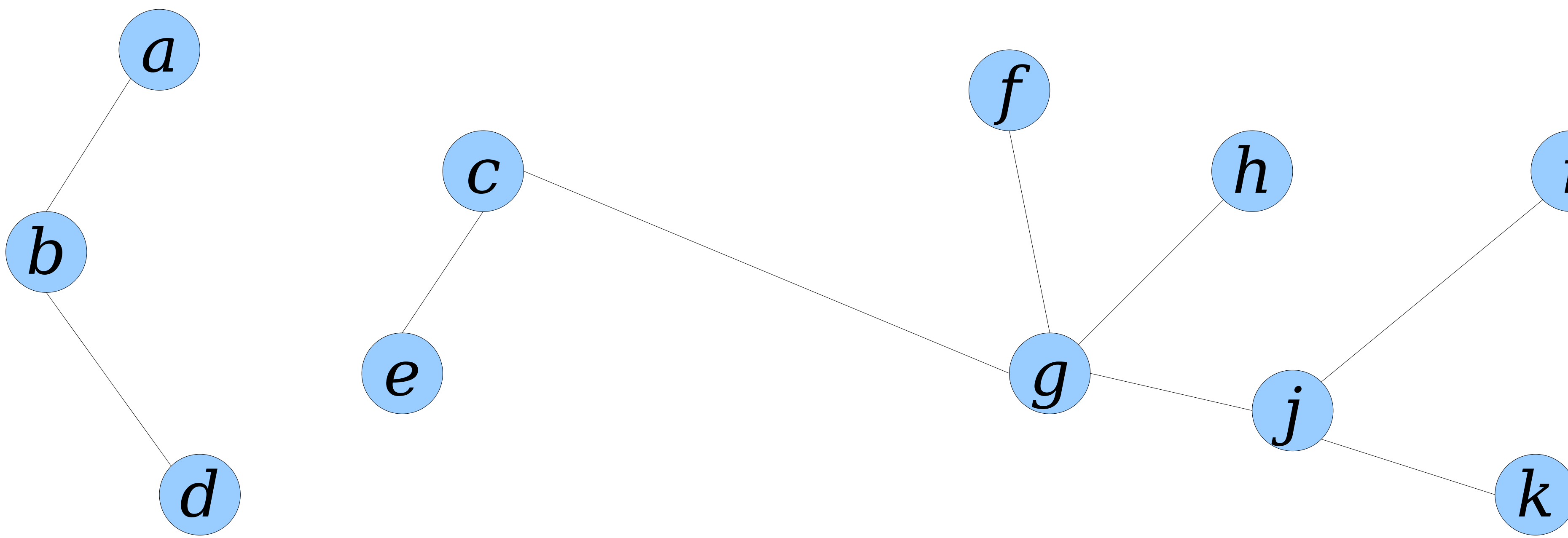

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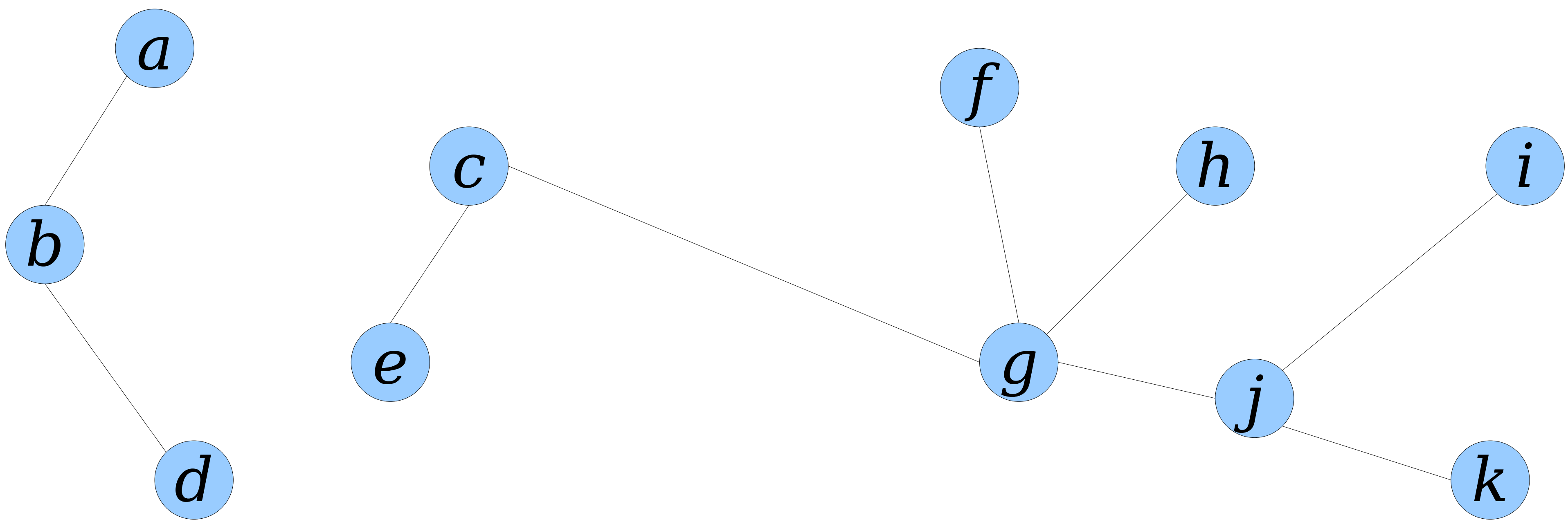

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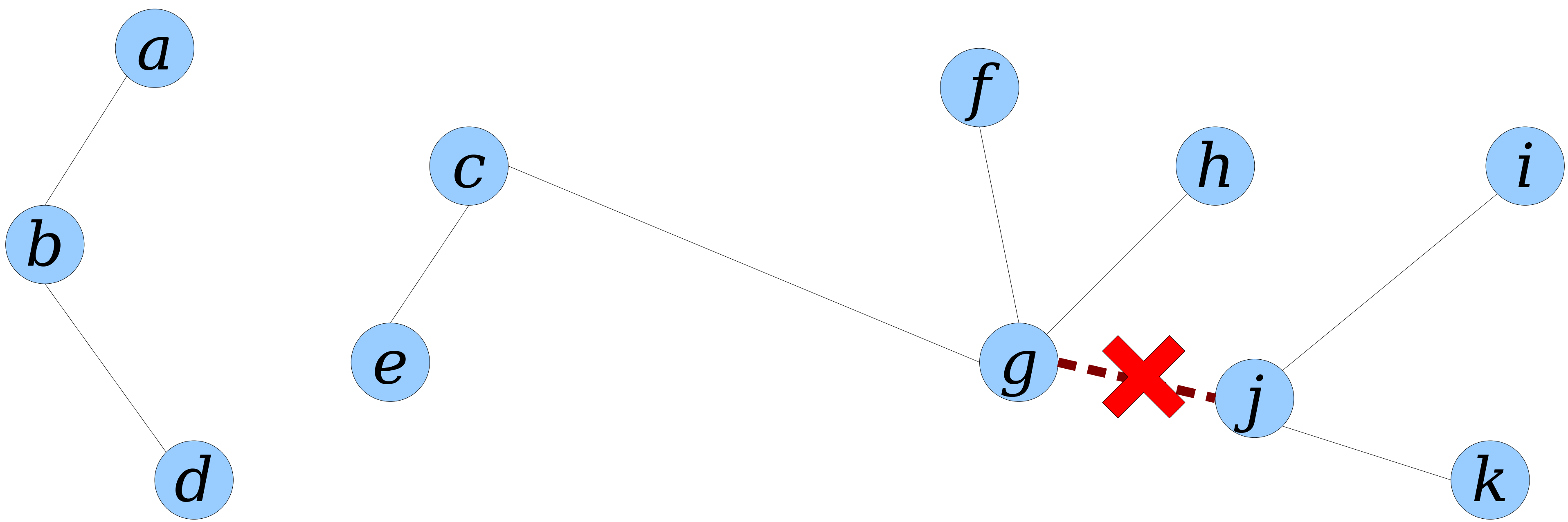

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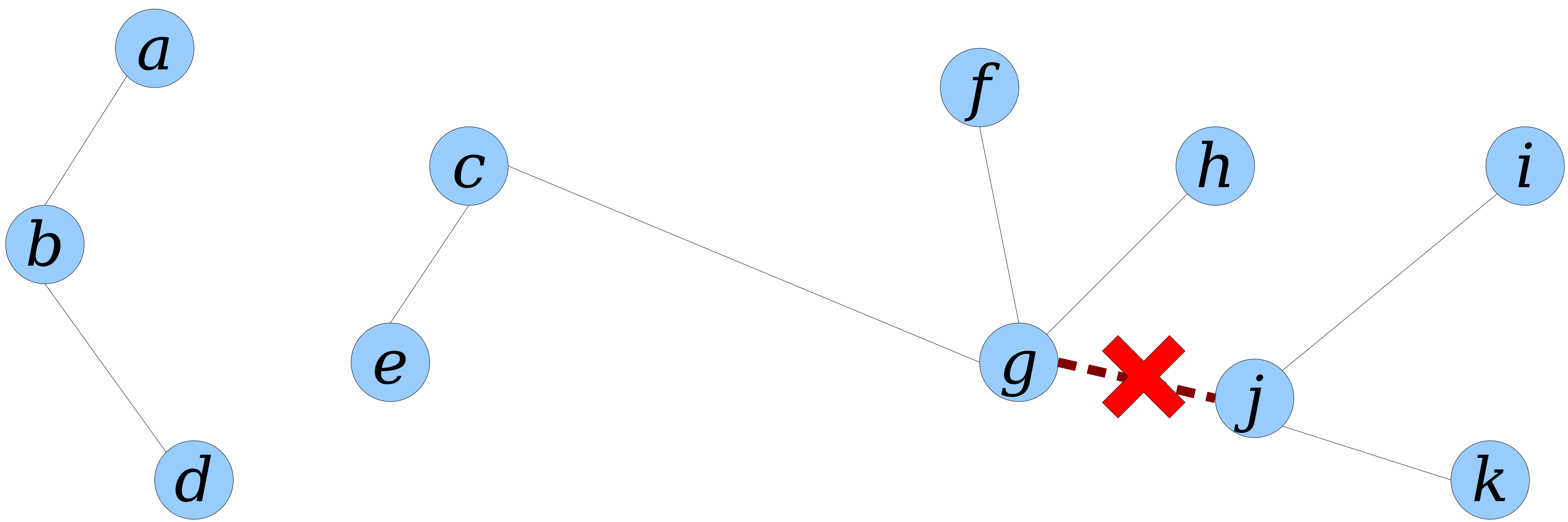

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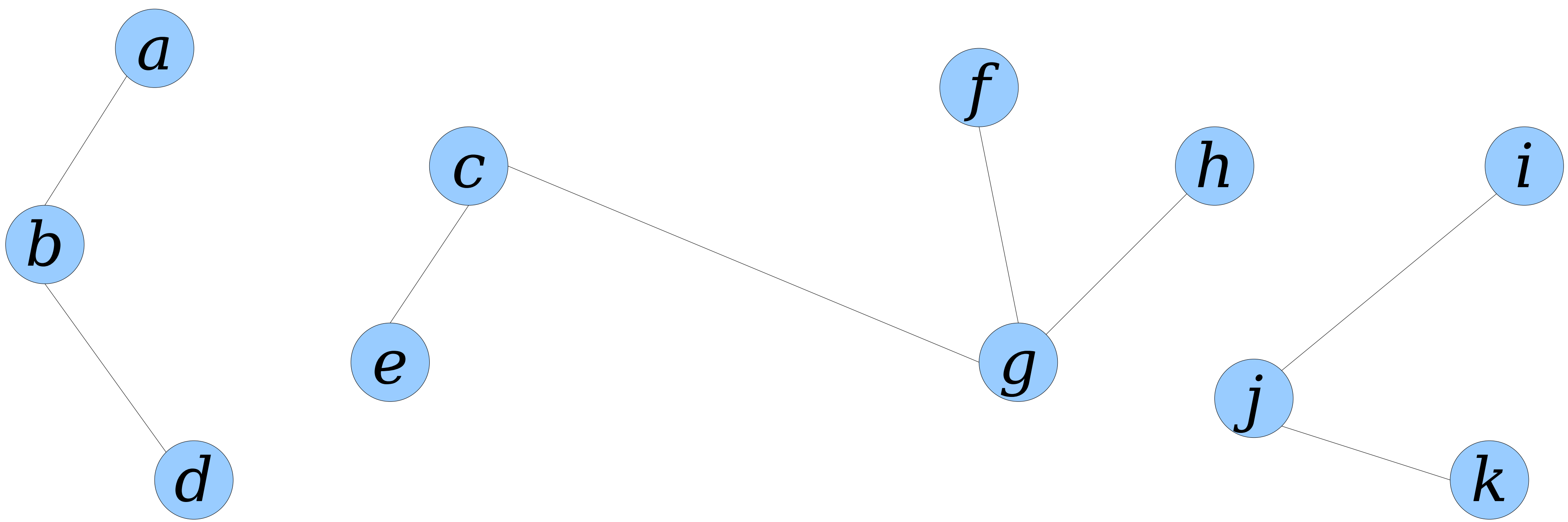

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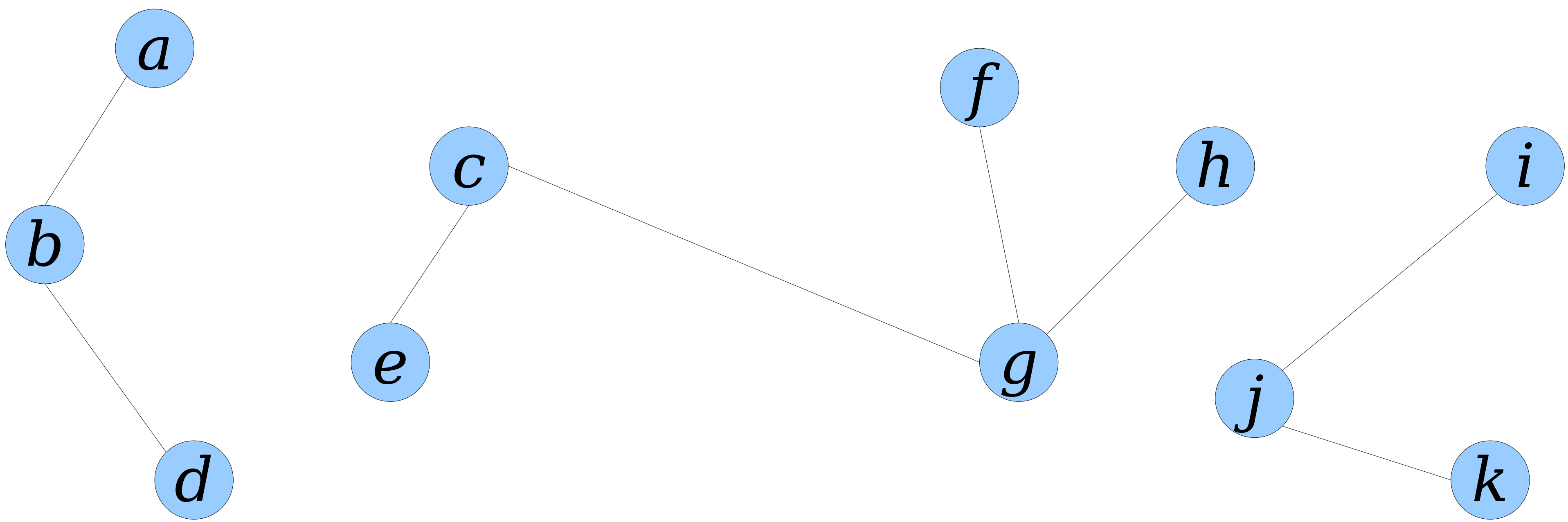

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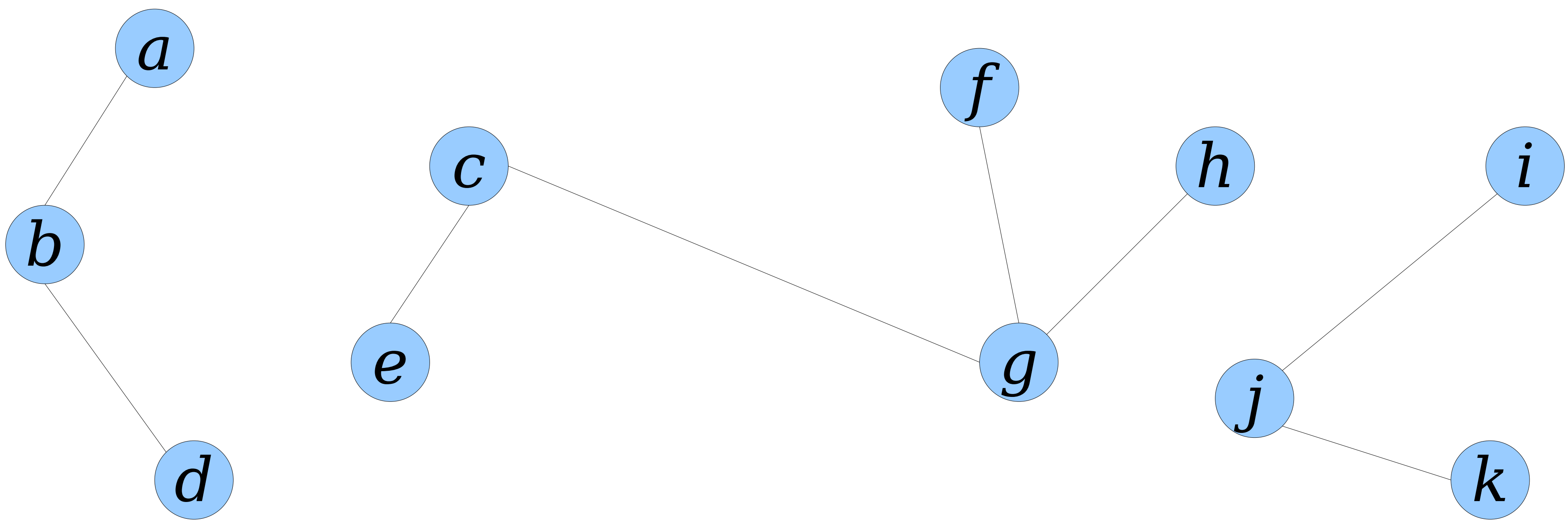

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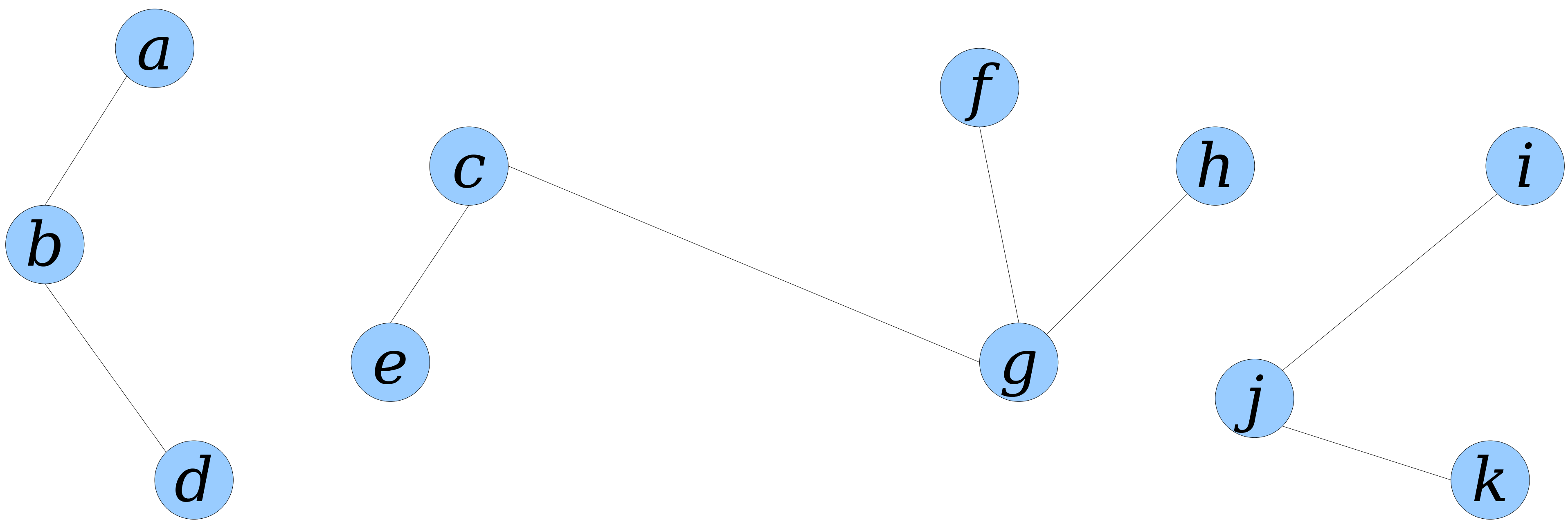

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Euler Tours and Dynamic Trees

- Given a tree T, executing ***cut***(u, v) cuts the edge uv from the tree (assuming it exists).
- To perform ***cut***(u, v):
  - Let E be the Euler tour containing uv and vu.
  - Remove uv and vu from E to form E_1, E_2, and E_3.
  - Then E_1E_3 and E_2 are Euler tours of the two new trees.

Checking Connectivity

- We also need a way to answer queries of the form ***are-connected***(*u*, *v*).
- This query focuses on *nodes*, but our Euler tours store *edges*.
- ***Cute Trick:*** Introduce a self-loop on each node that represents the node itself. Add that to each tour as a proxy for the node itself.
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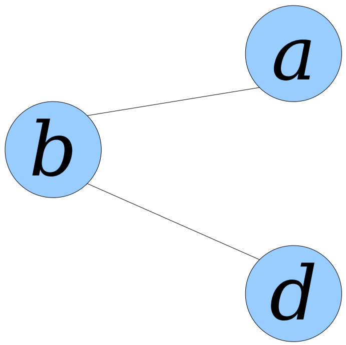

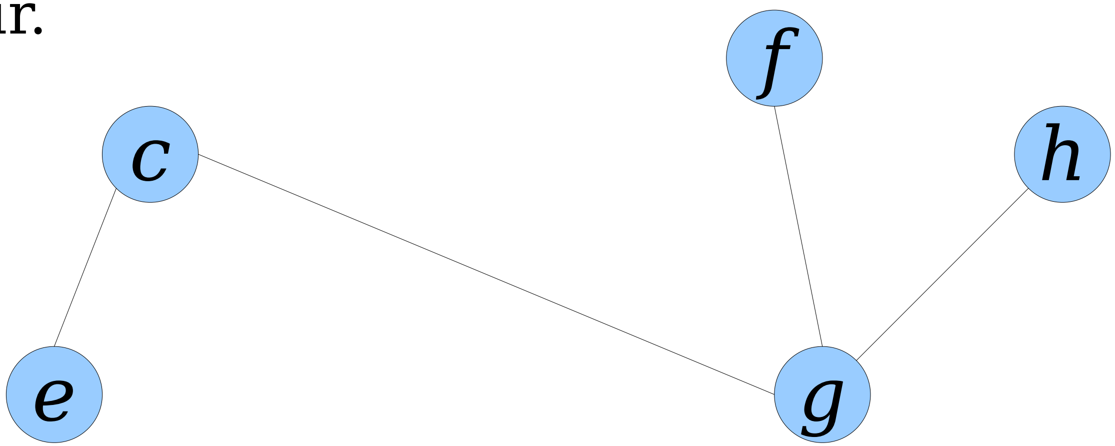

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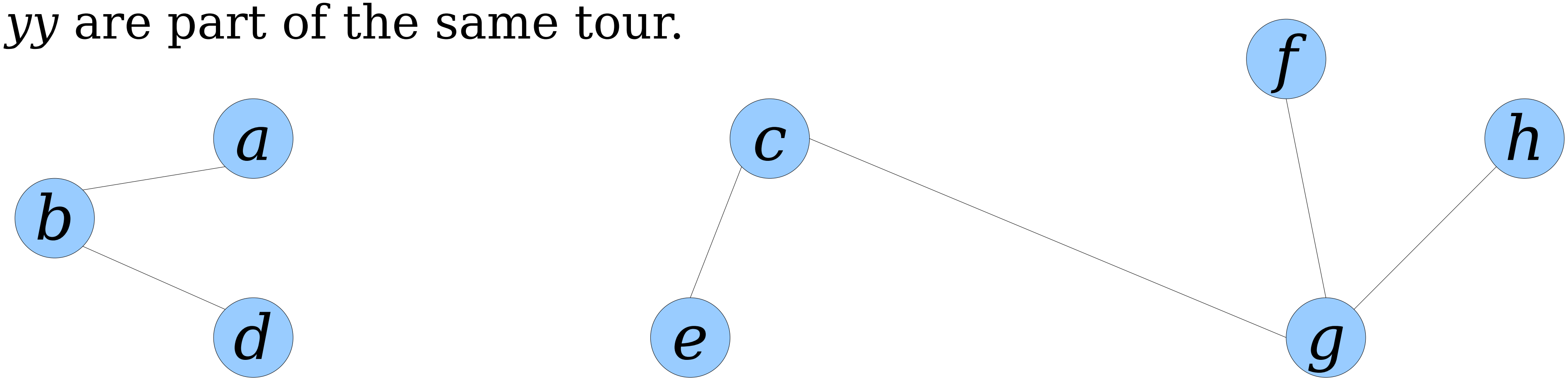

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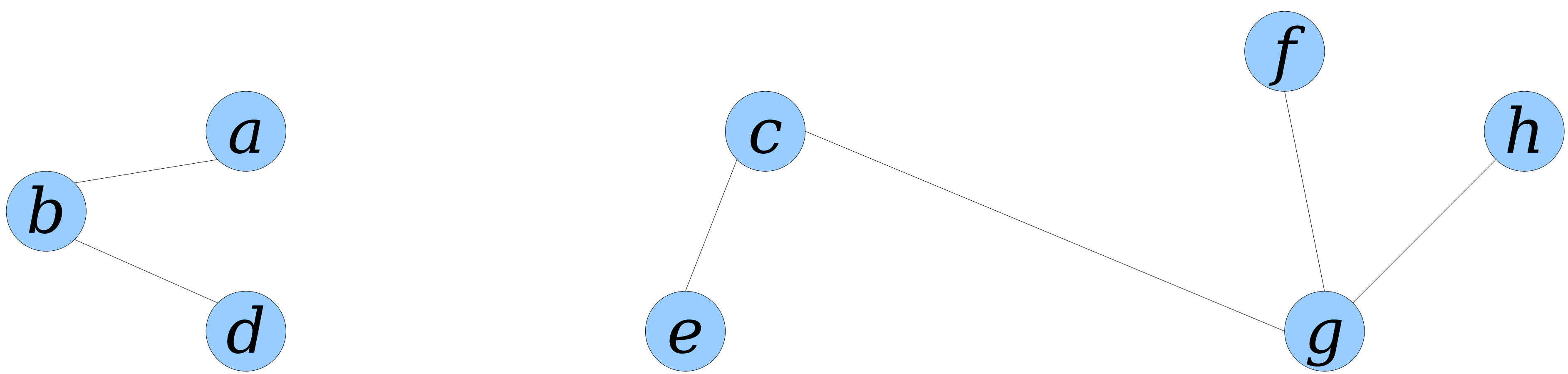

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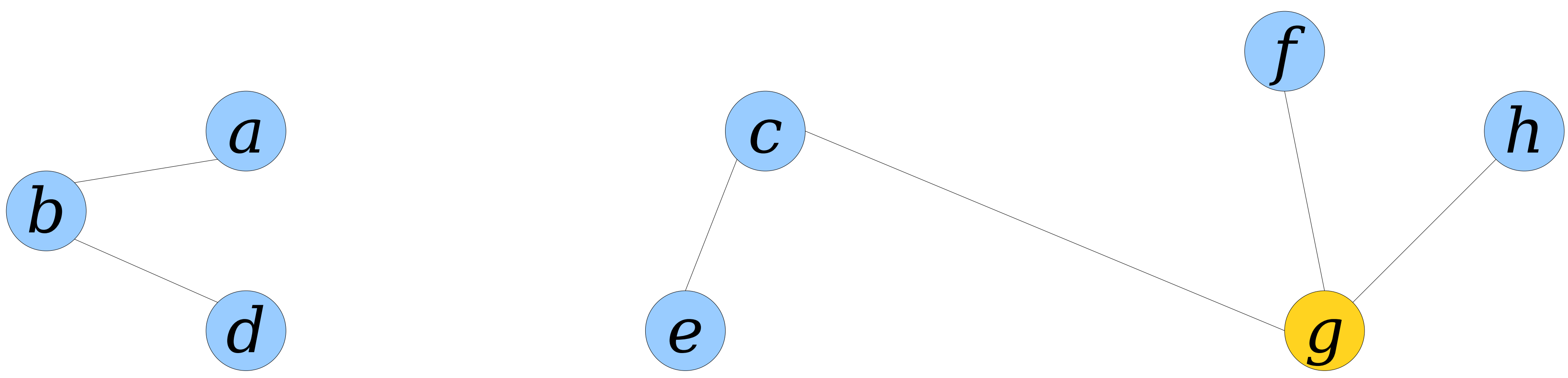

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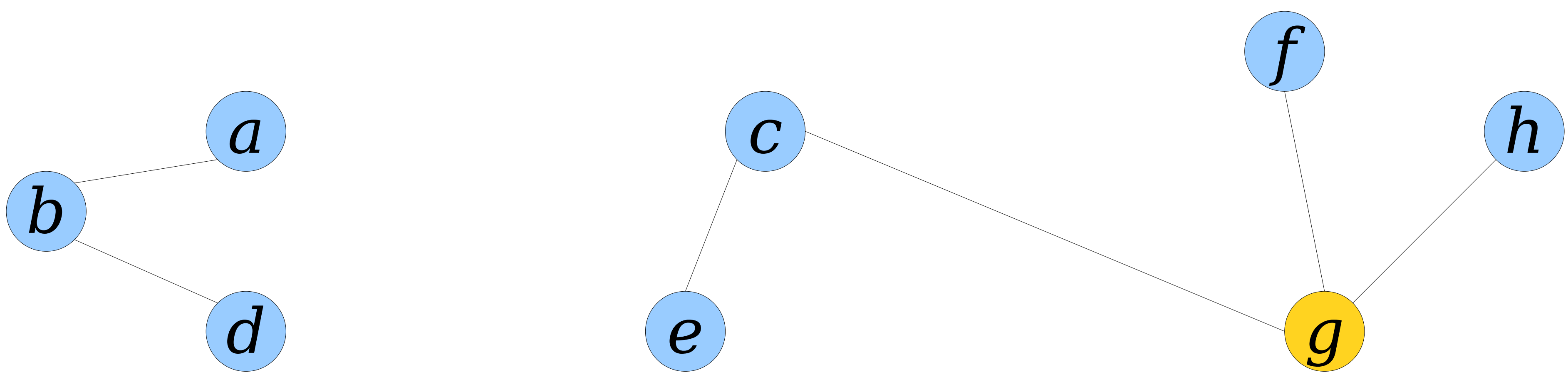

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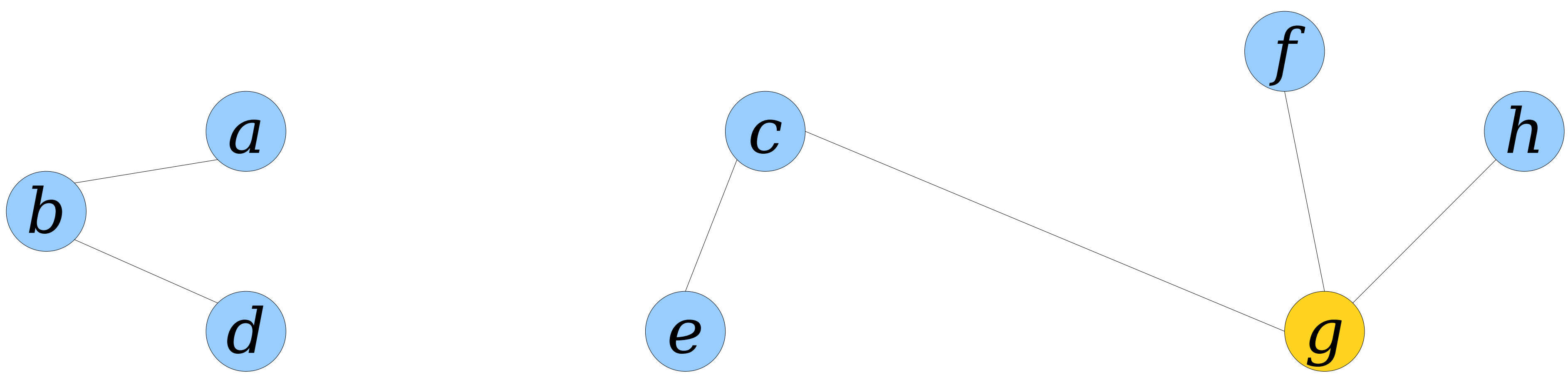

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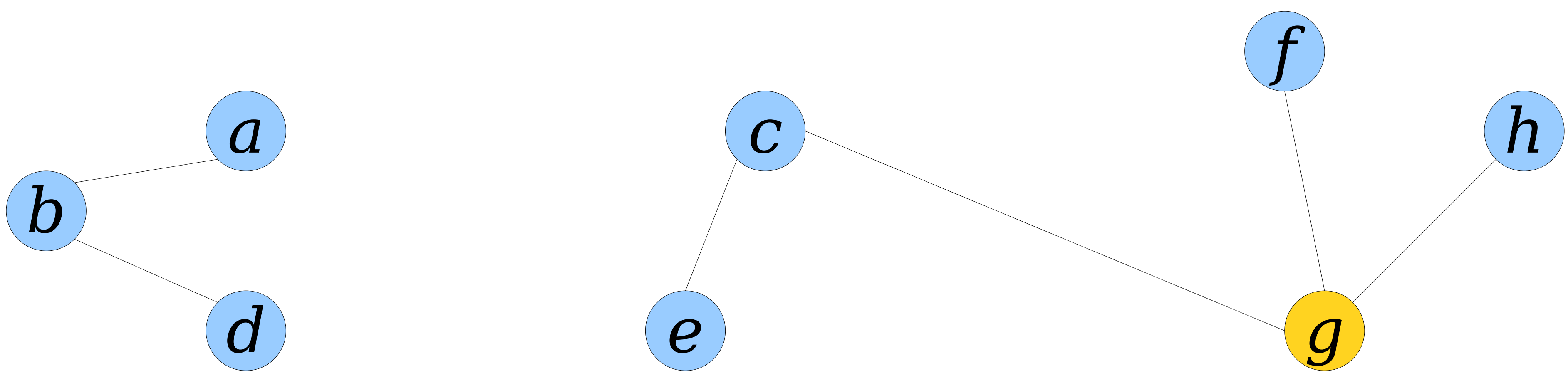

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Putting It All Together

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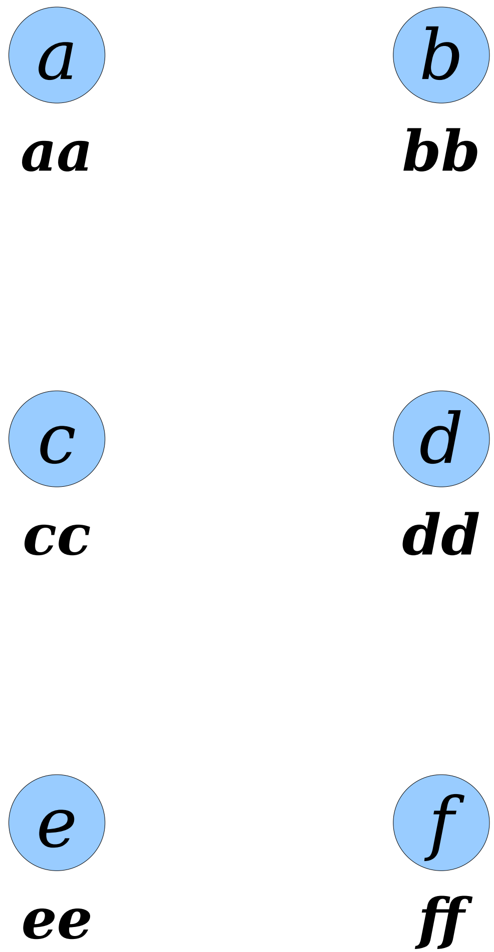

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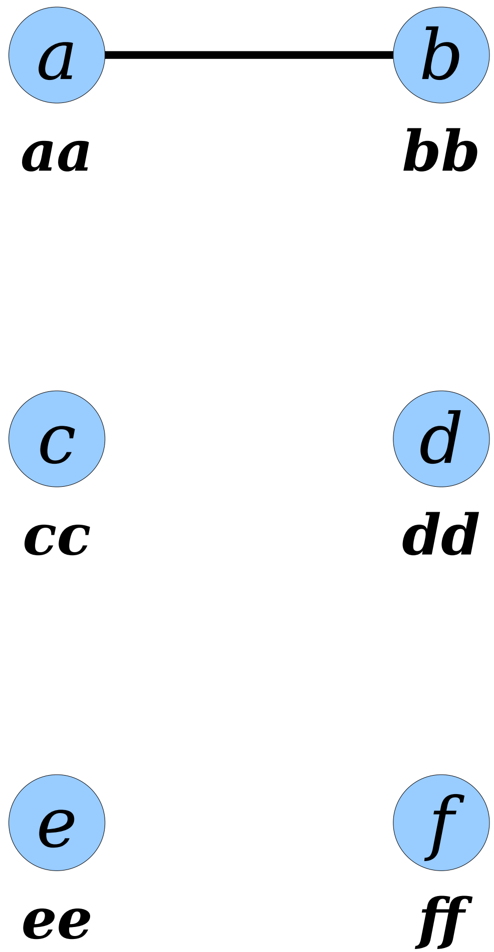

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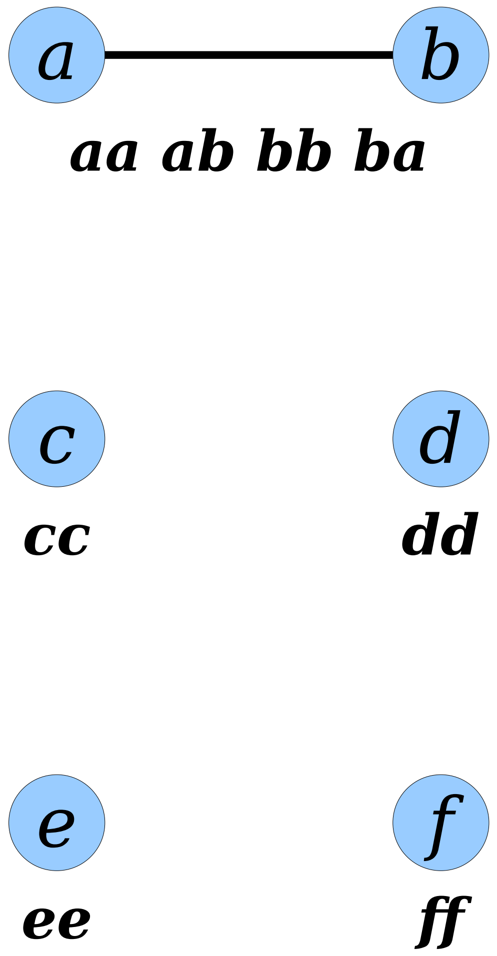

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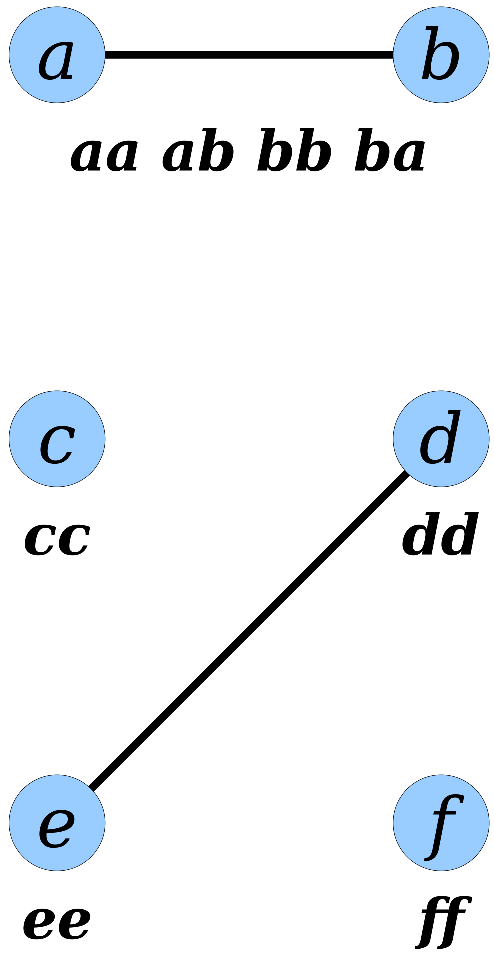

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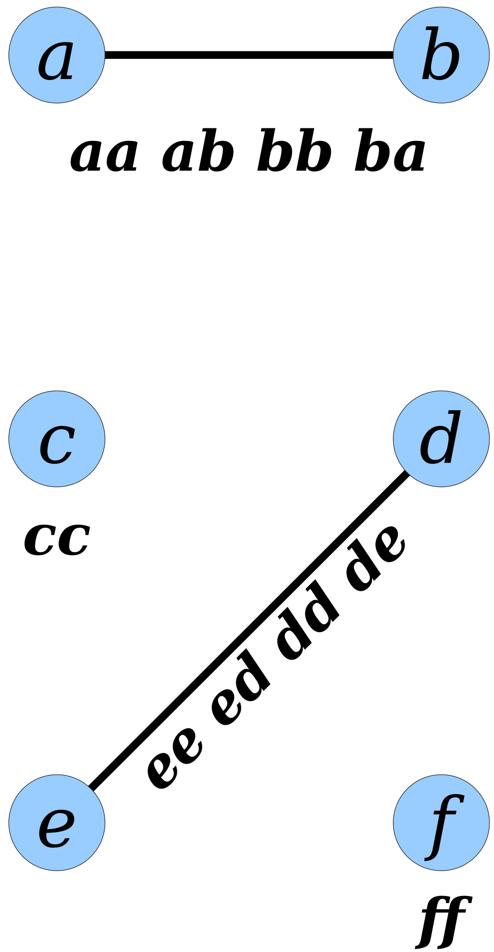

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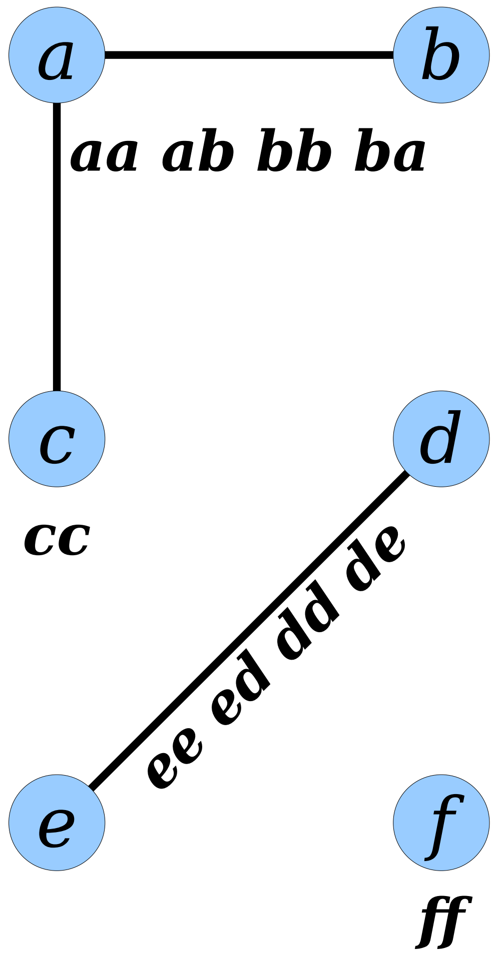

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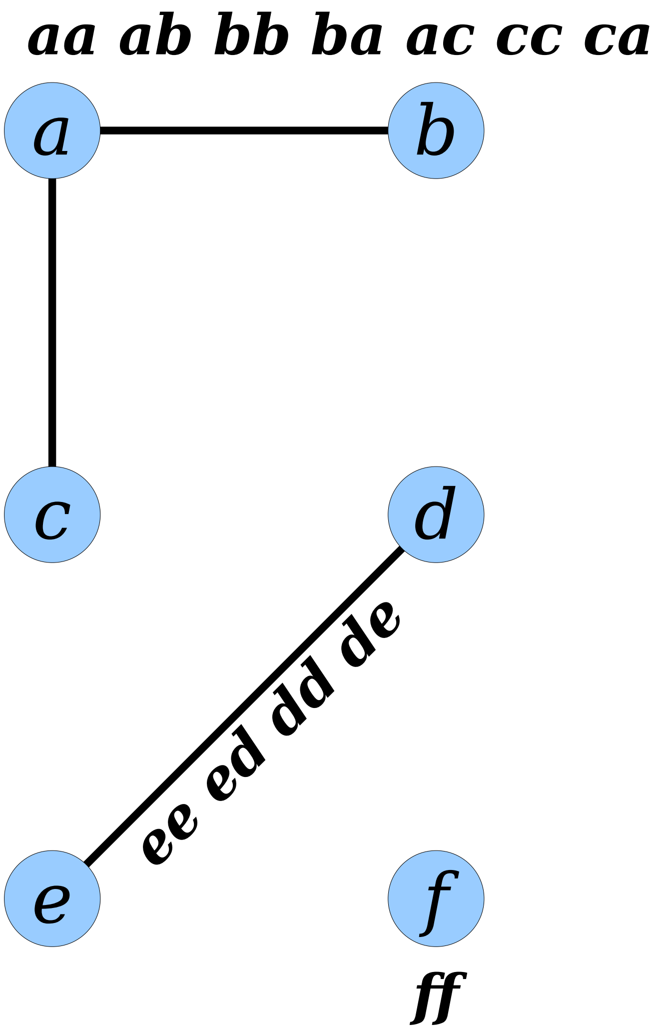

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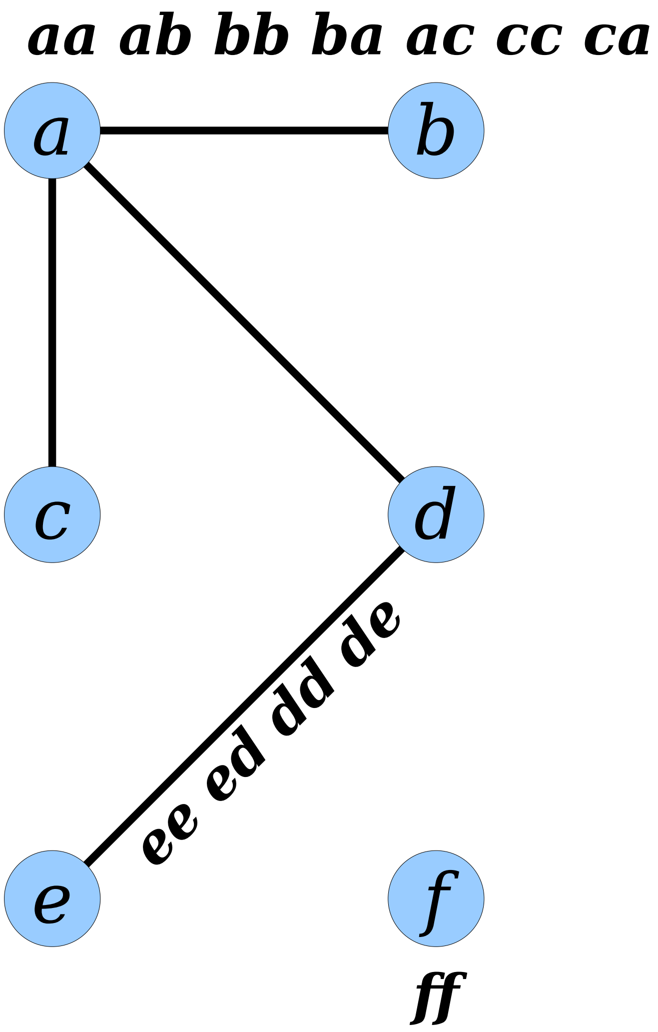

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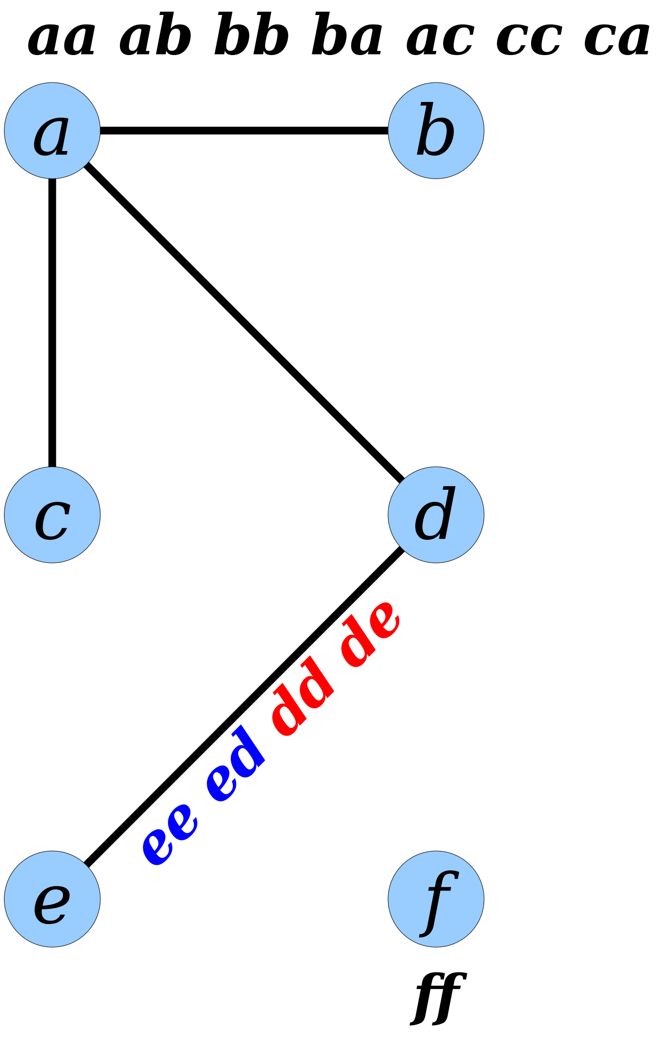

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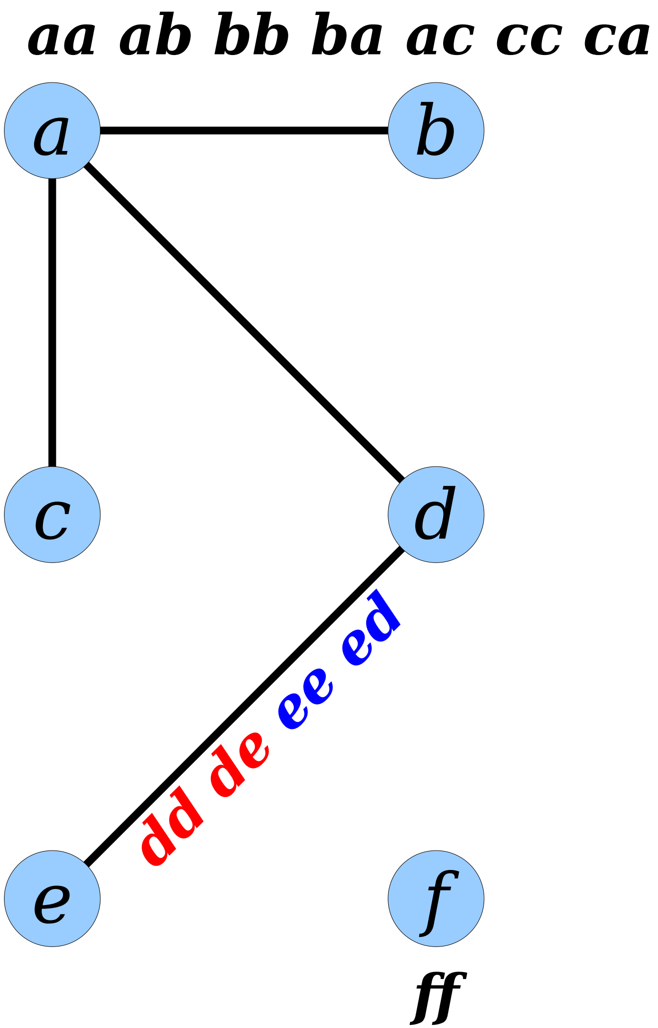

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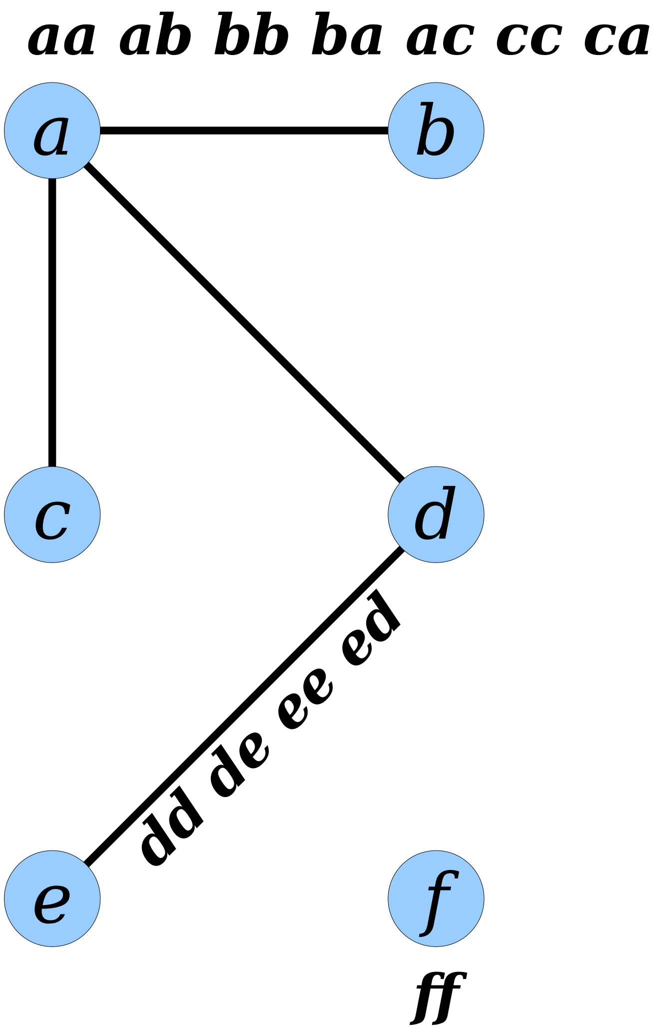

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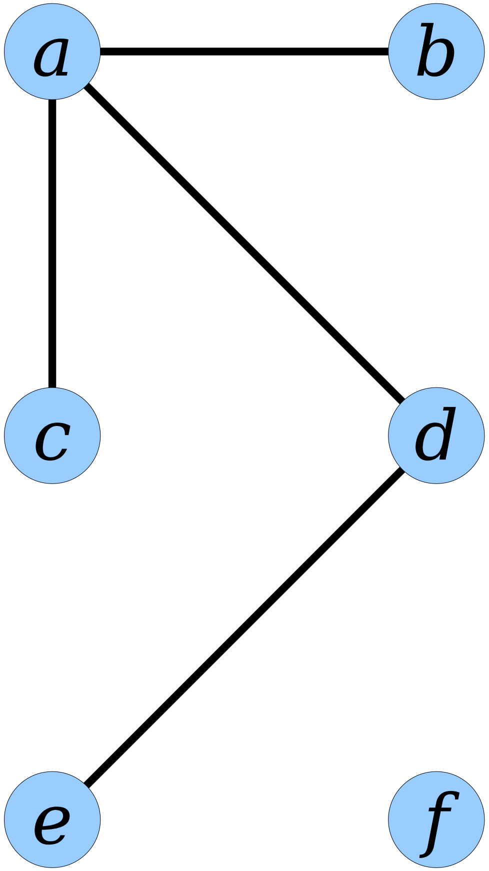

ff

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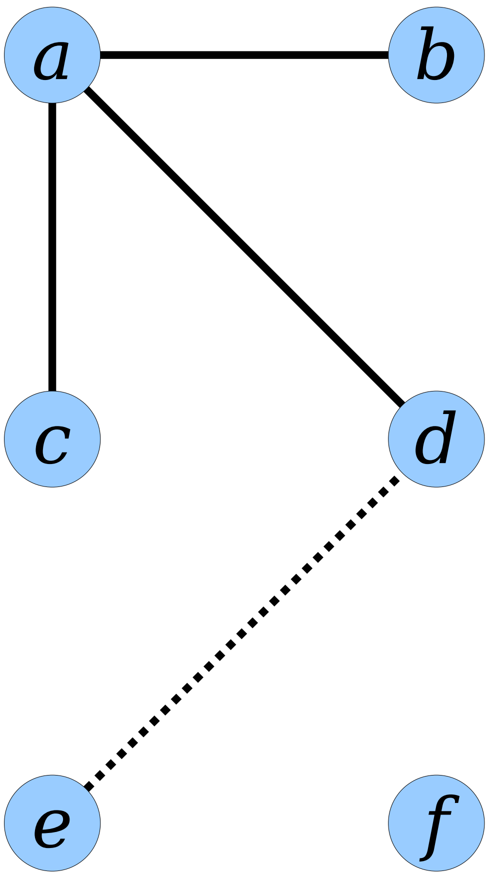

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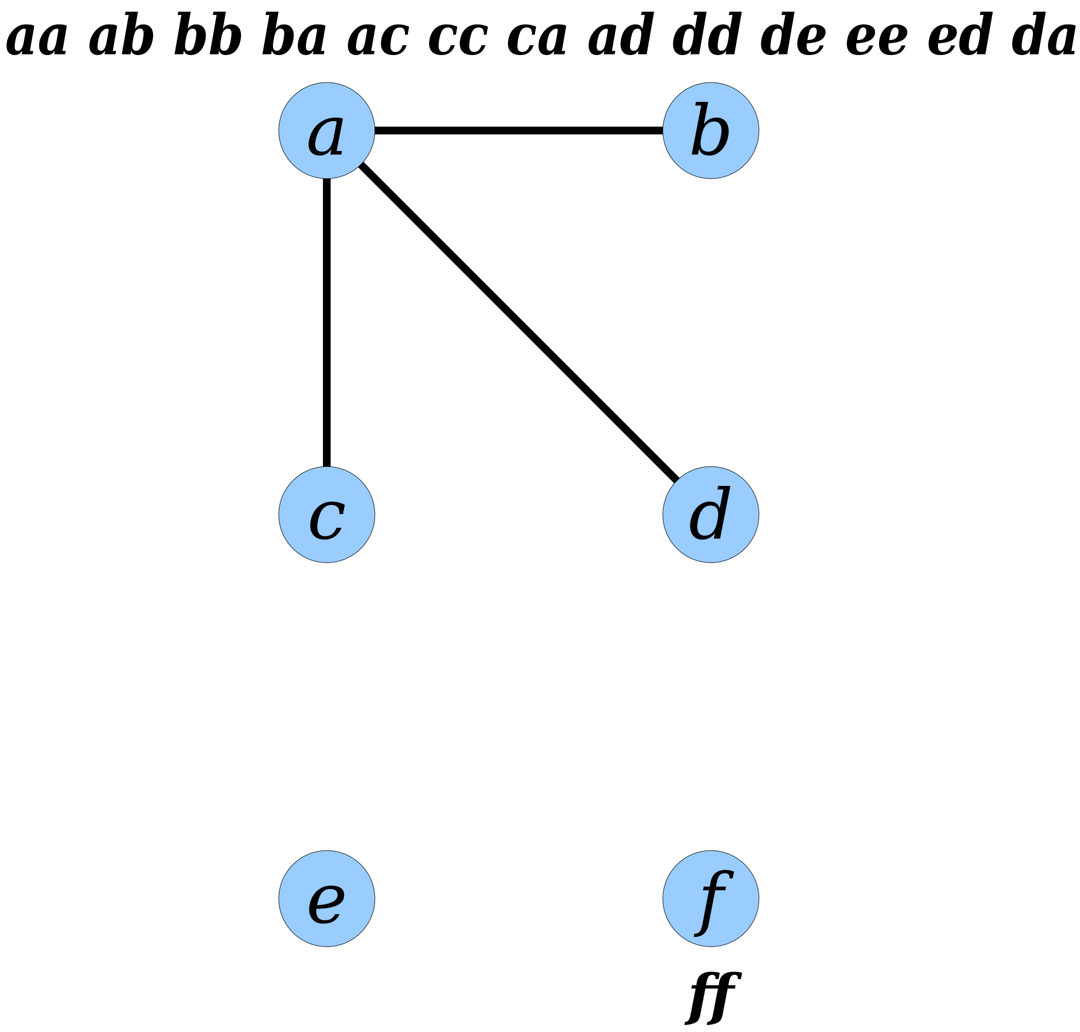

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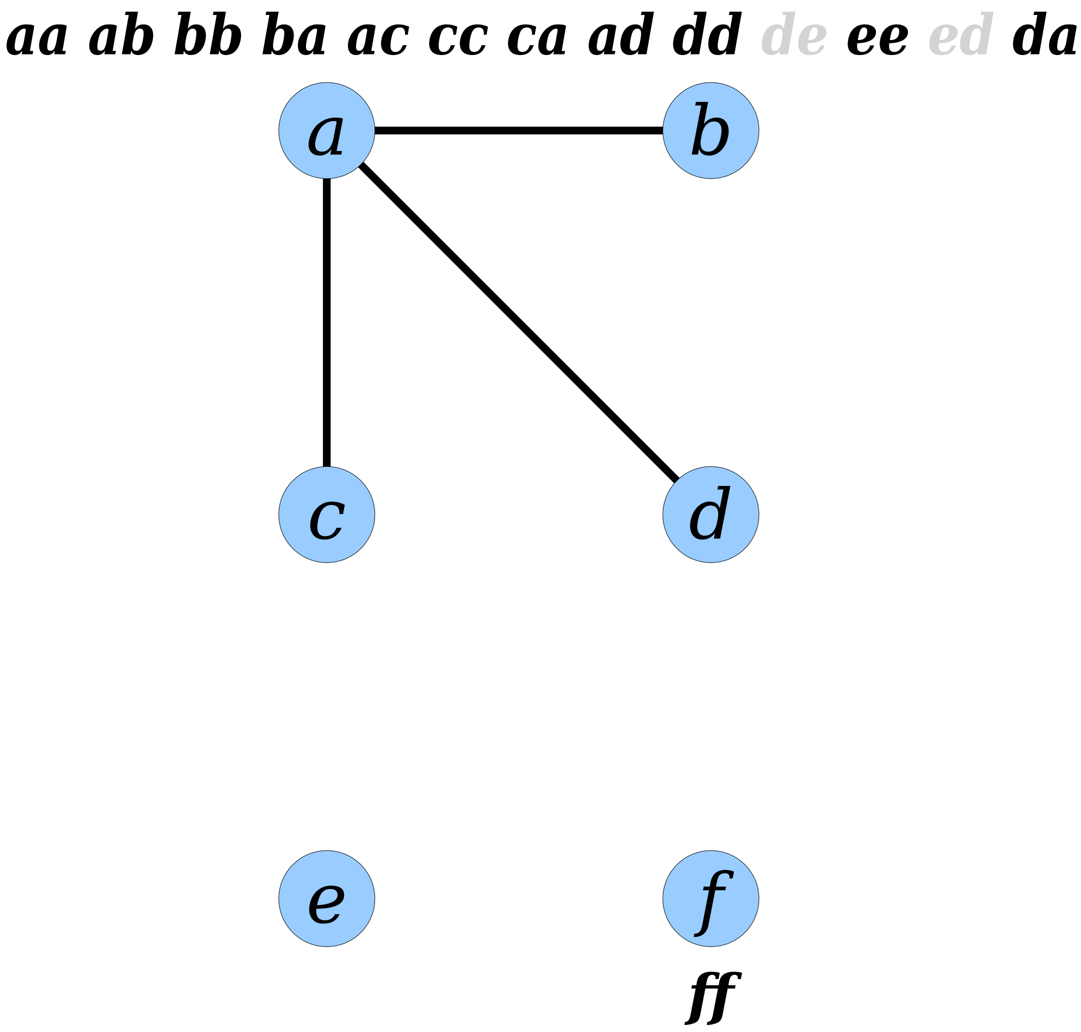

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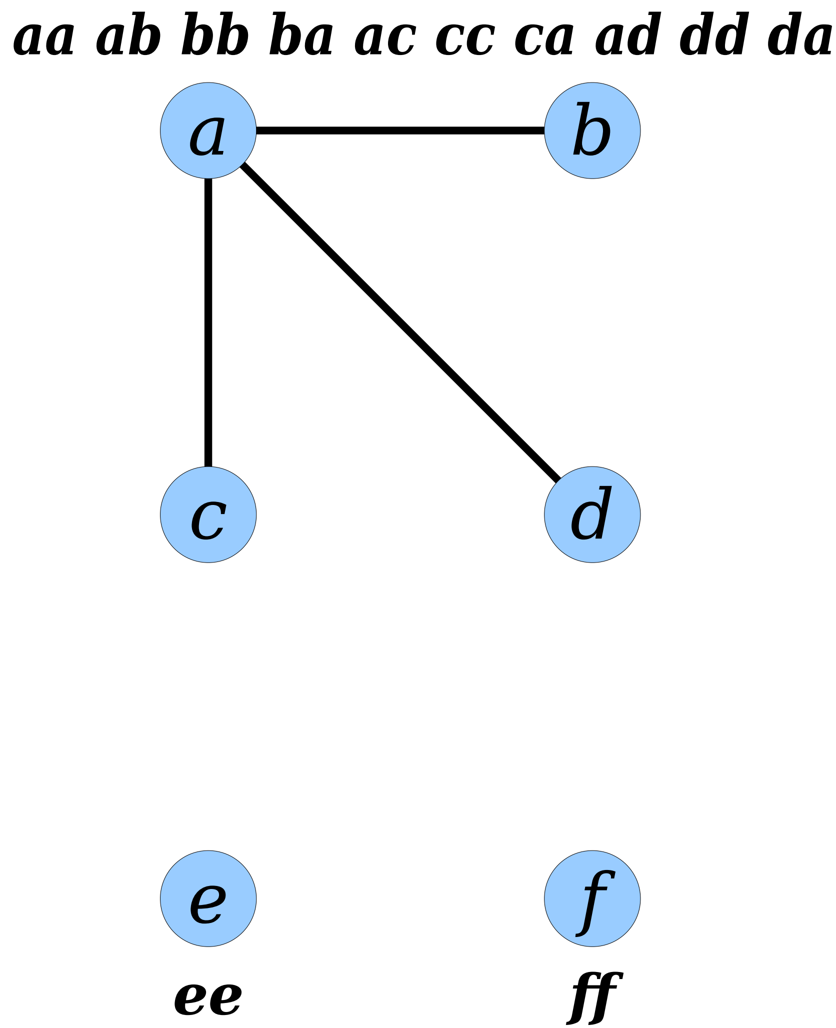

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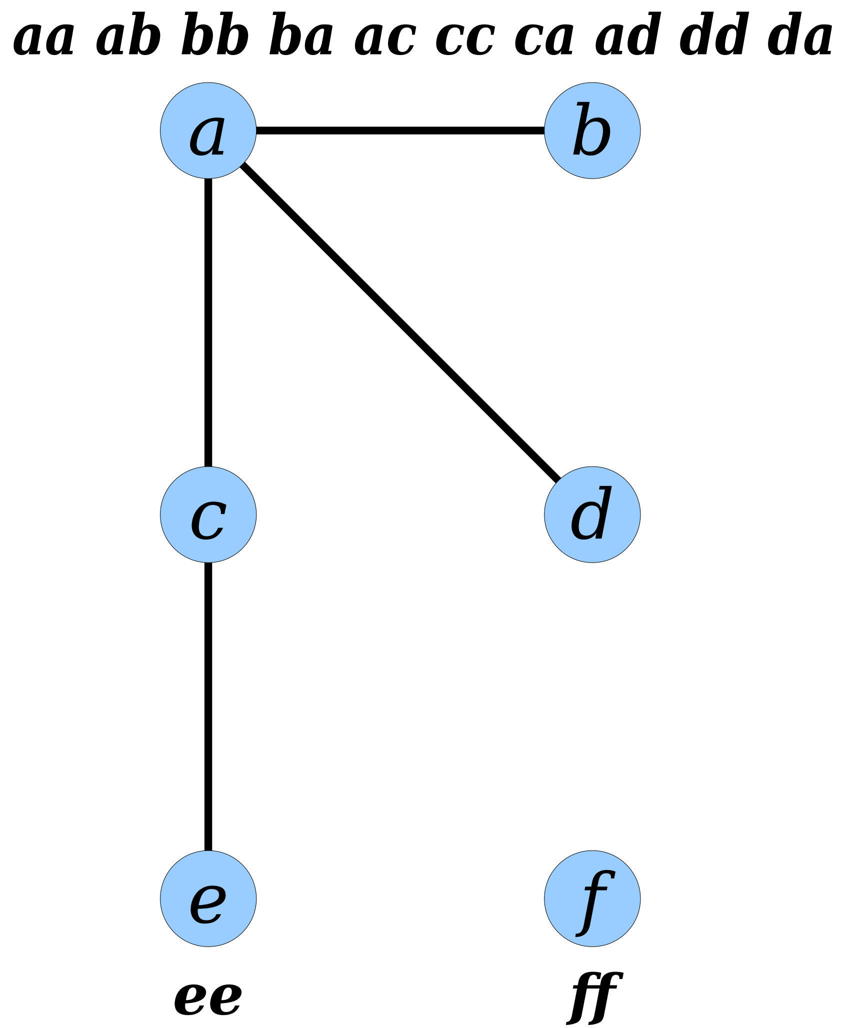

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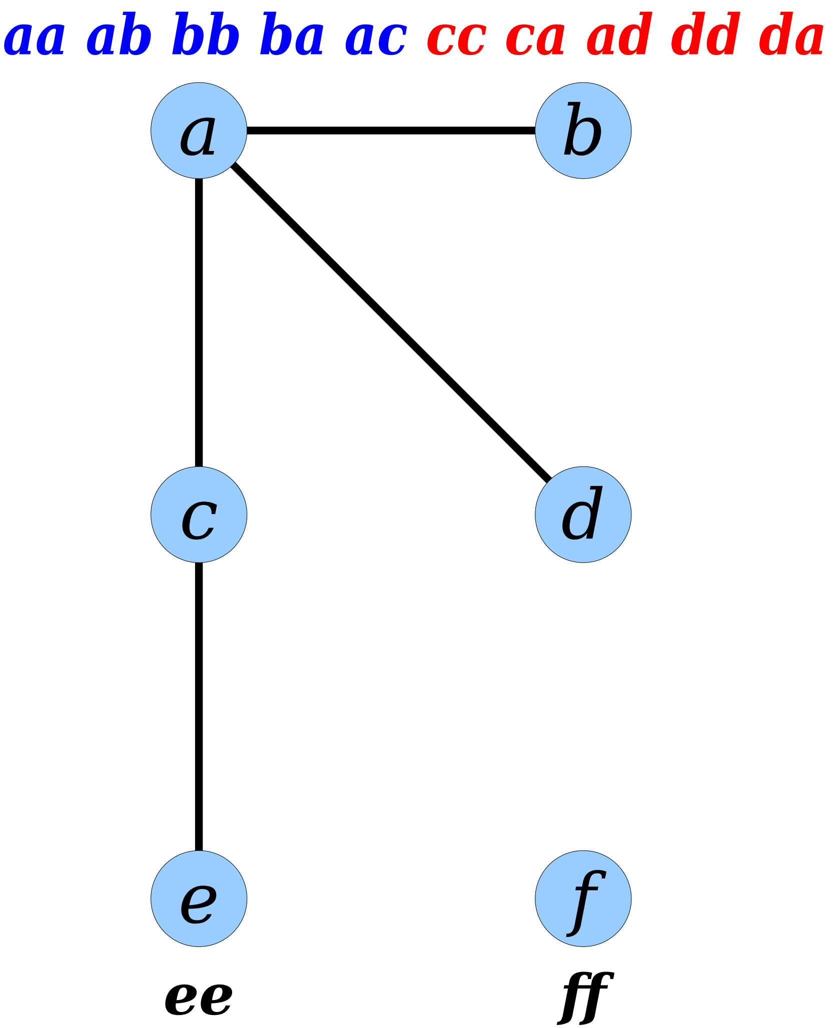

Putting It All Together

- To ***link***(x, y):
  - Rotate xx and yy to the fronts of their tours T_x and T_y.
  - Join the tours together as $T_x\ xy\ T_y\ yx$.
- To ***cut***(x, y):
  - Delete the edges xy and yx from the tour T to form tours T_1, T_2, T_3.
  - Regroup the tours as $T_1\ T_3$ and T_2.
- To answer ***are-connected***(x, y):
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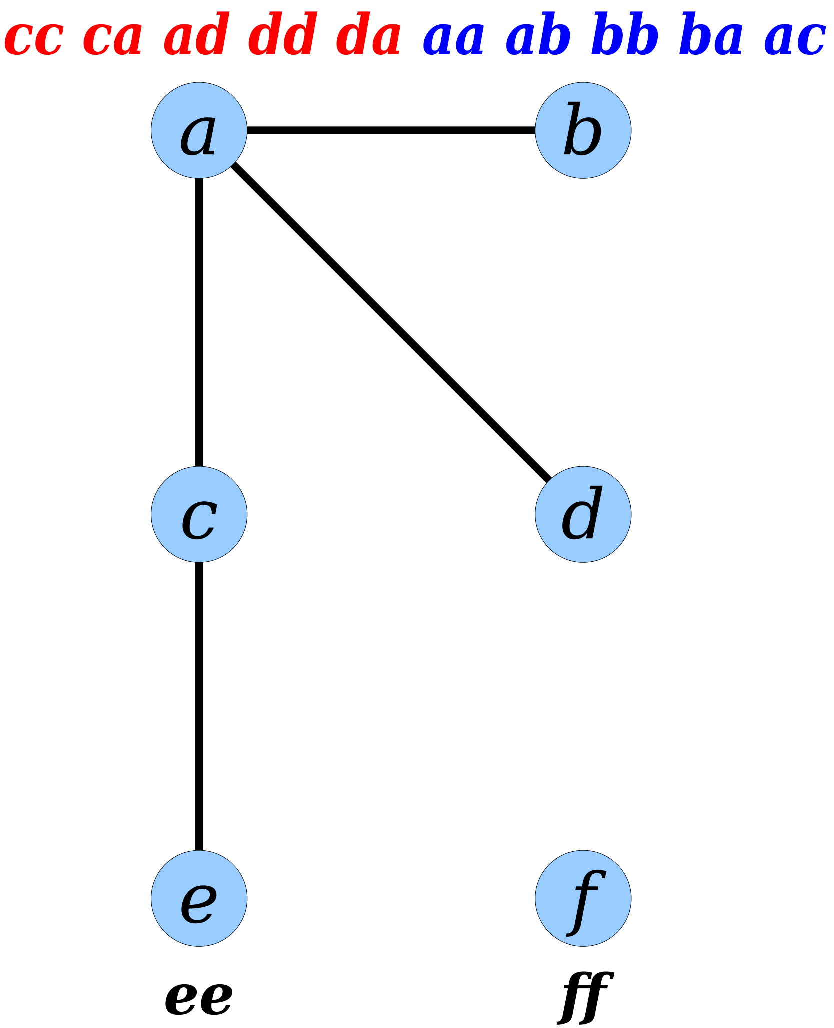

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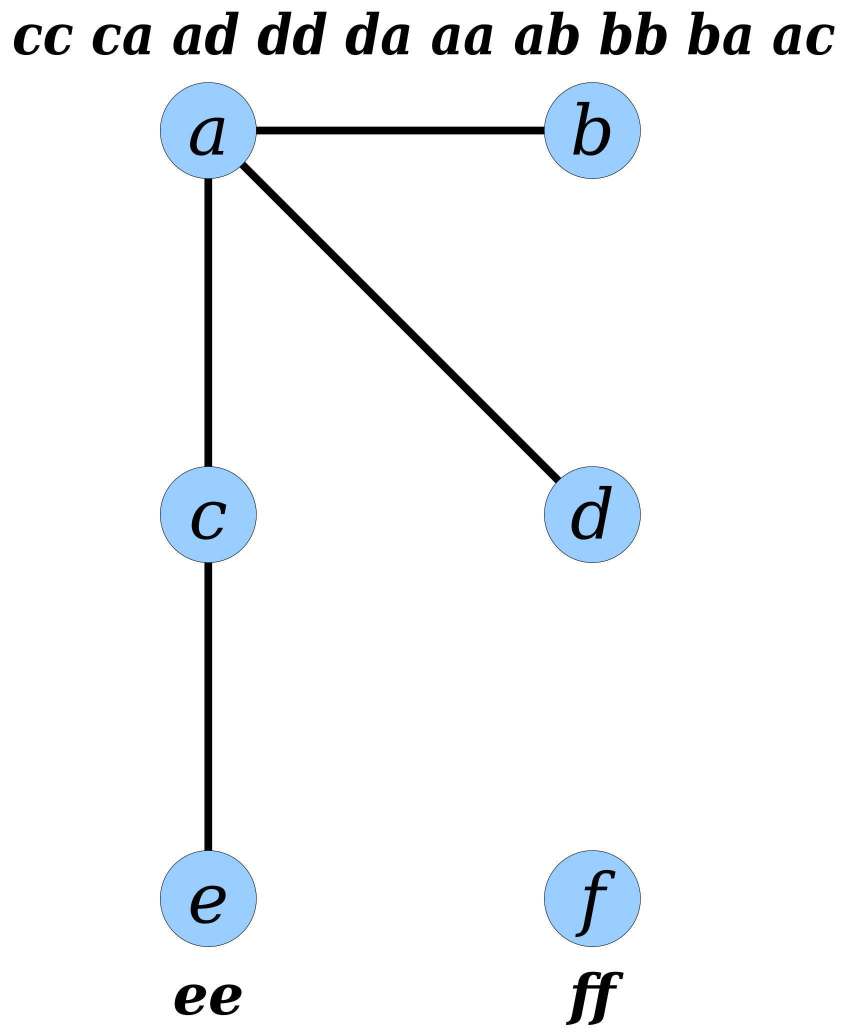

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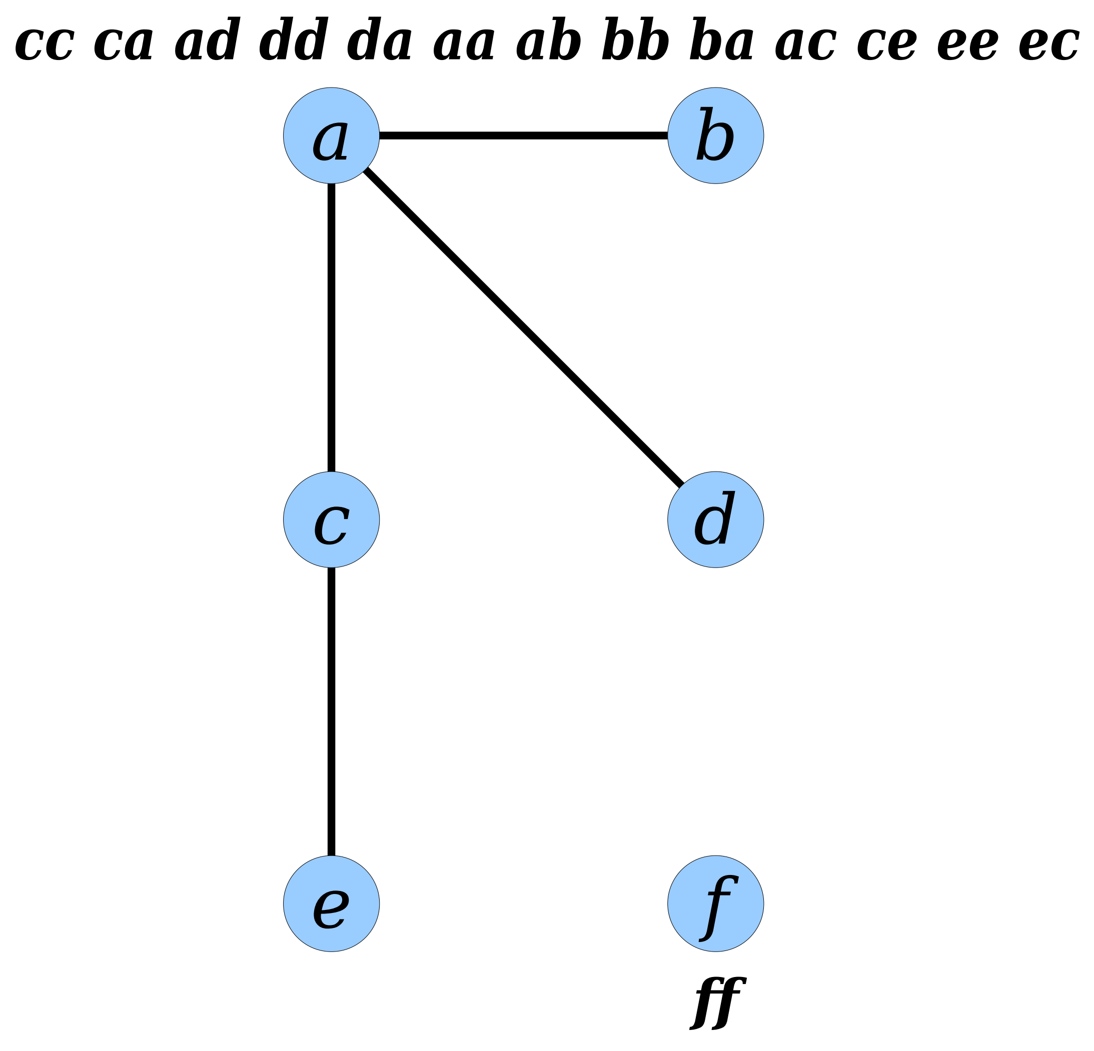

Implementing This Approach

The Story So Far

- We’ve seen how to implement ***reroot***, ***link***, ***cut***, and ***are-connected*** in terms of operations on Euler tours.
- The efficiency of those operations depend on how we choose to encode our sequences.
- ***Question:*** What data structure should we use to store those sequences?

Representation Issues

- We need a representation that lets us perform the following operations:
  - Locate specific edges (***reroot***, ***link***, ***cut***, ***are-connected***).
  - Split a sequence at a point (***reroot***, ***cut***).
  - Join two sequences together (***reroot***, ***link***).
  - Remove an edge from a sequence (***cut***).
  - Append an edge to a sequence (***link***).
  - Check if two edges are in the same sequence (***are-connected***).
- What data structures might be appropriate here?

Answer at

https://pollev.com/cs166spr23

Representation Issues

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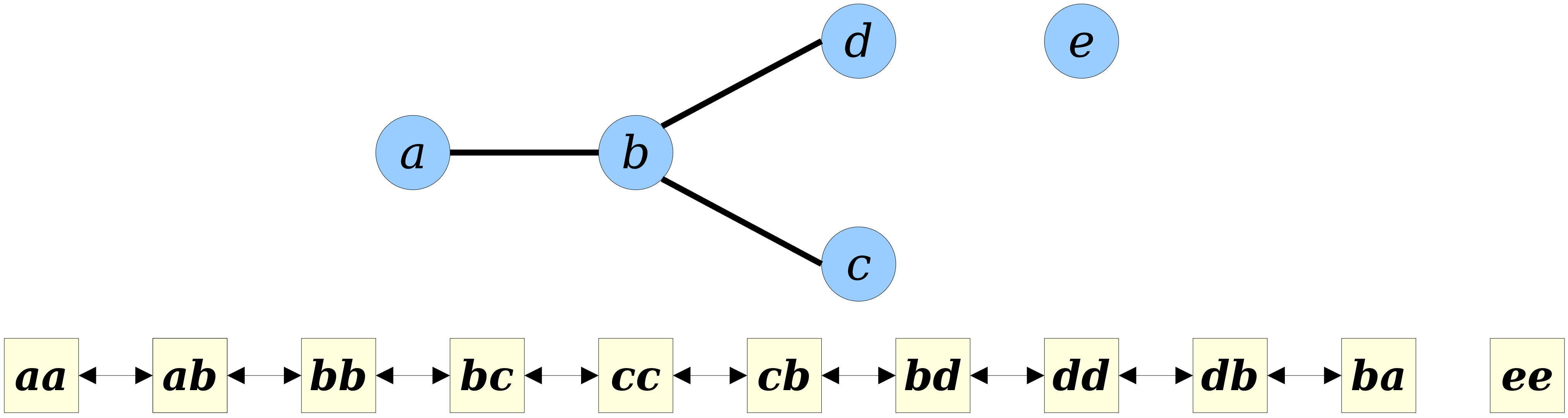

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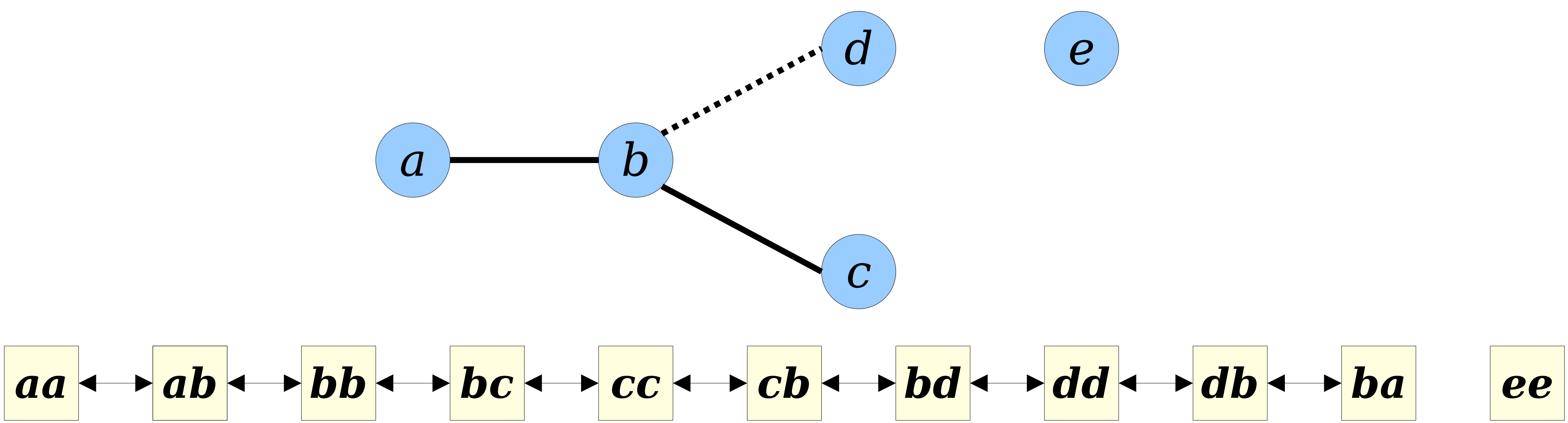

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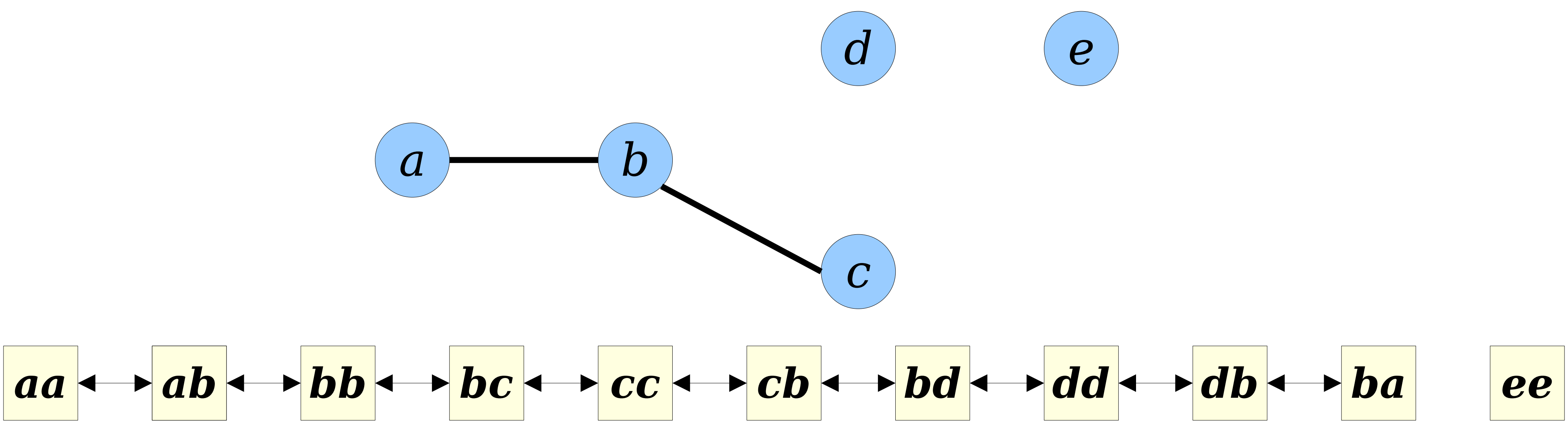

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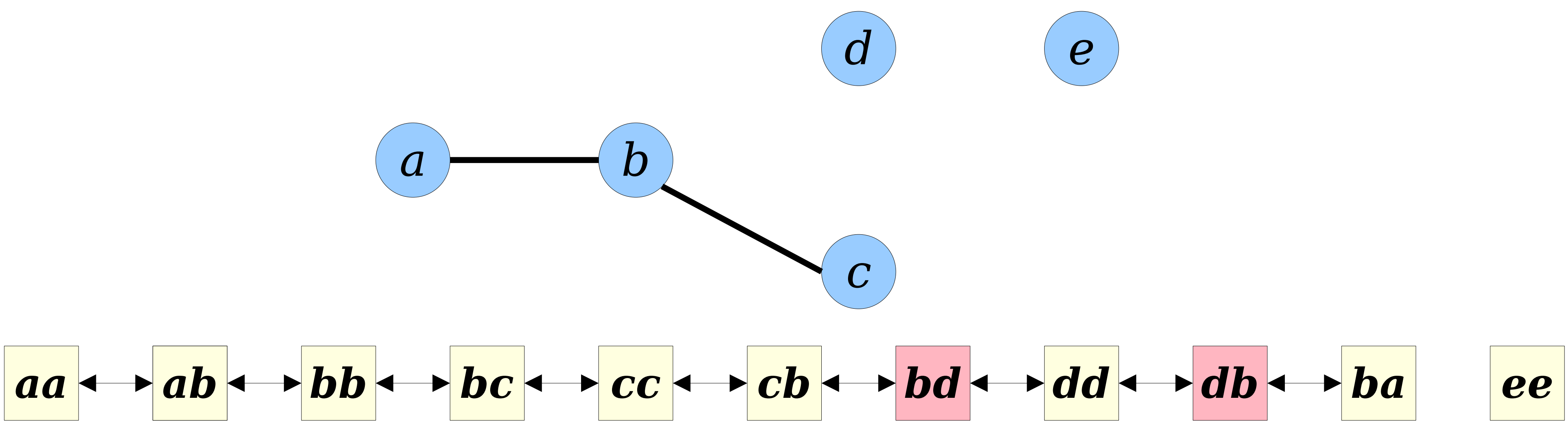

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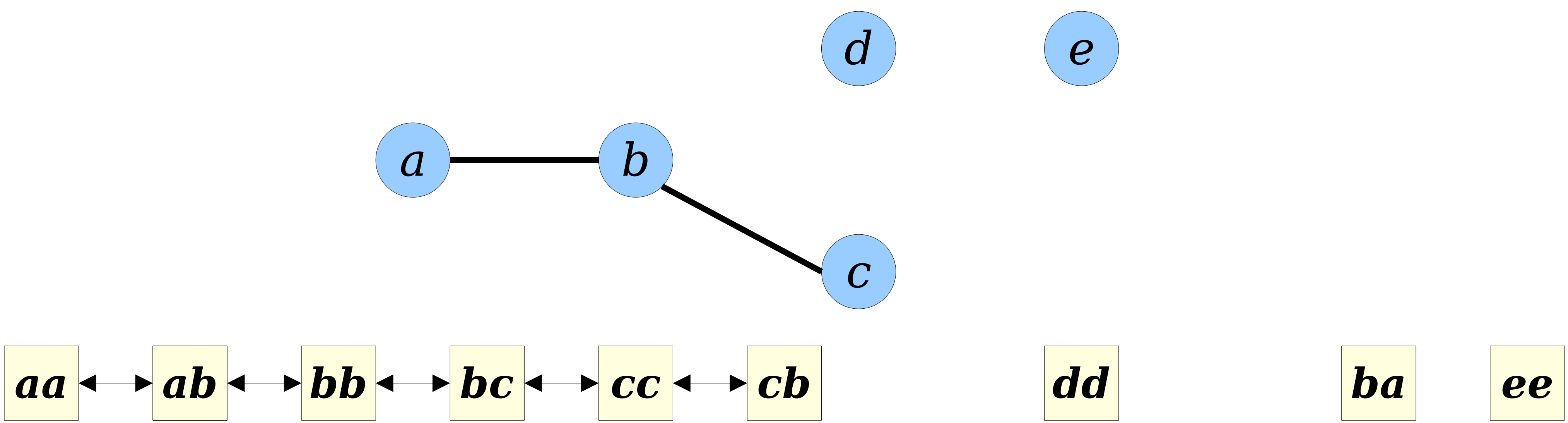

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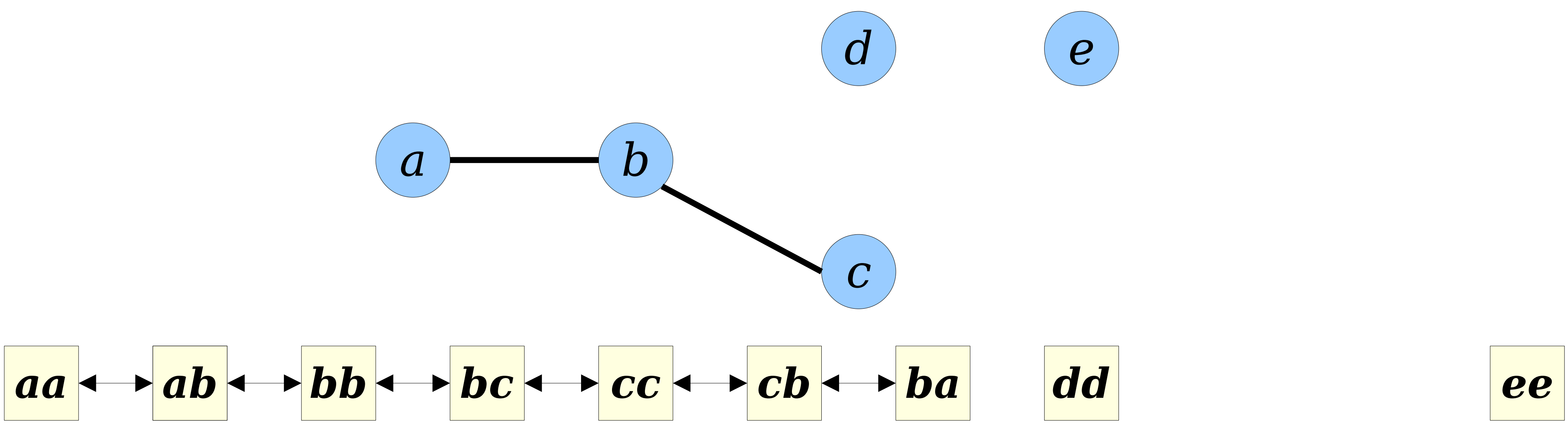

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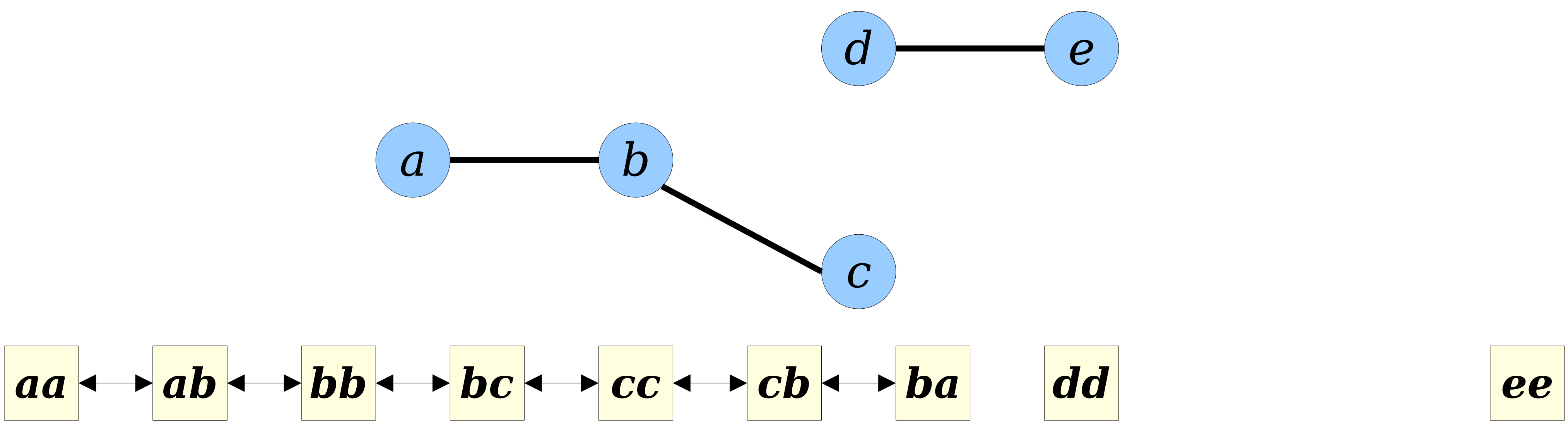

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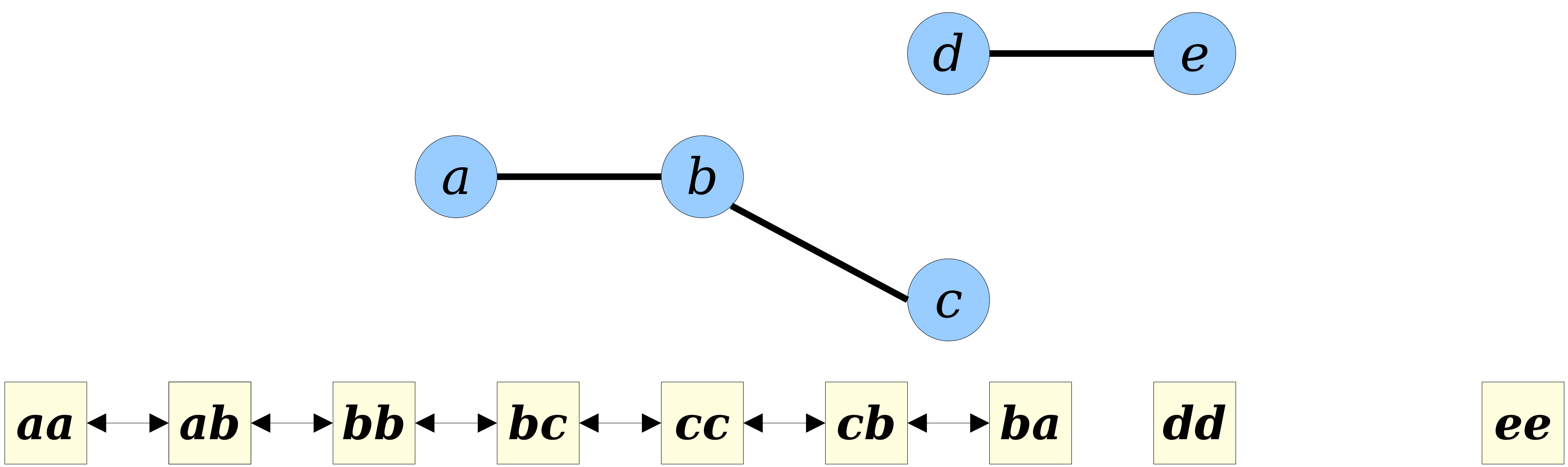

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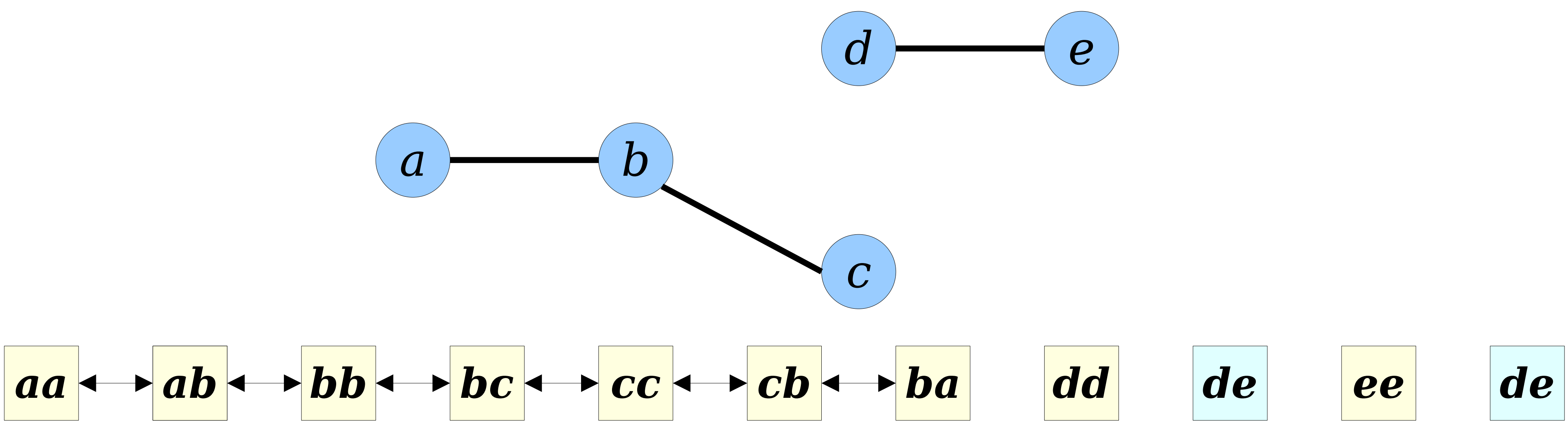

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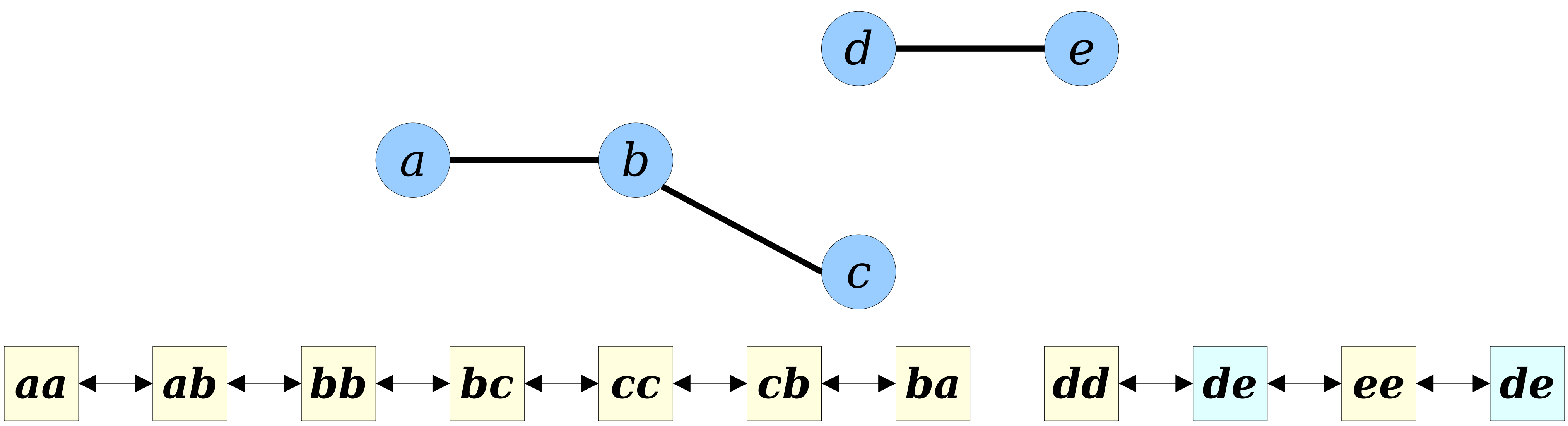

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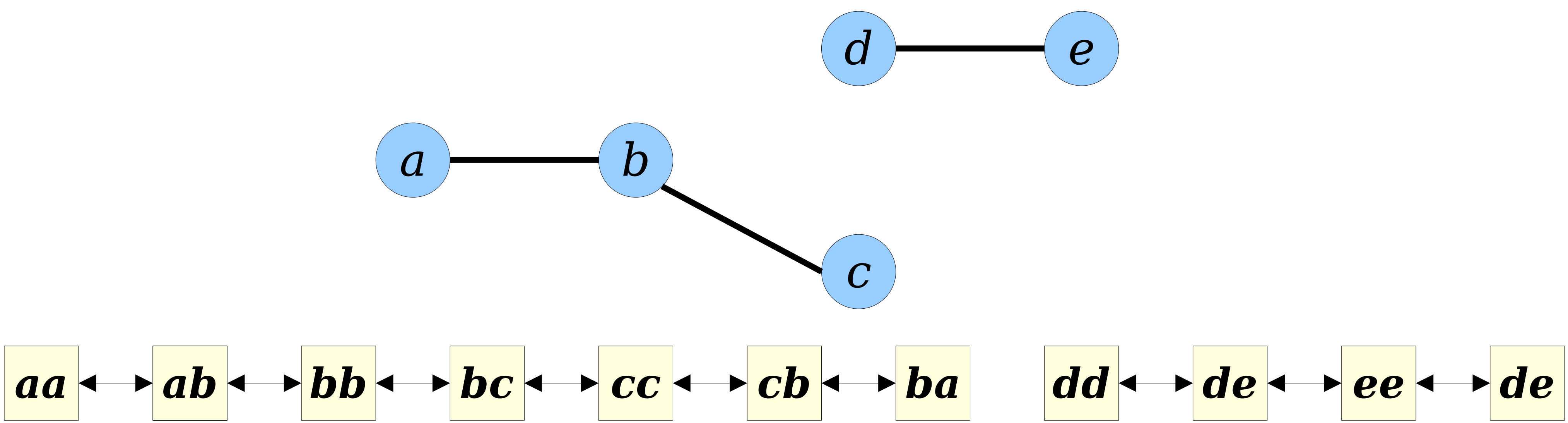

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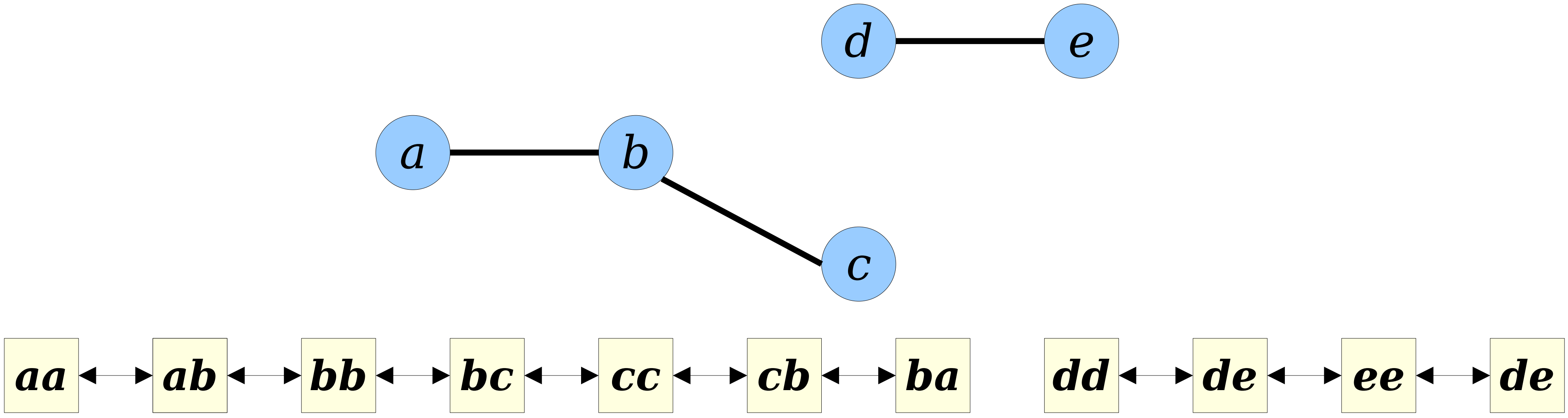

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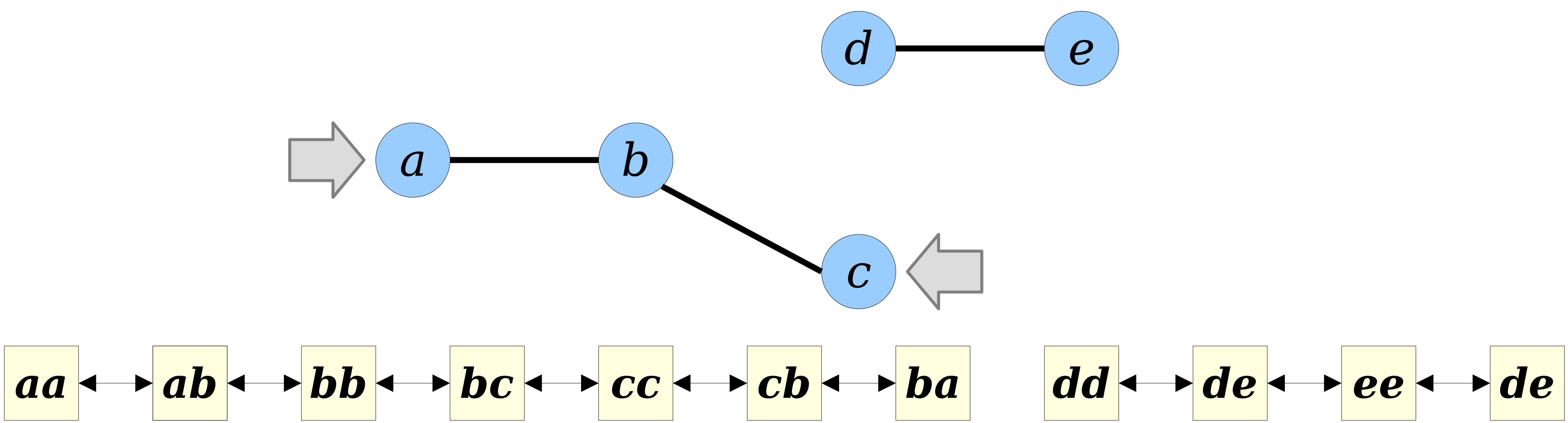

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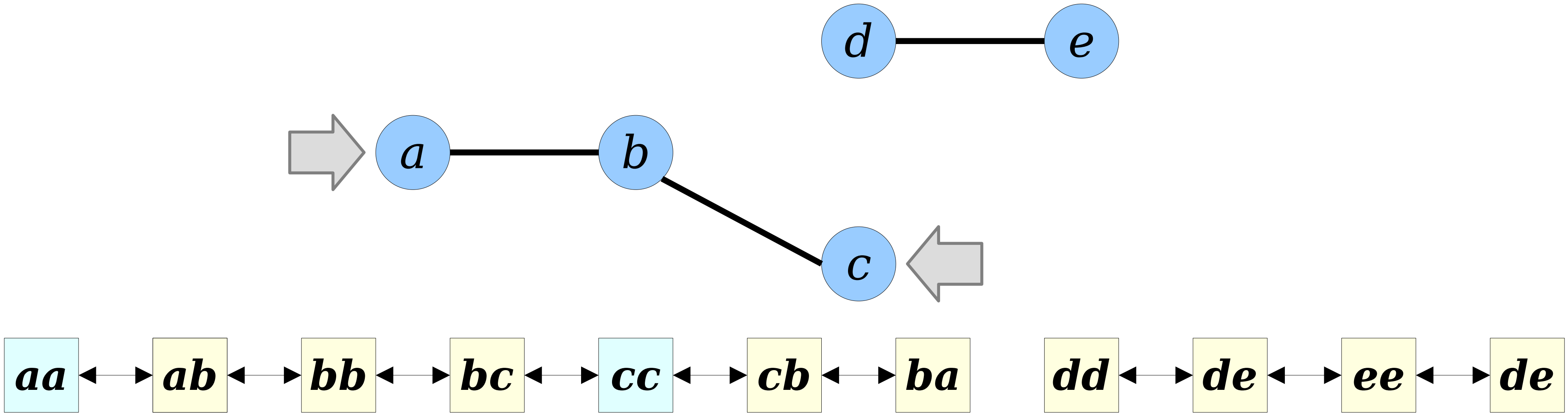

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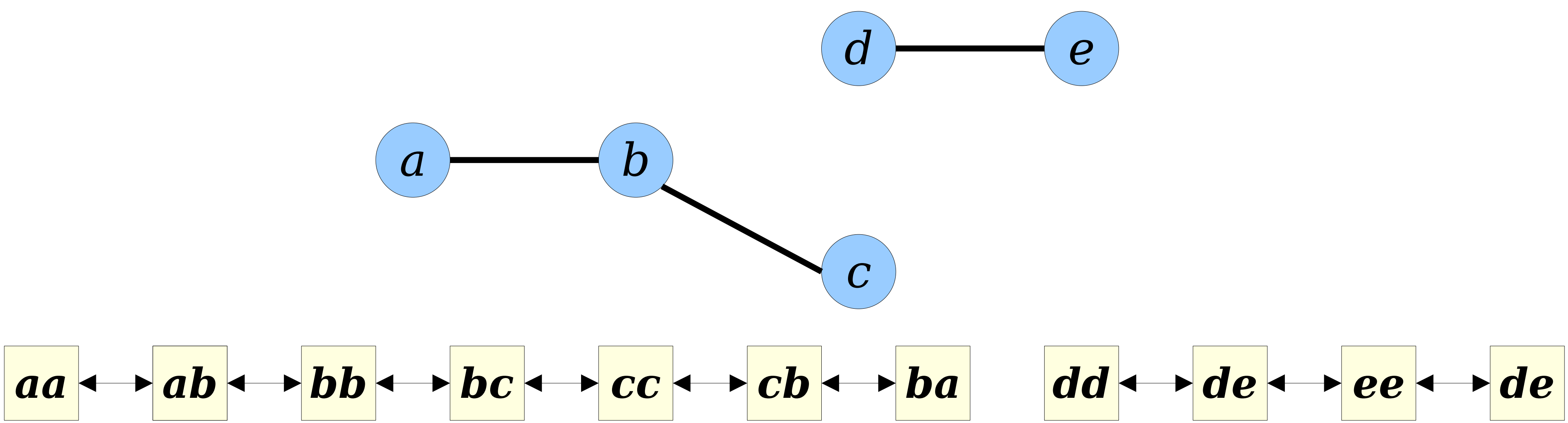

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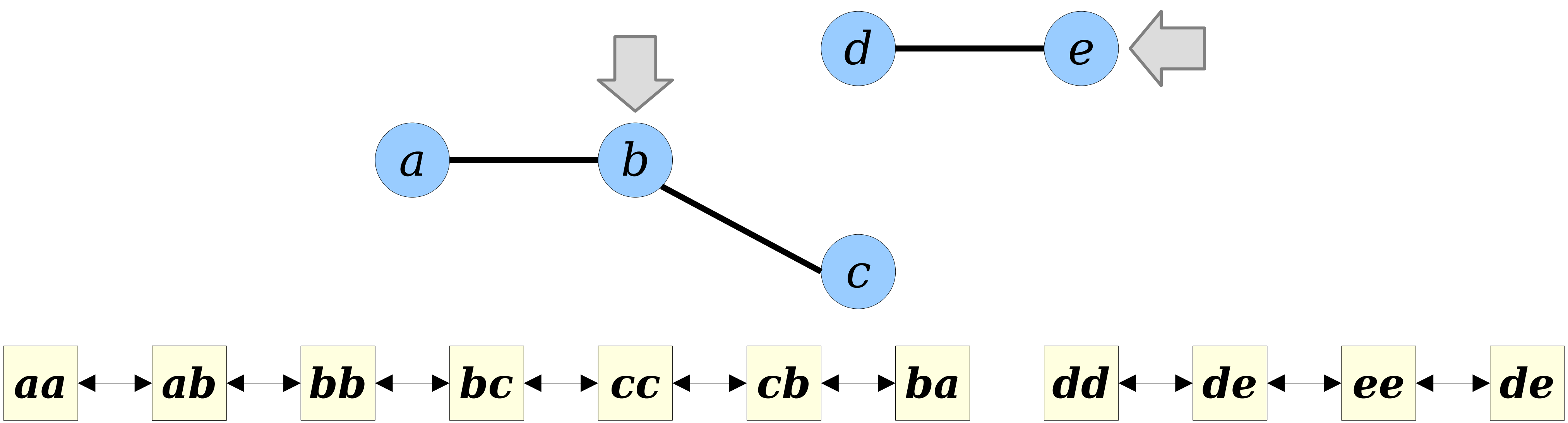

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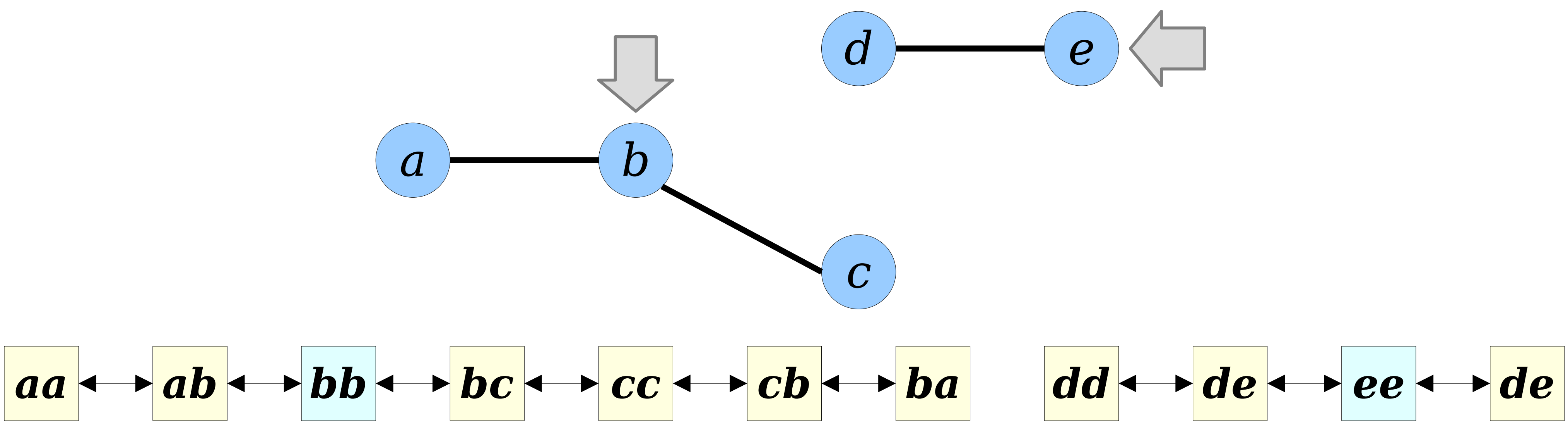

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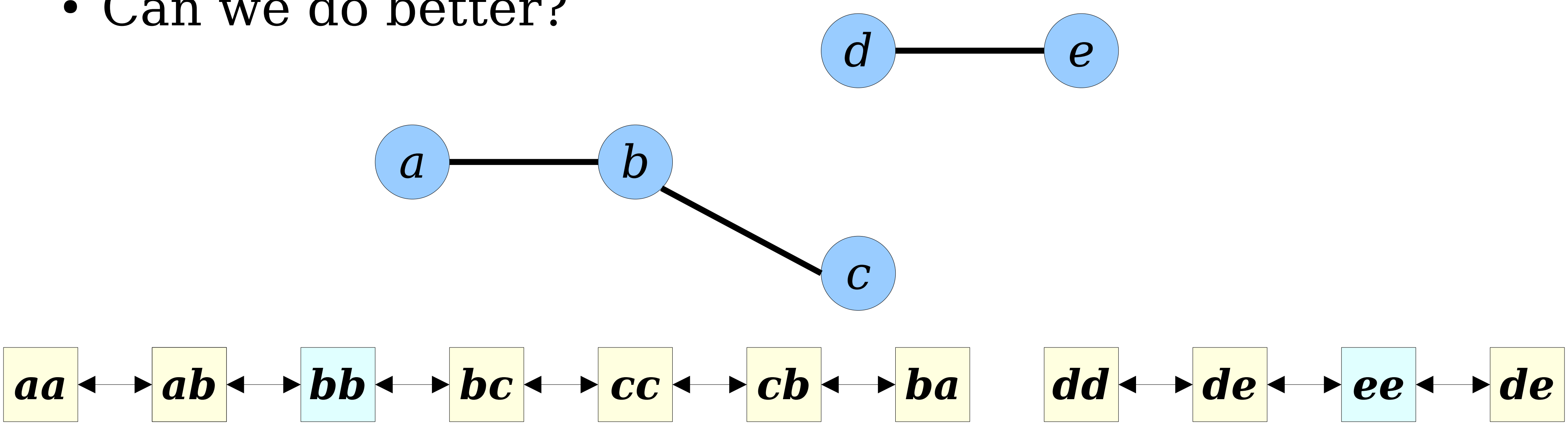
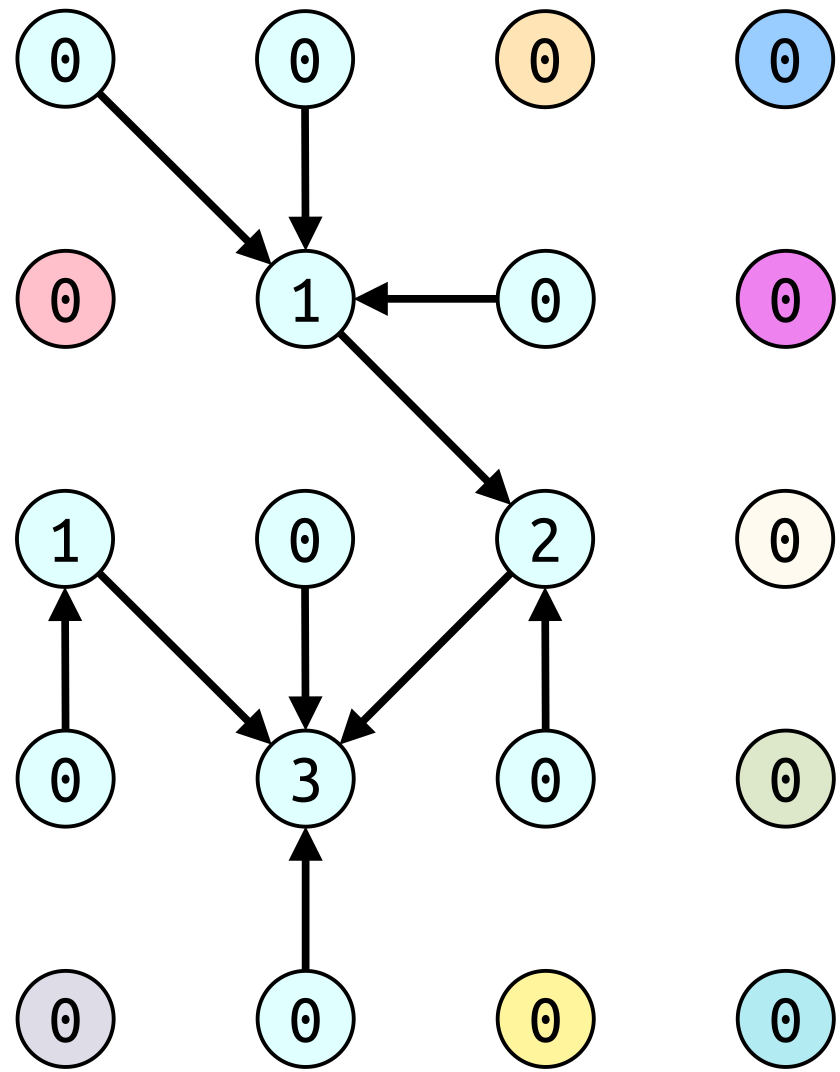
- Can we do better?



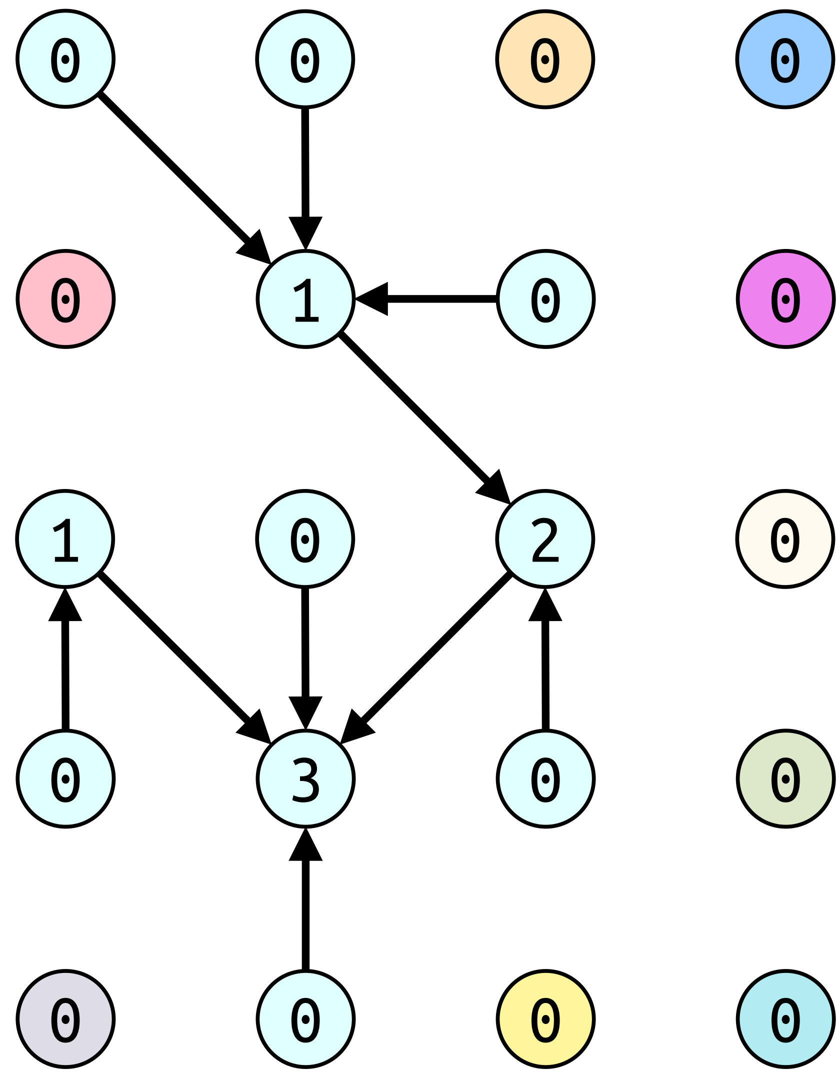
Representation Issues

- In incremental connectivity, we selected a representative for each CC.
- We then had elements store parent pointers that formed a path to the representative.
- Could we do something like that here?



Representation Issues

- The idea of using trees to store representatives is a good one.
  - If the trees are wide and flat, it won't take too long to find the representative.
  - If we don't have to update "too many" pointers when CC's change, our operations can run quickly.
- The trees we used last time won't (immediately) work here.
  - We have to store the elements of the tour in sequential order. There was no such notion of order in disjoint set forests.
  - In disjoint-set forests, linked items can never be cut, allowing for some clever optimizations.
- What's another tree we can use?



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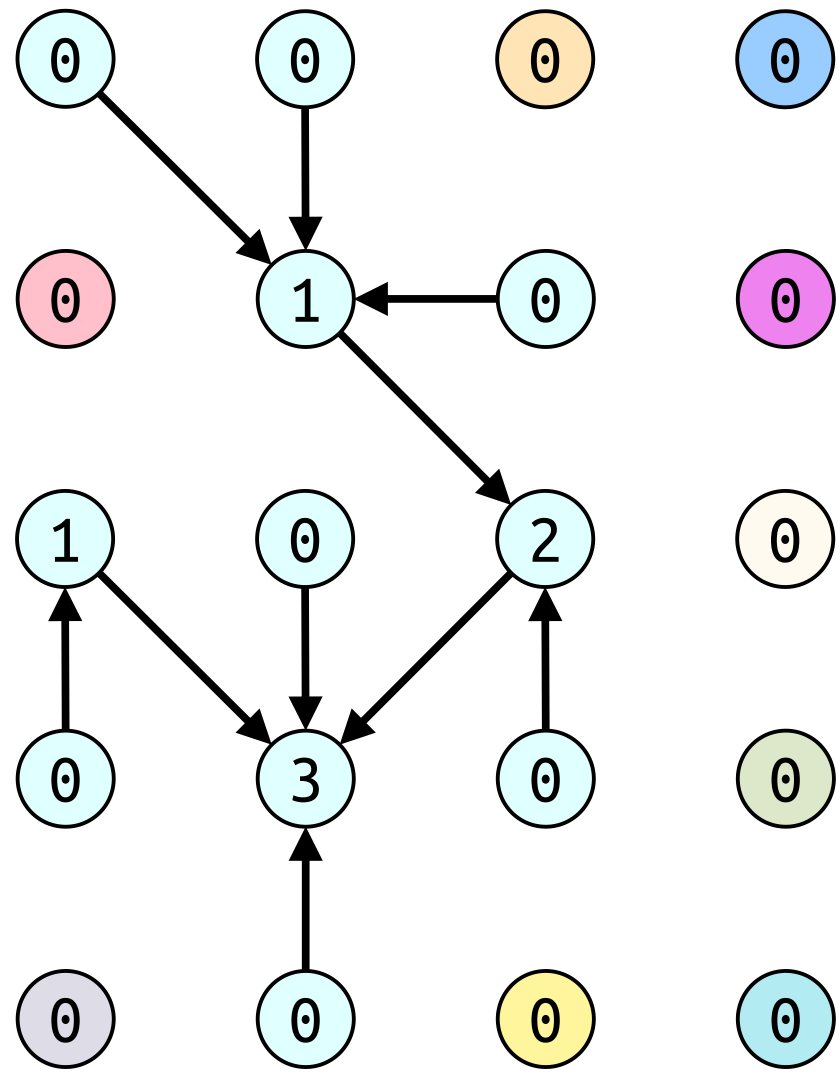
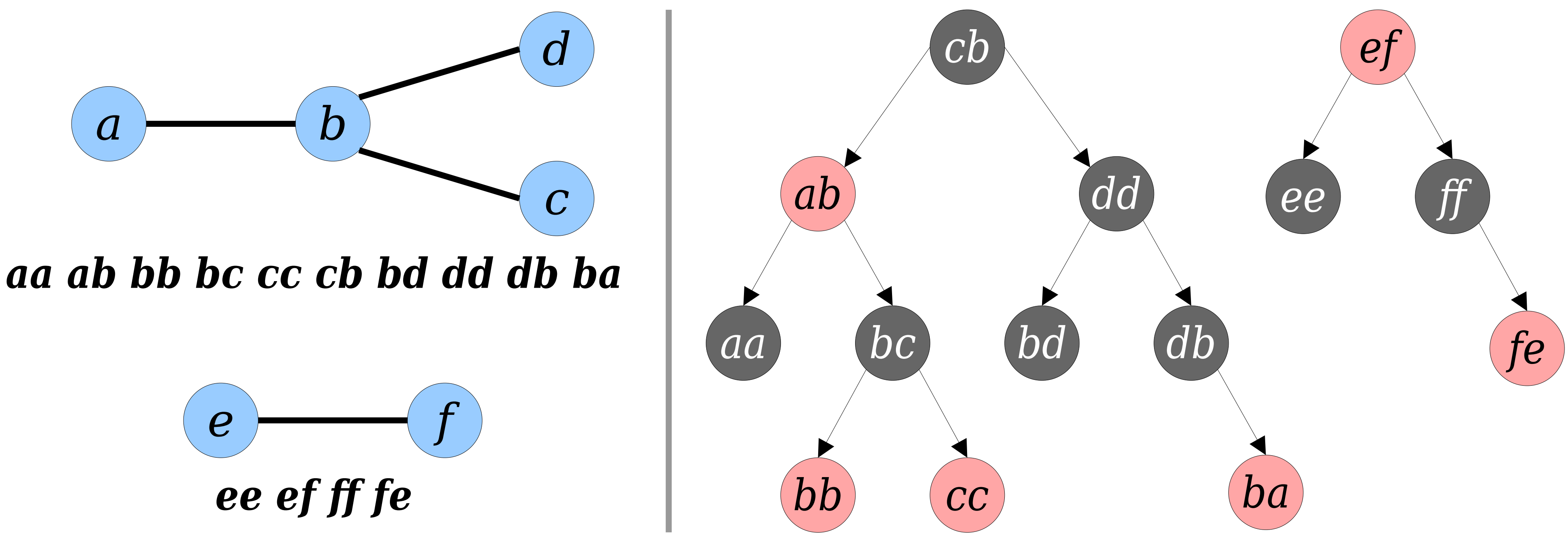


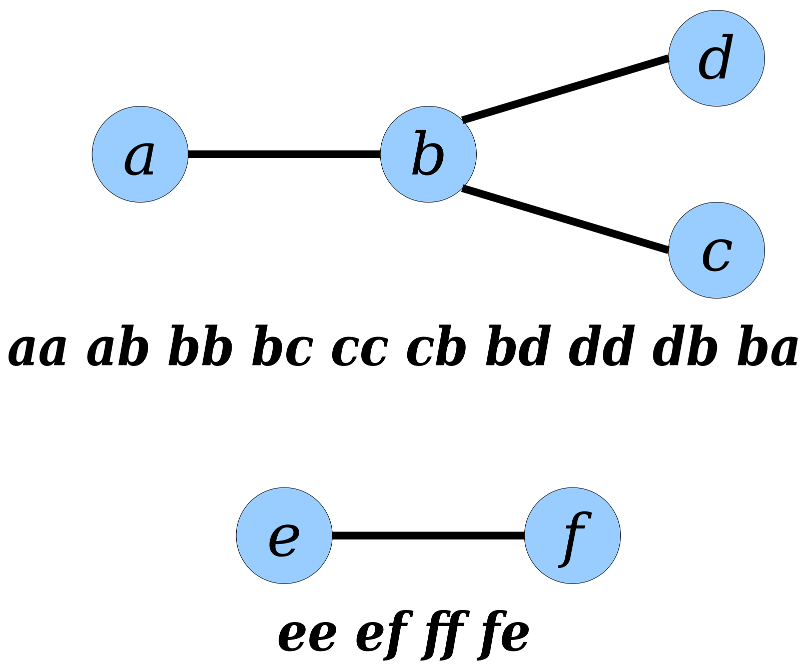
Binary Search(less) Trees

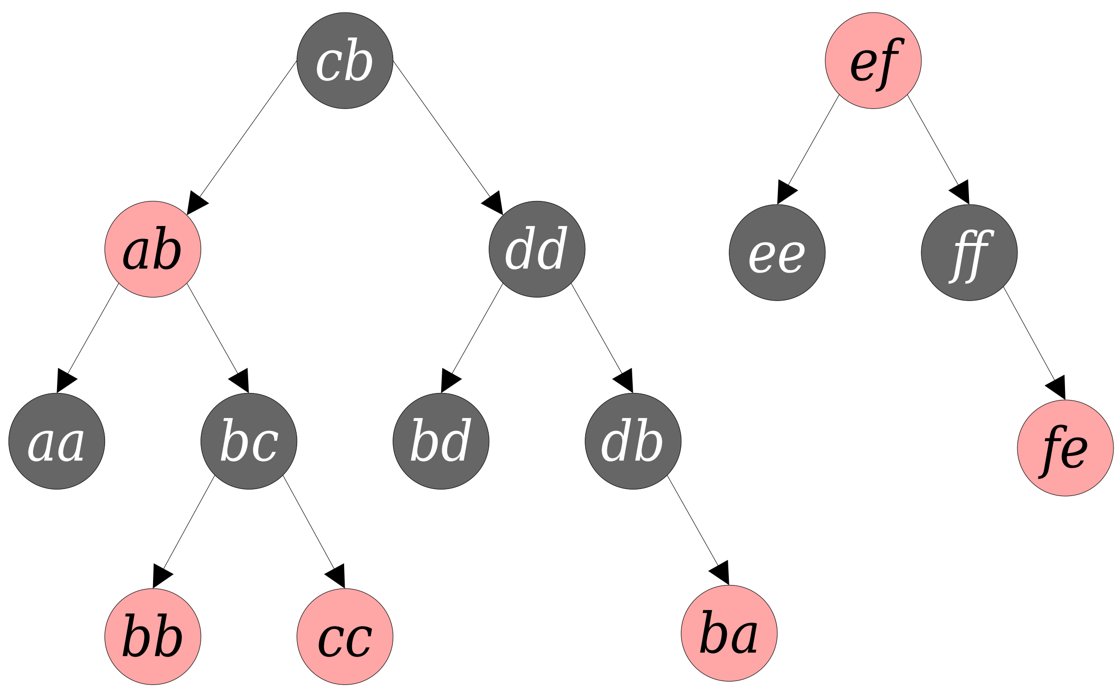
- ***Idea 2:*** Store our sequences in a balanced BST, sorted by their position within the sequence.
- We'll use the *shape* and *algorithm* of a BST, but won't have the ability to conventionally search the tree top-down.
- We'll rely on the fact that we have external pointers that let us jump to items within the BST.



Binary Search(less) Trees

- We can now answer ***are-connected***(x, y) in time O(log n).
  - Find xx and yy using our auxiliary lookup table.
  - Walk up from xx and yy to the roots of their trees.
  - See if they're the same root.





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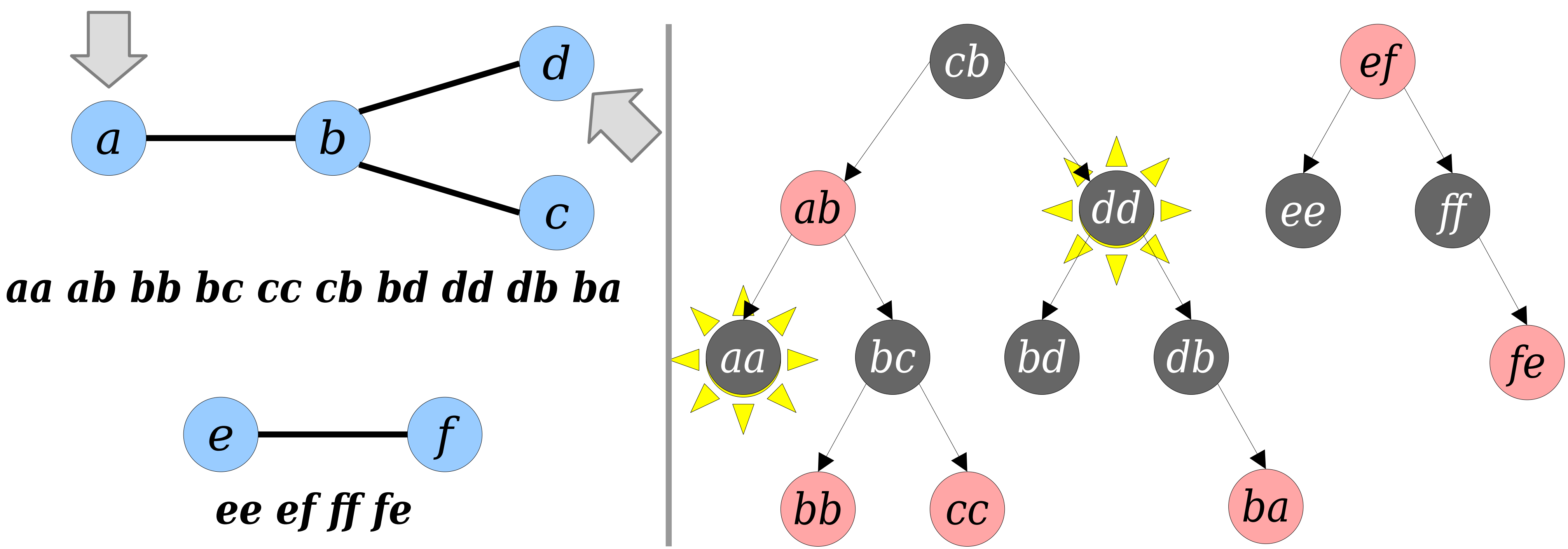

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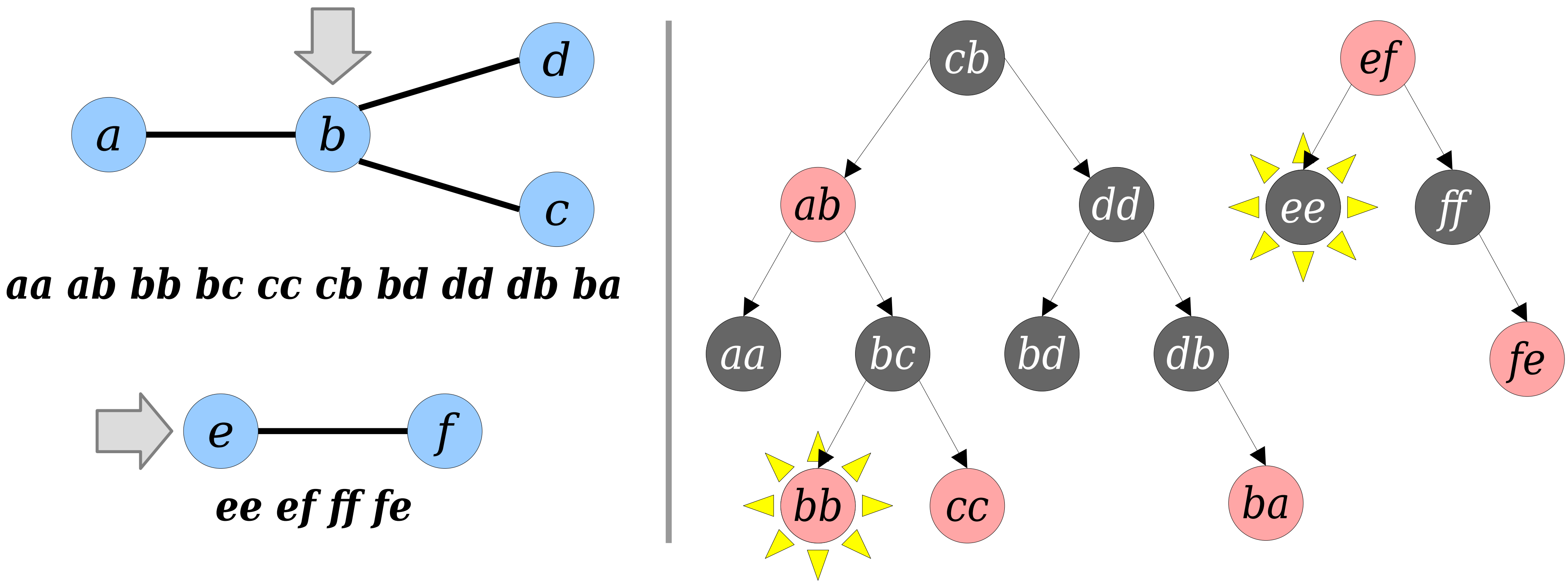

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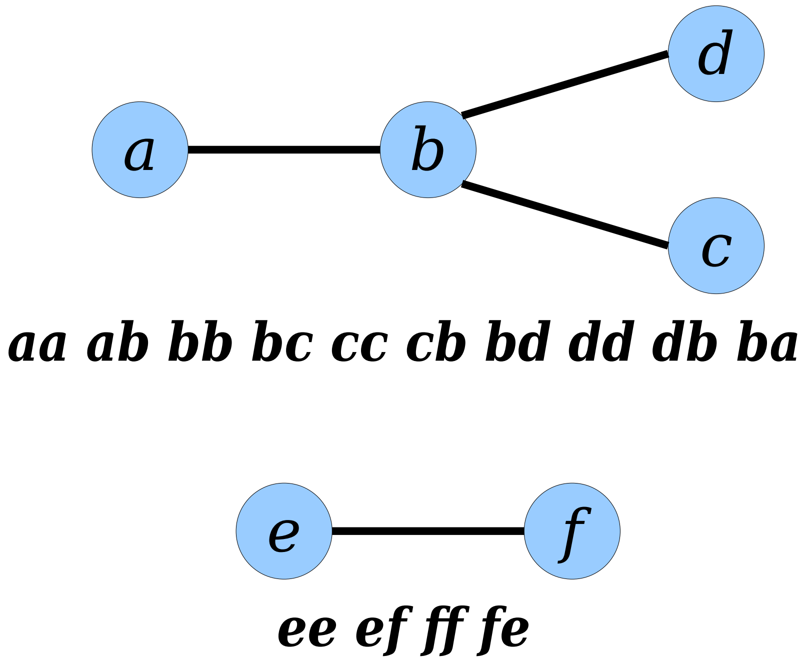

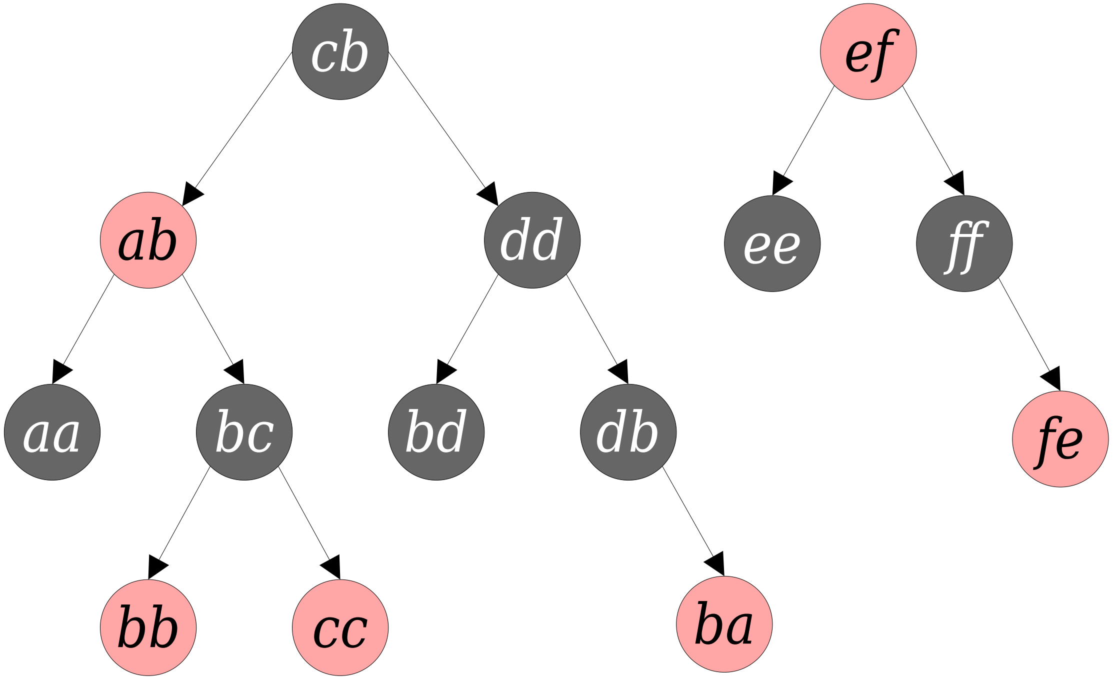

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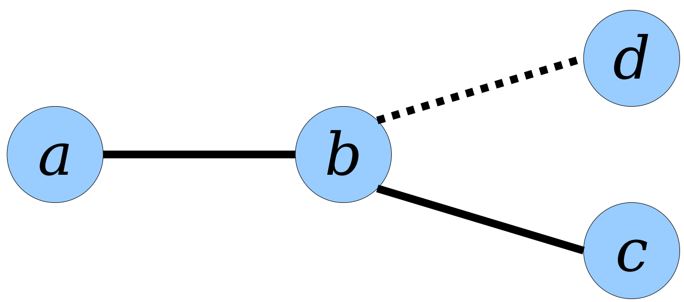

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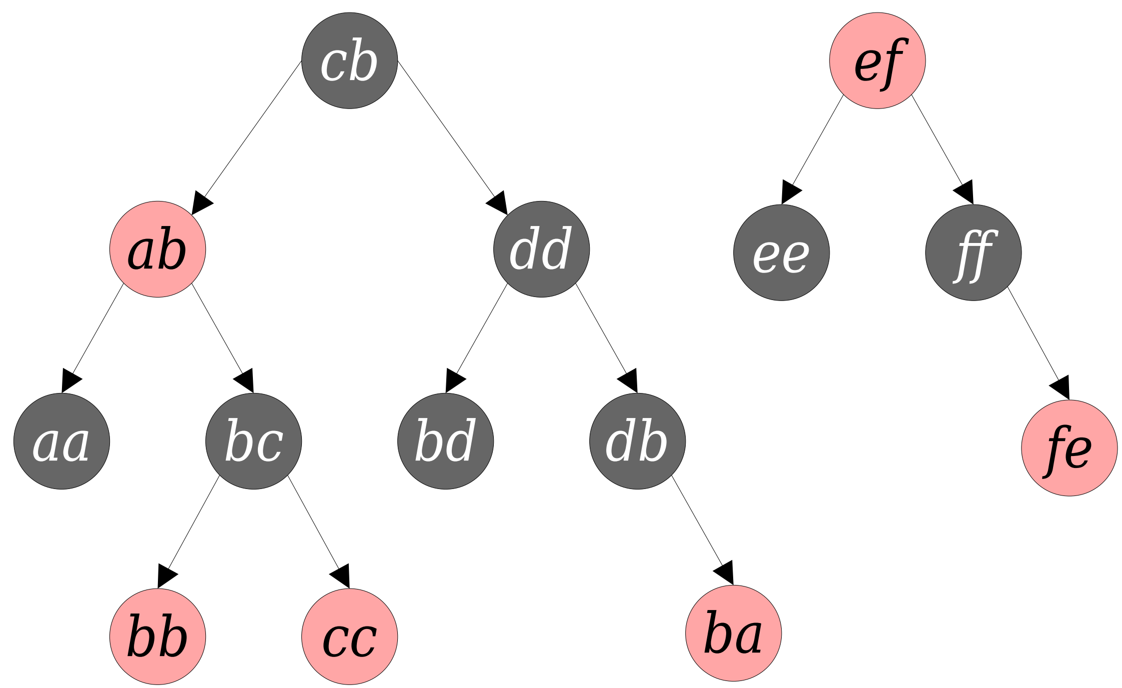

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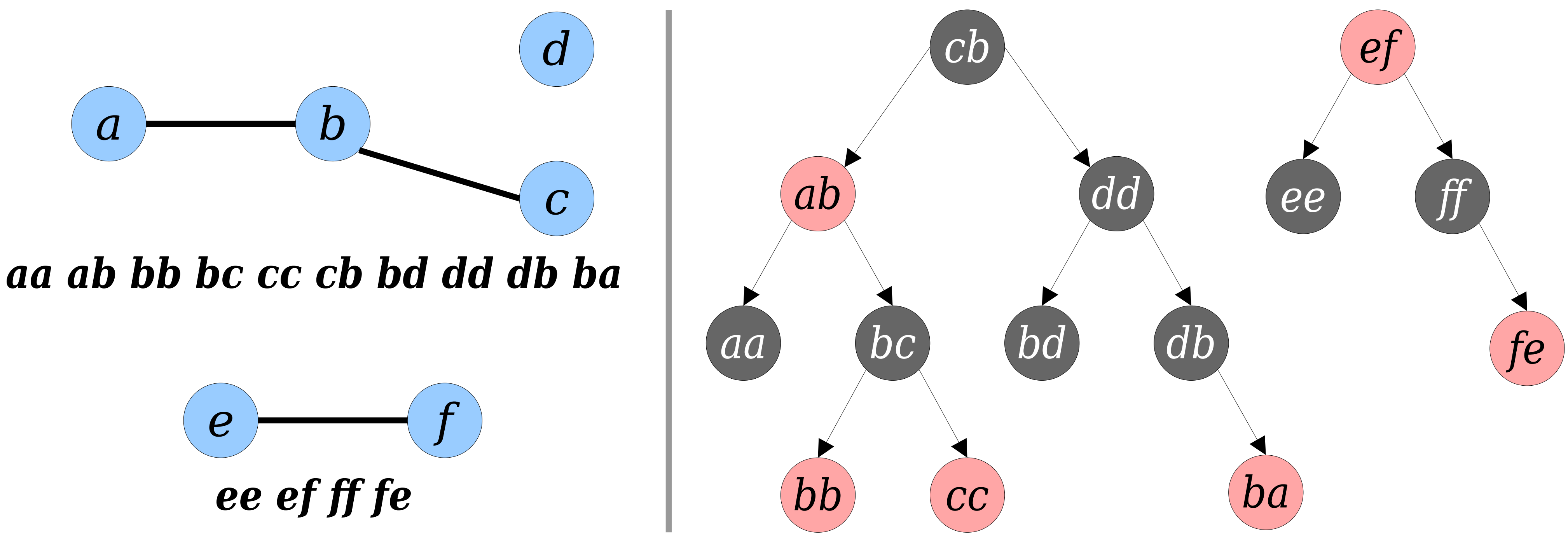

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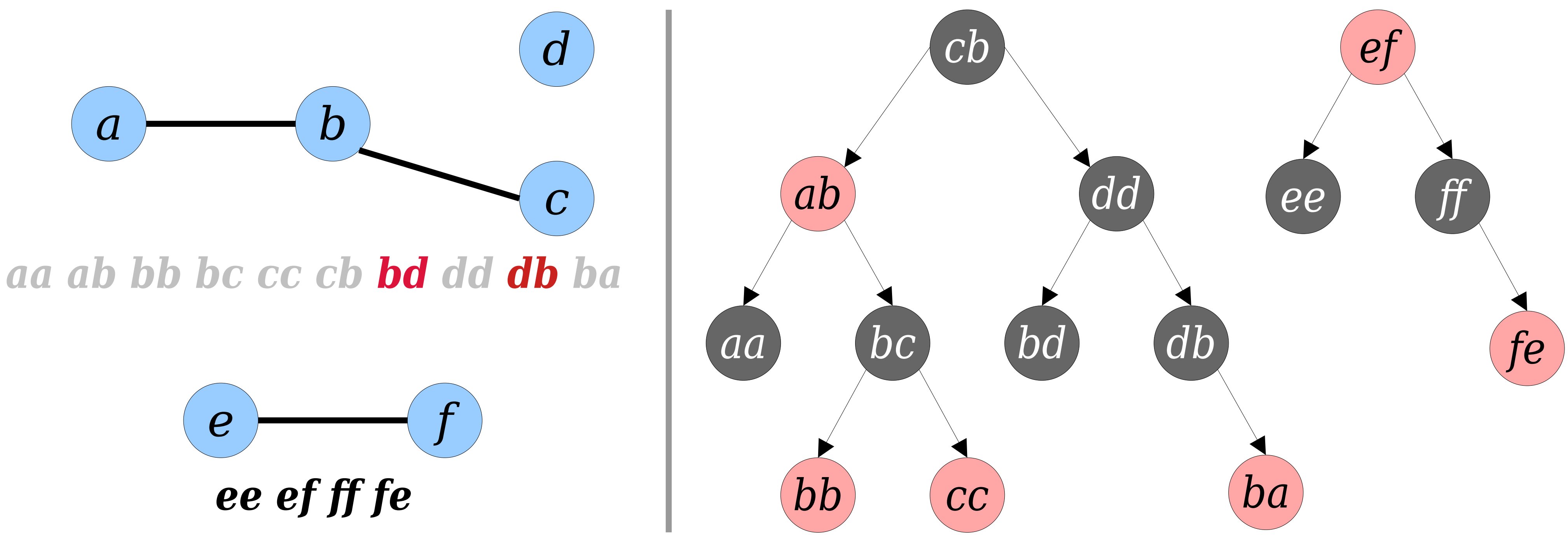

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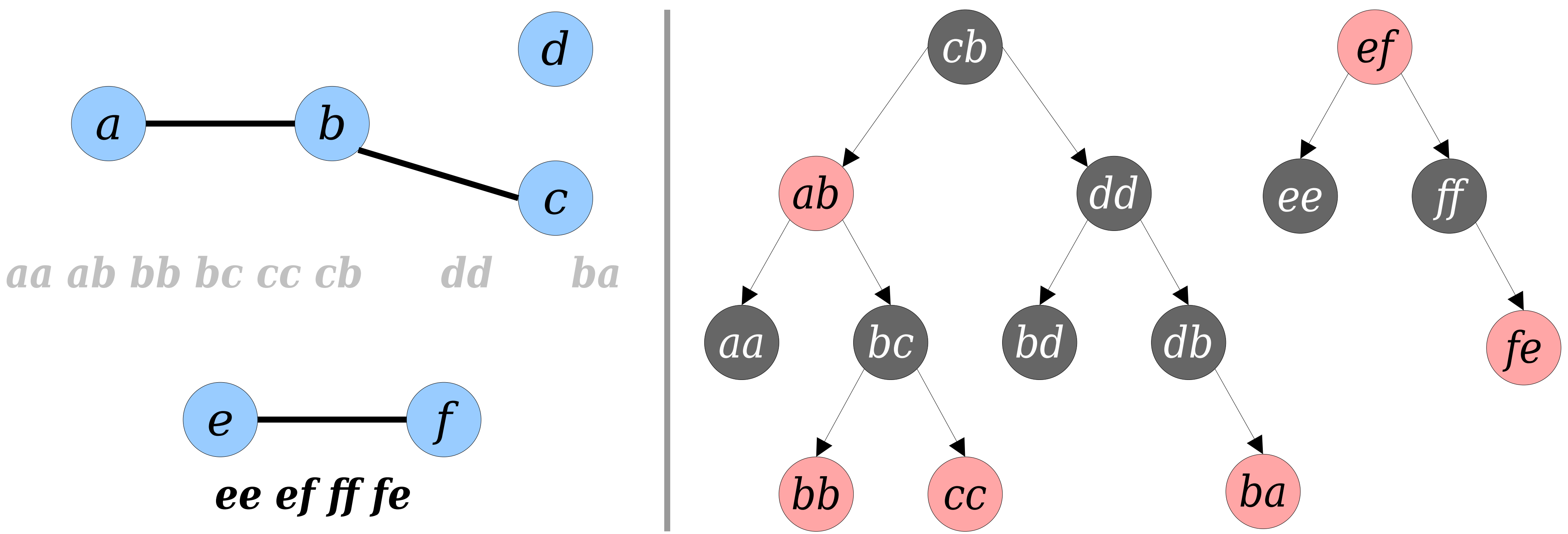

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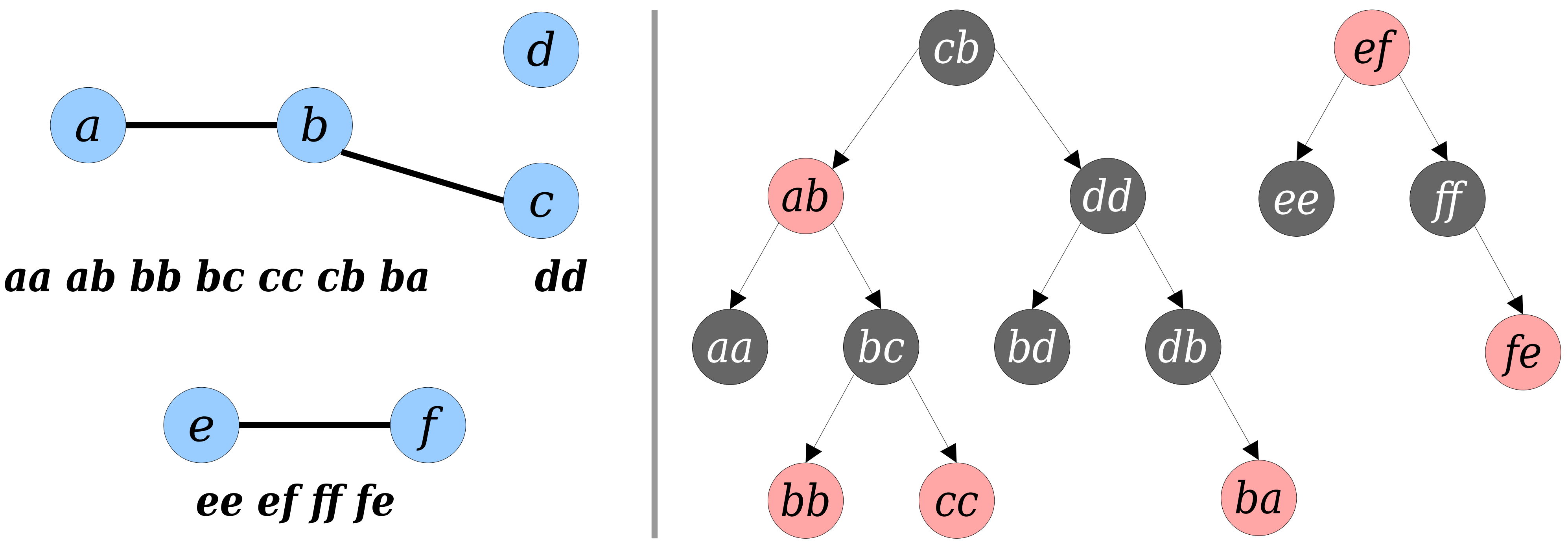

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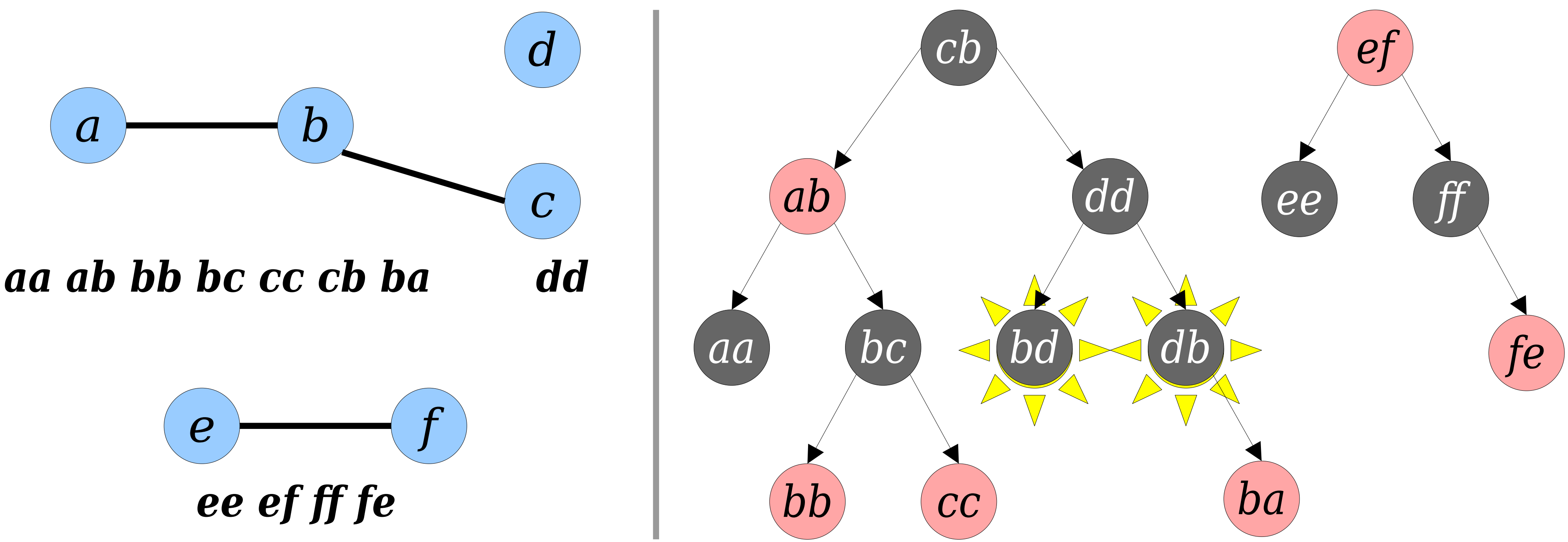

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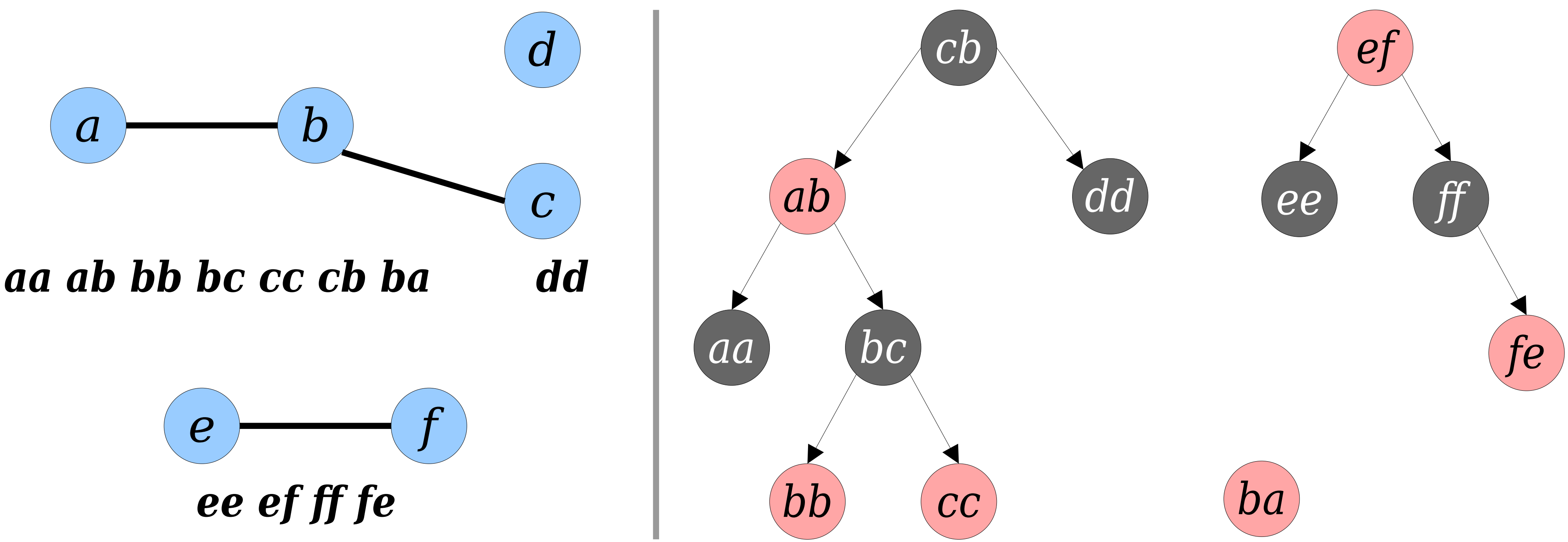

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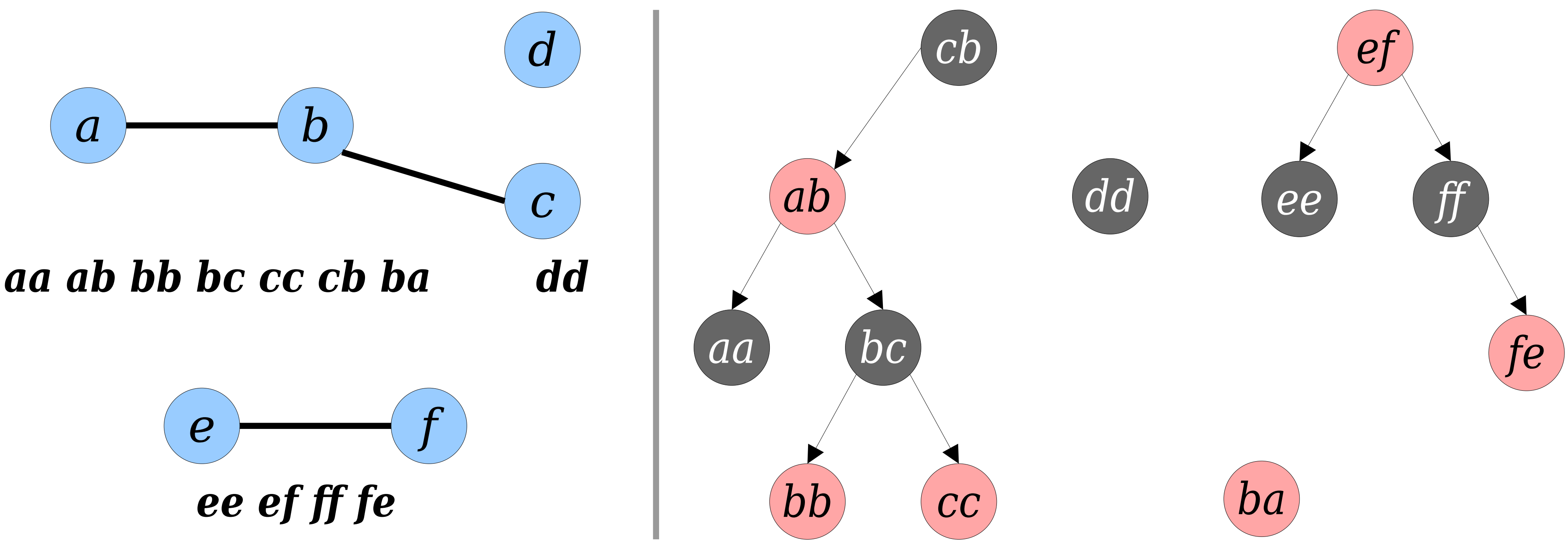

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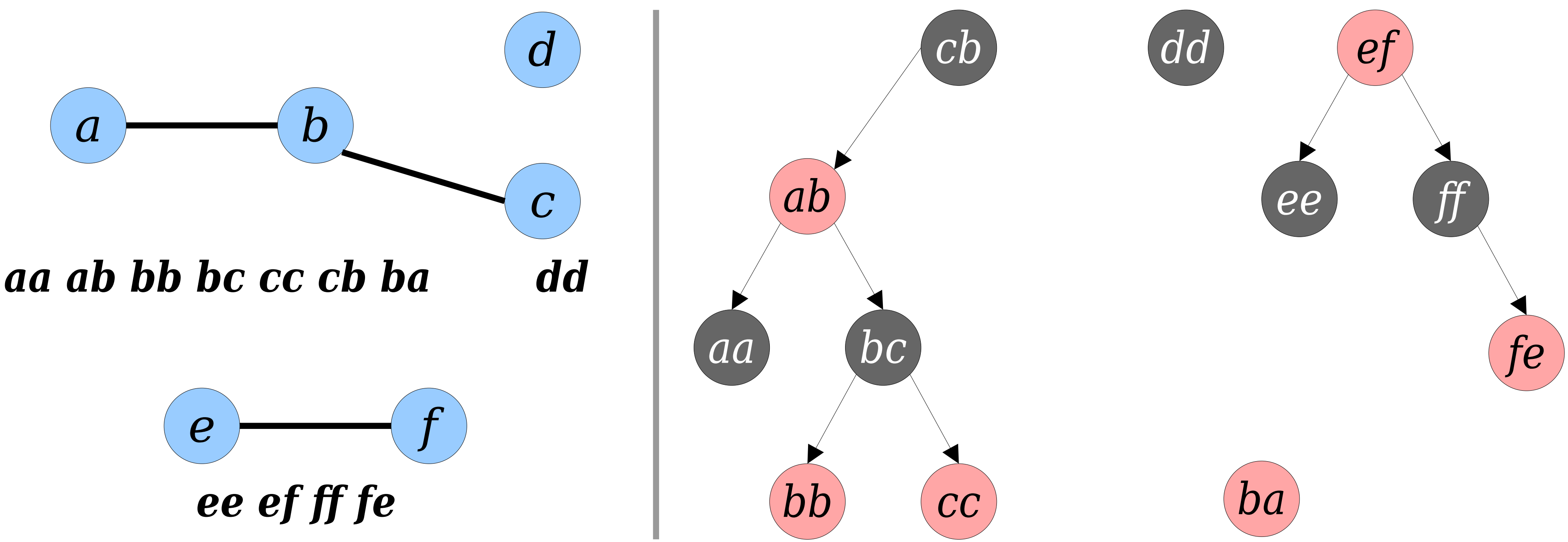

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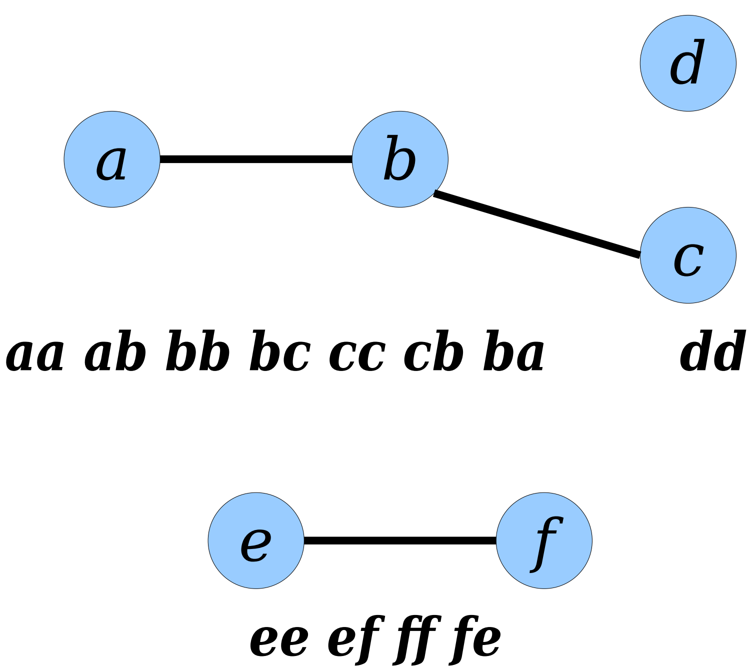

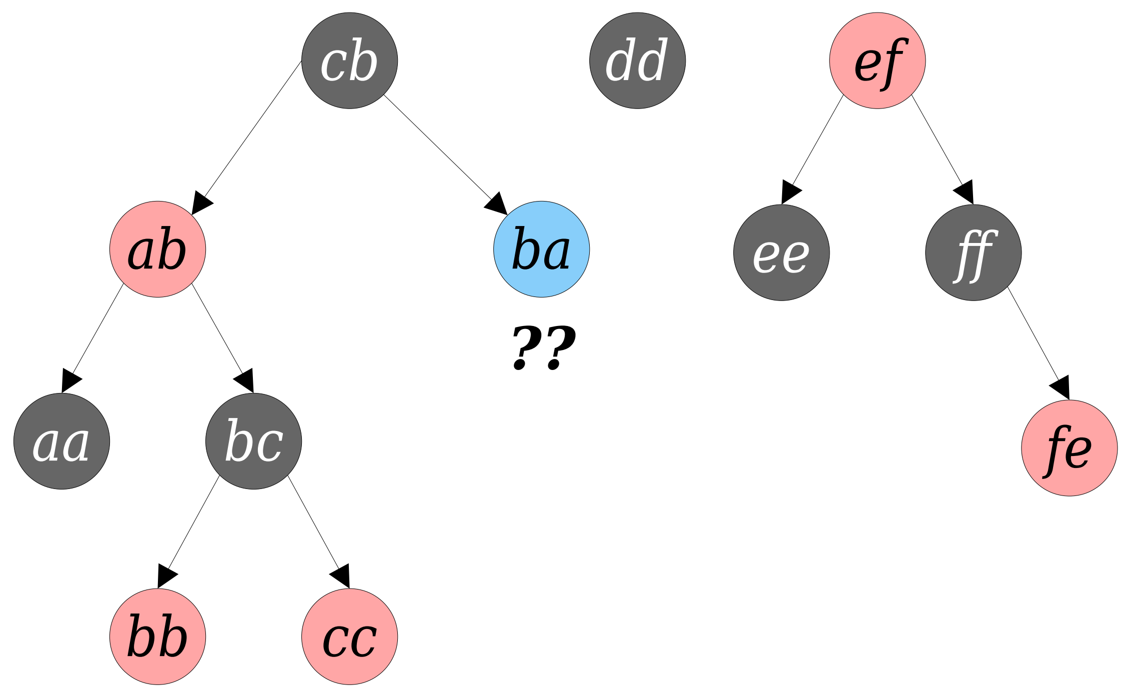

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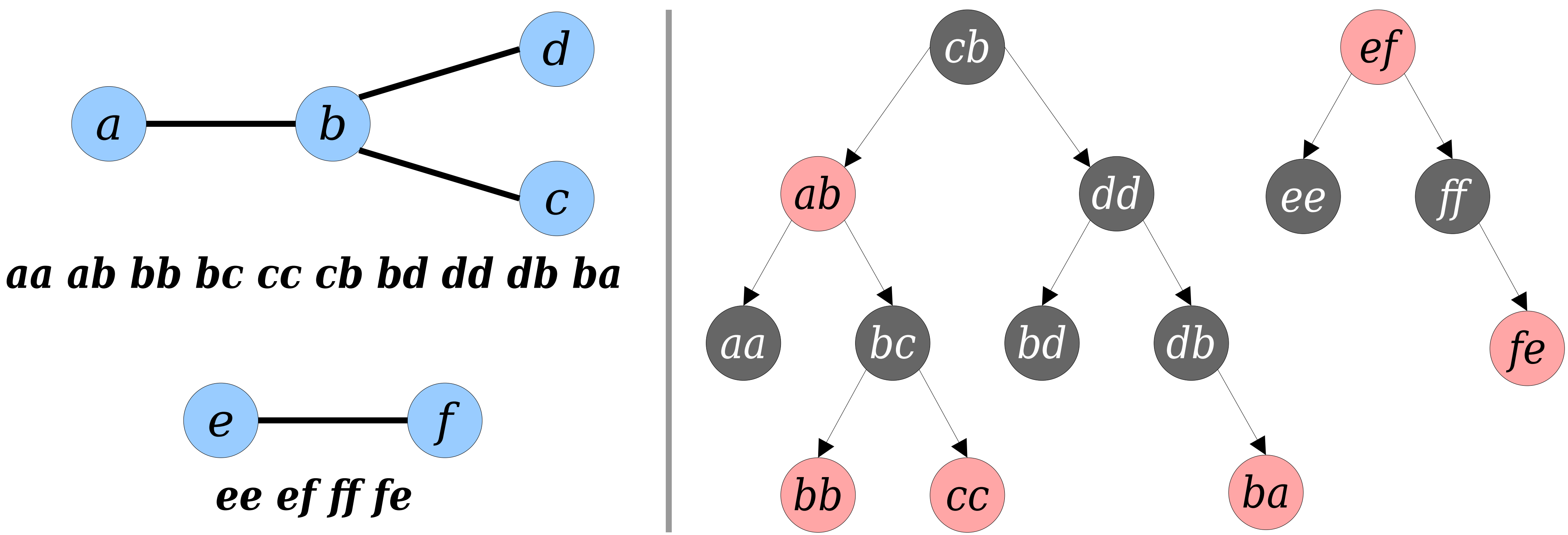

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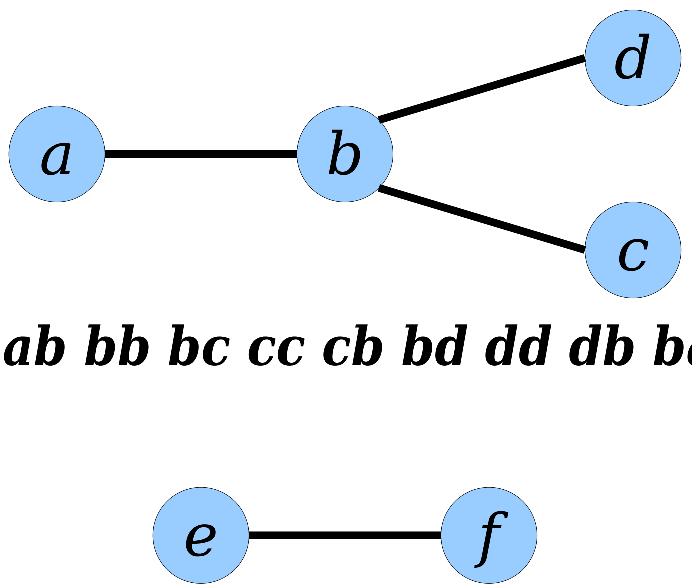
- ***Answer:*** Use splay trees! They support these operations in amortized time O(log n).



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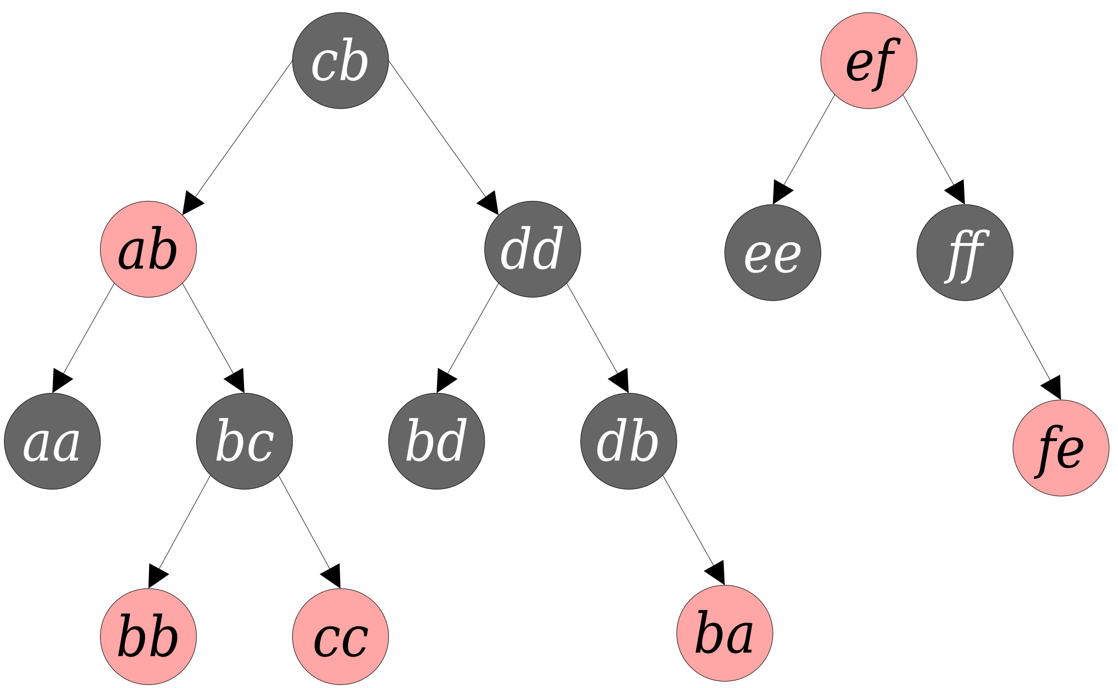

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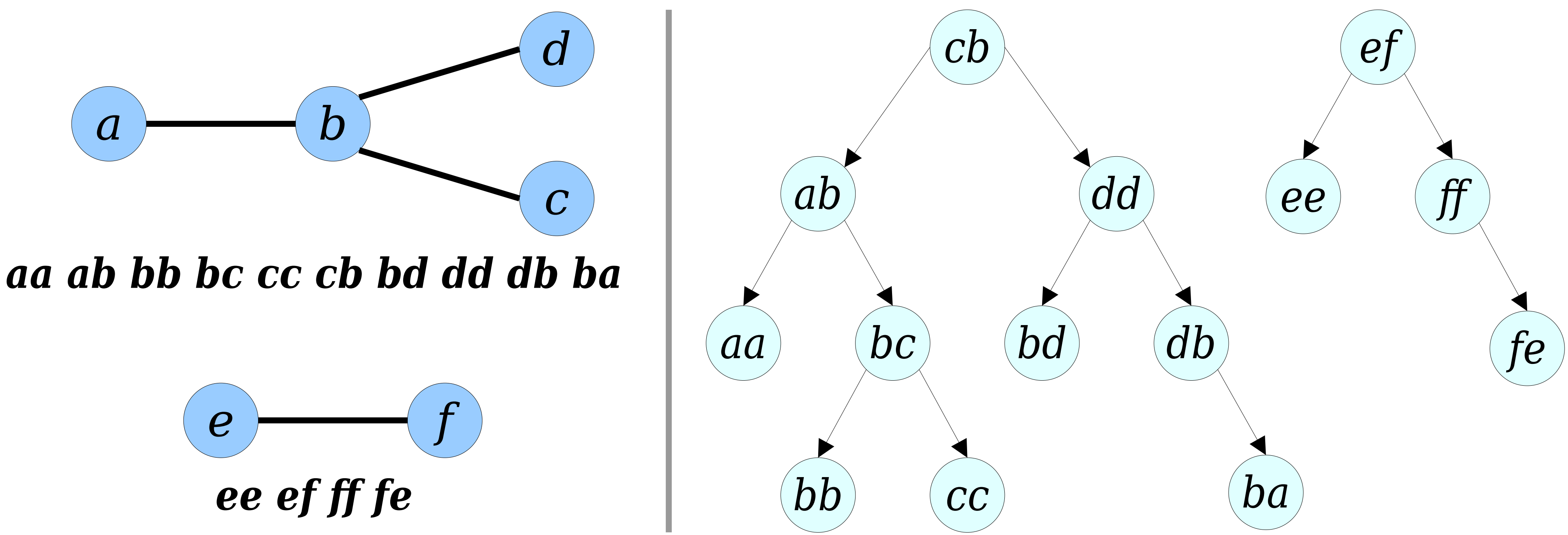

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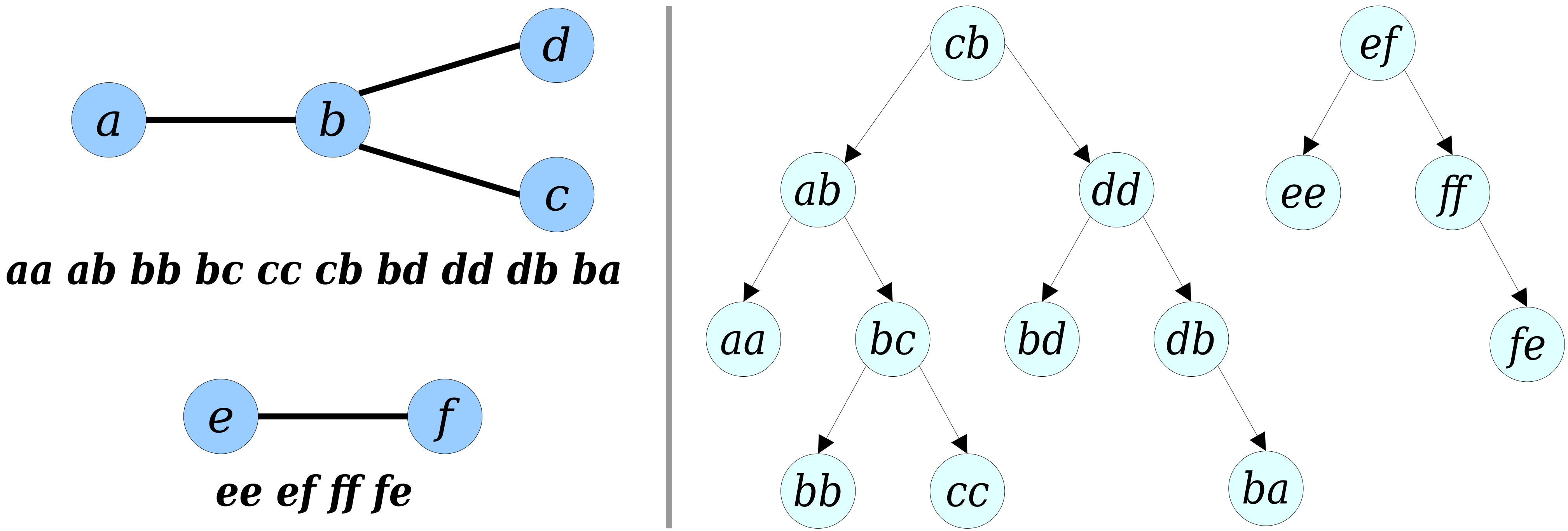

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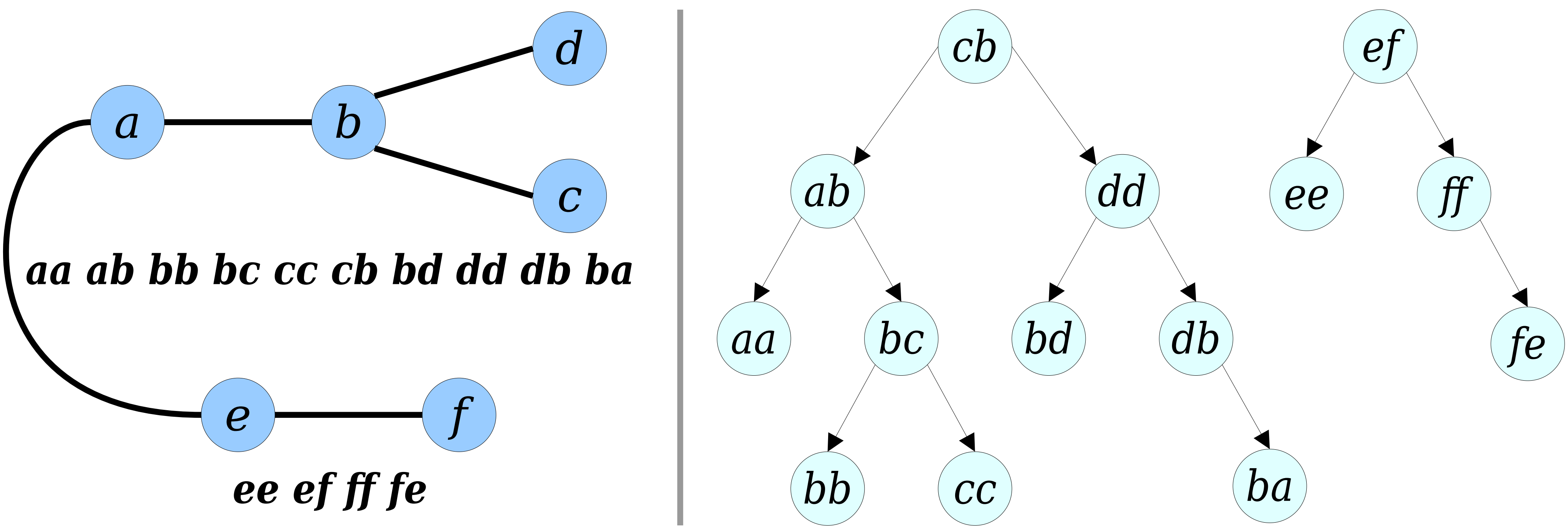

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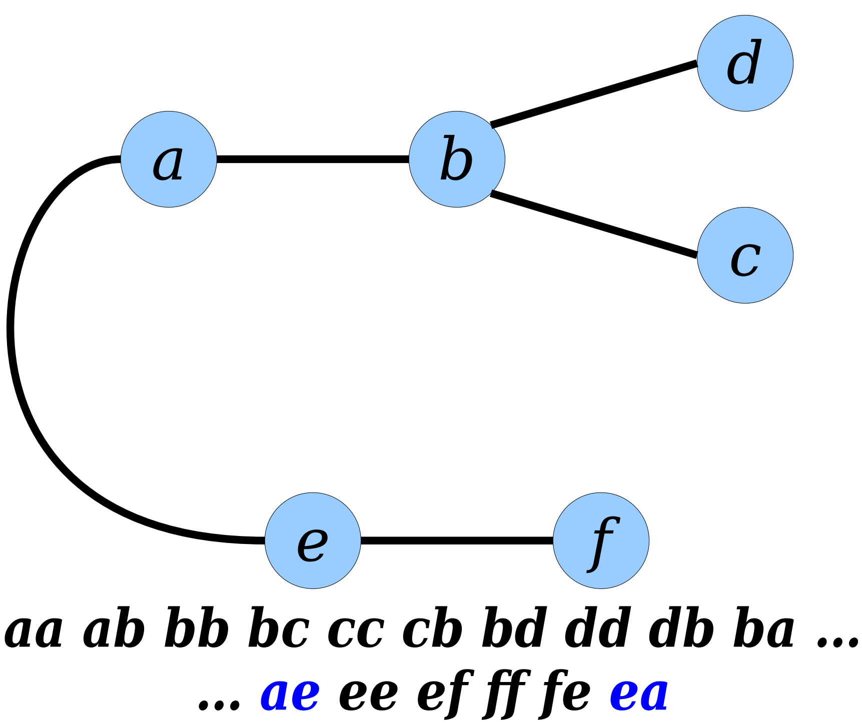

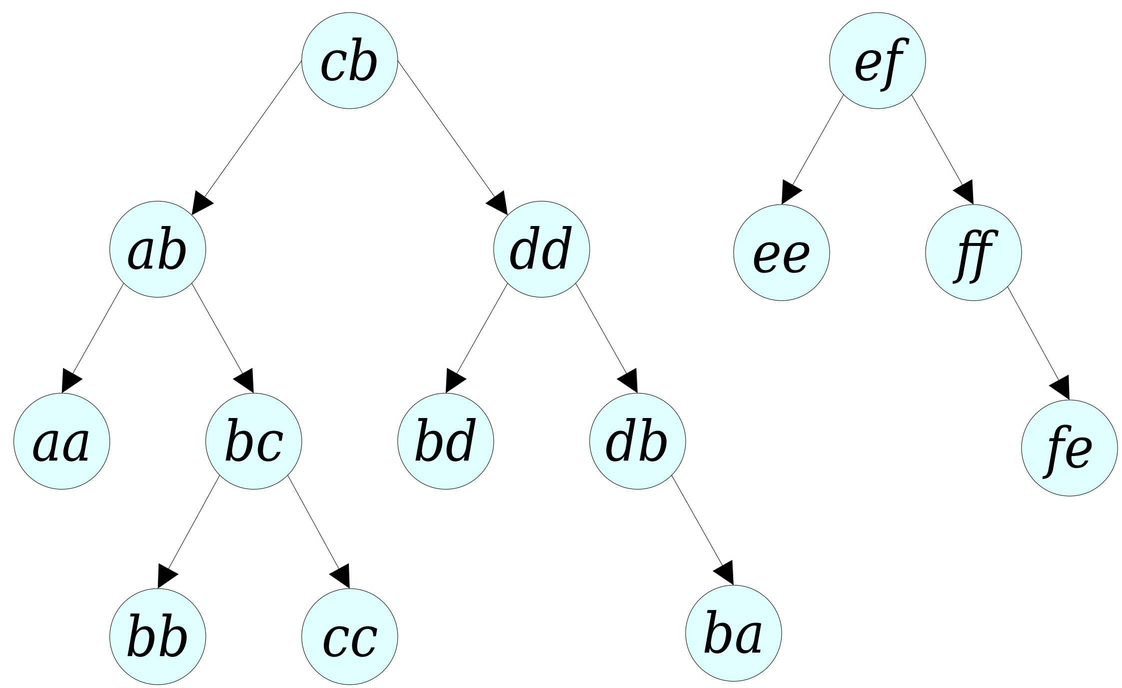

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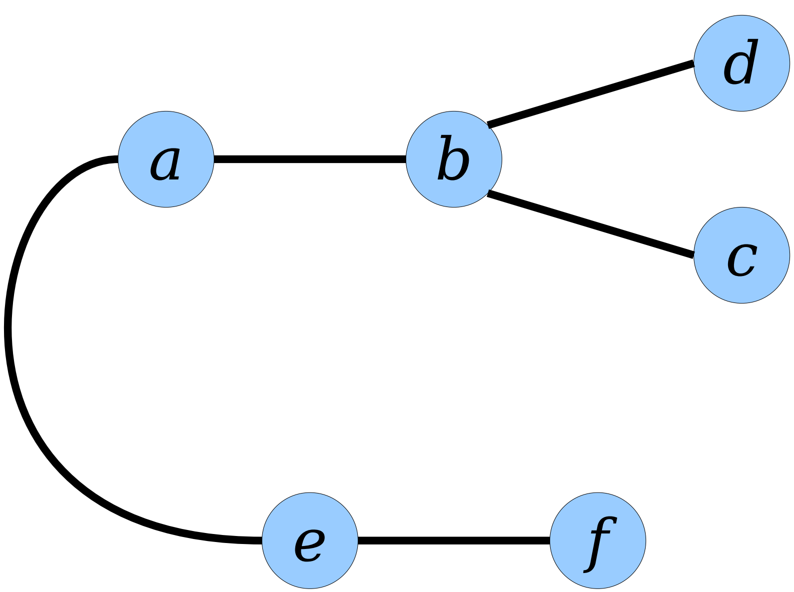

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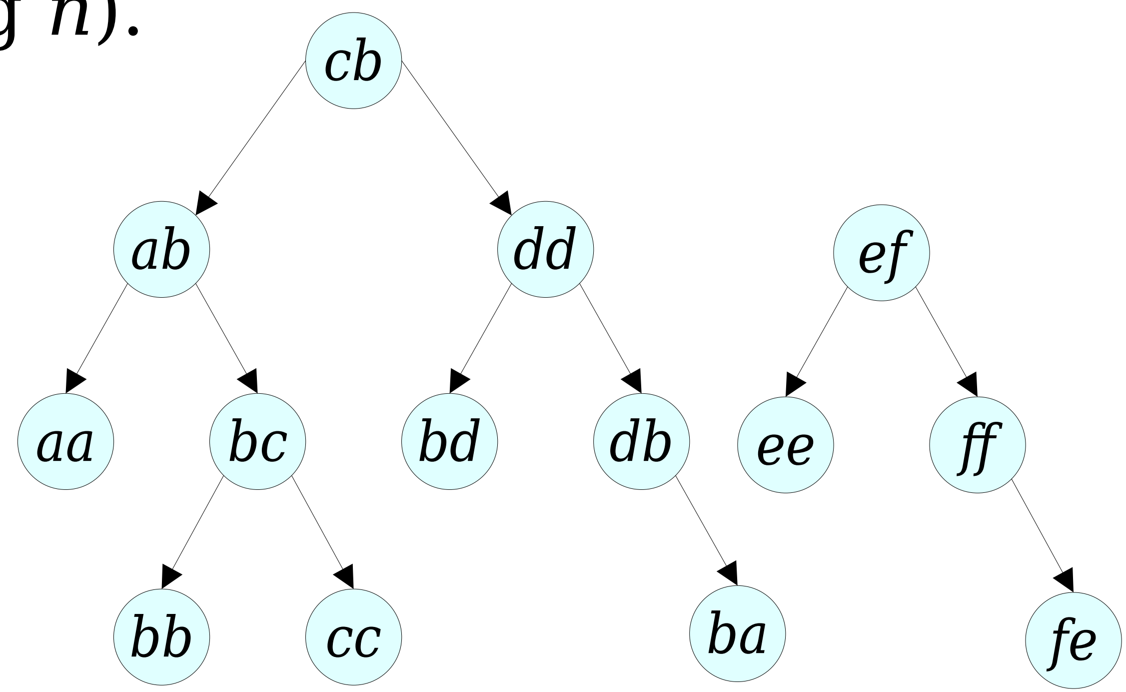

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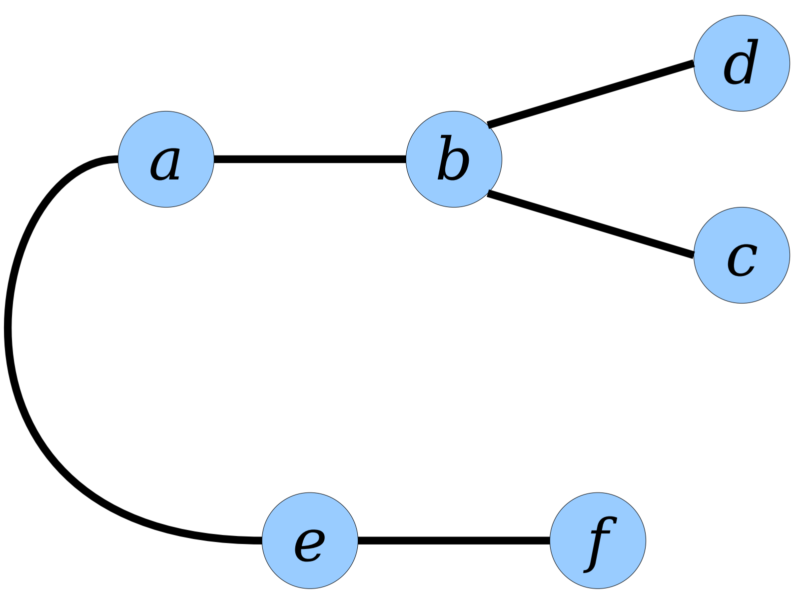

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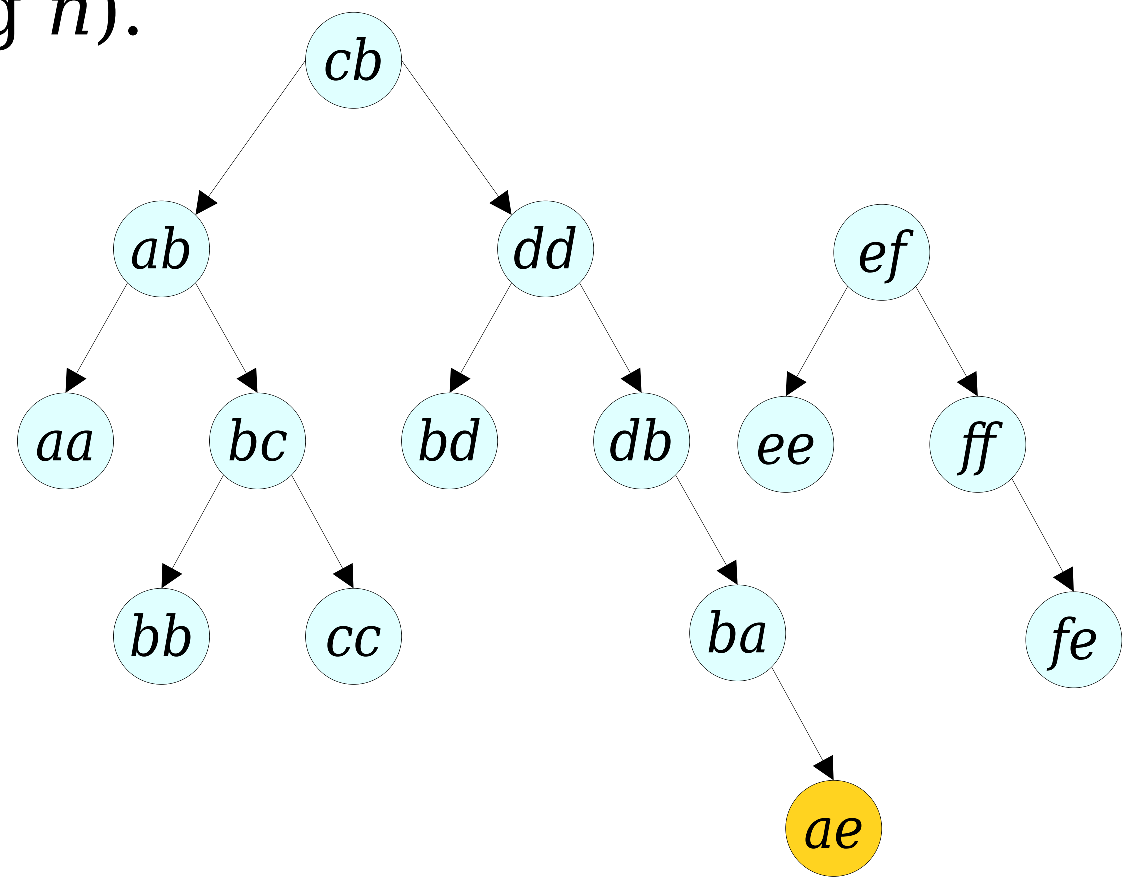

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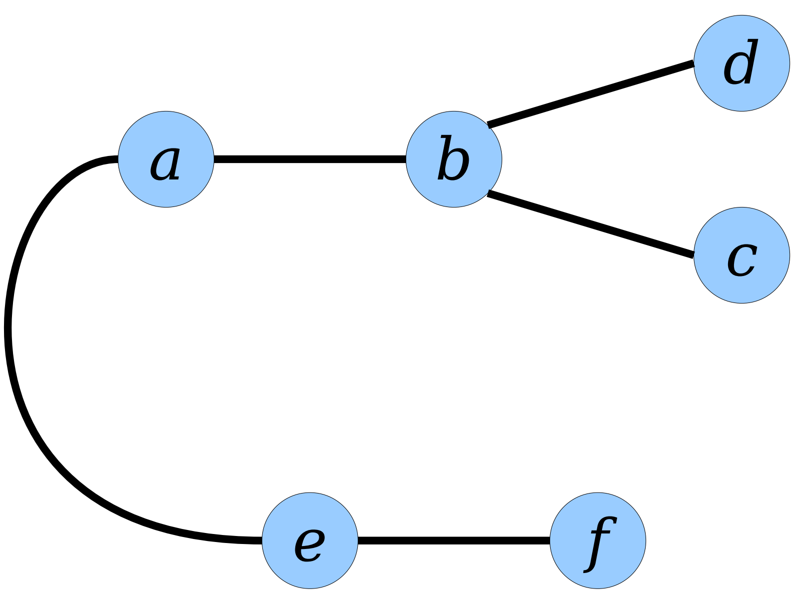

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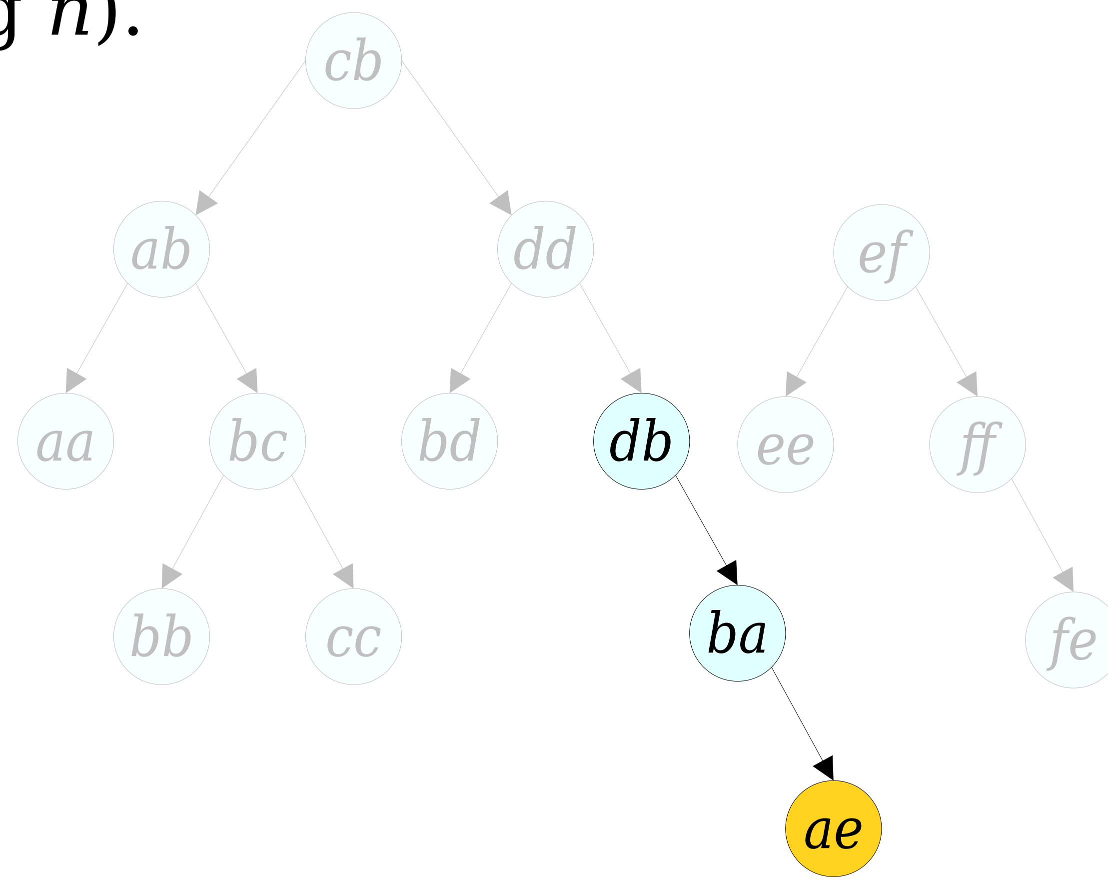

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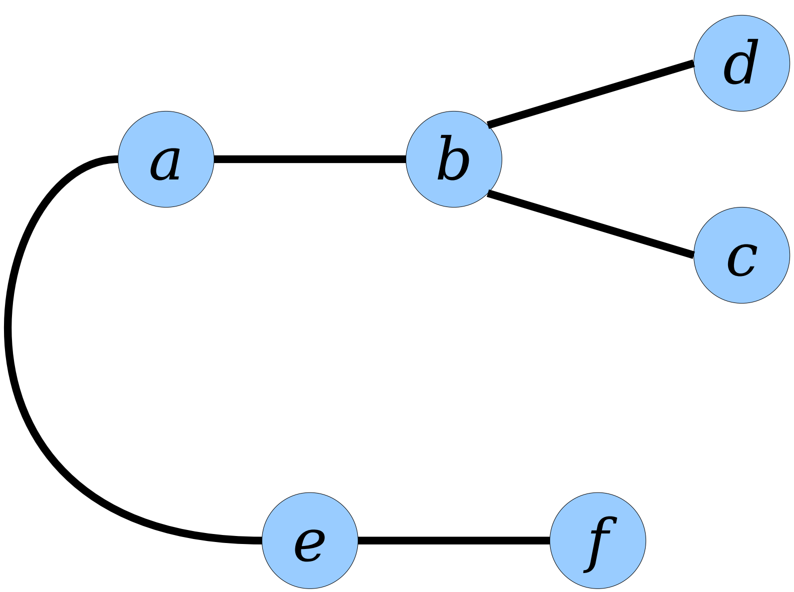

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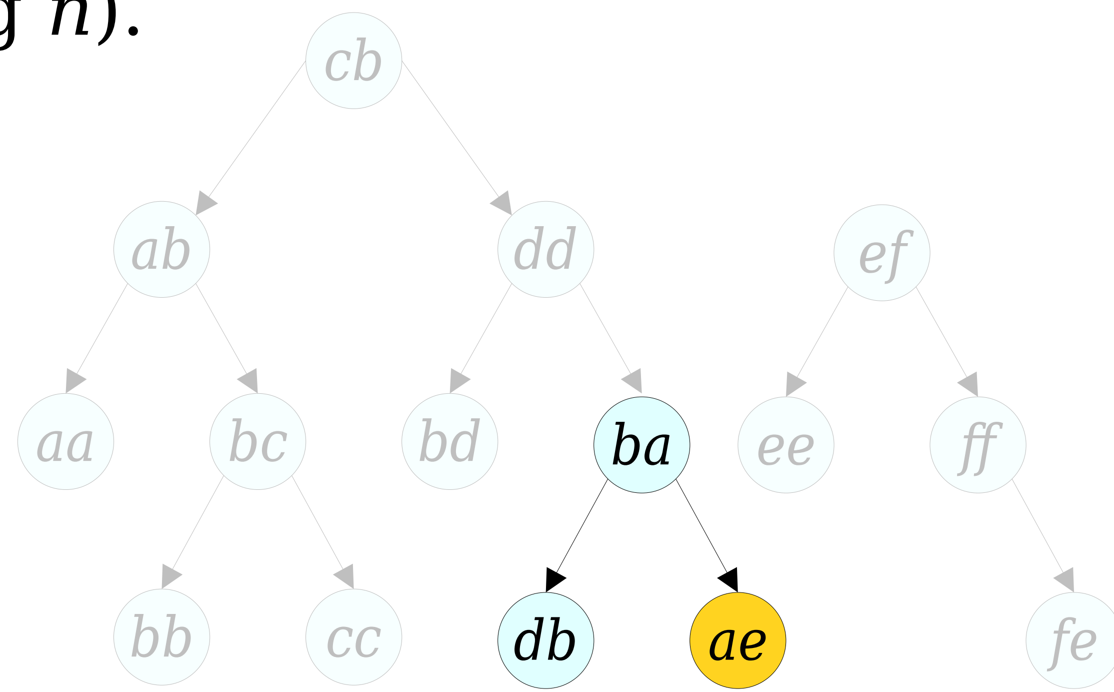

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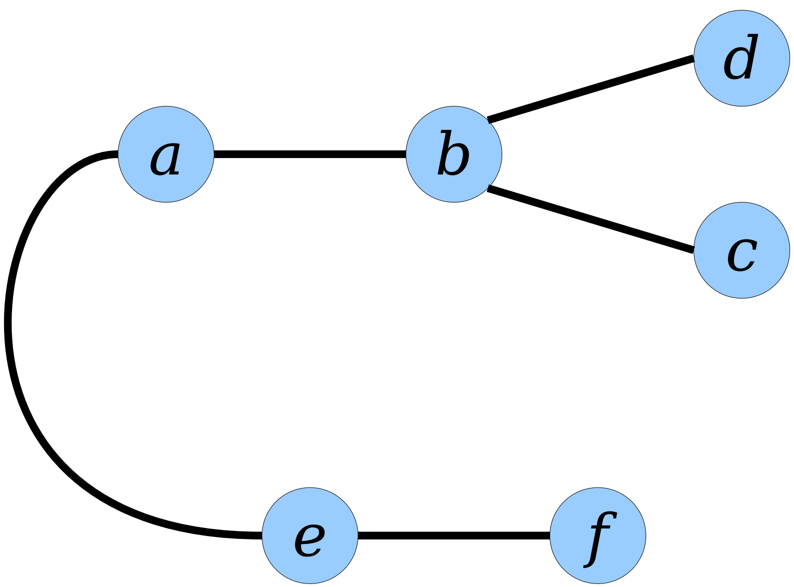

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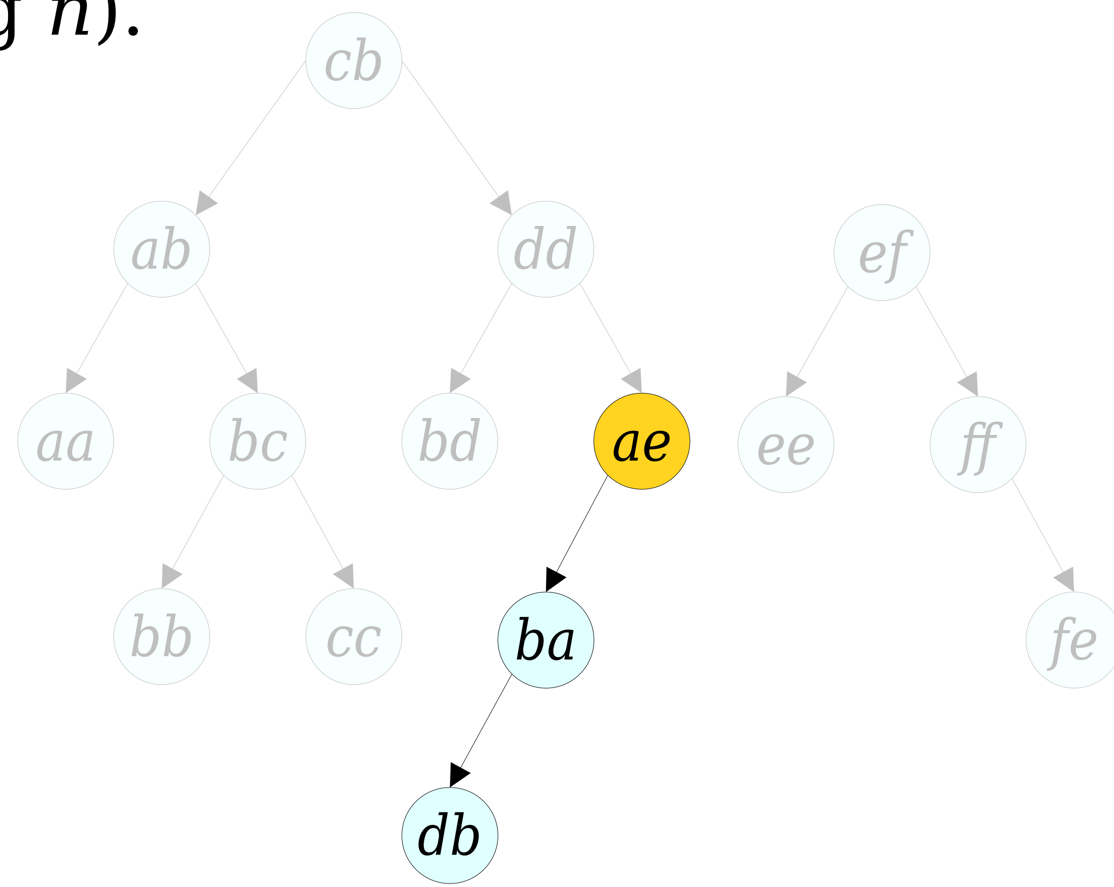

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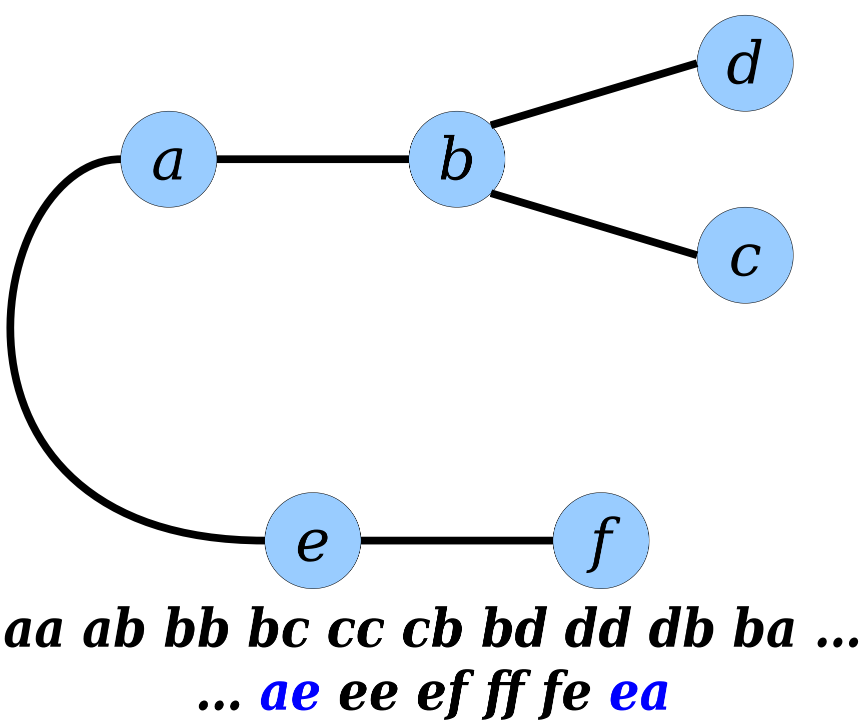

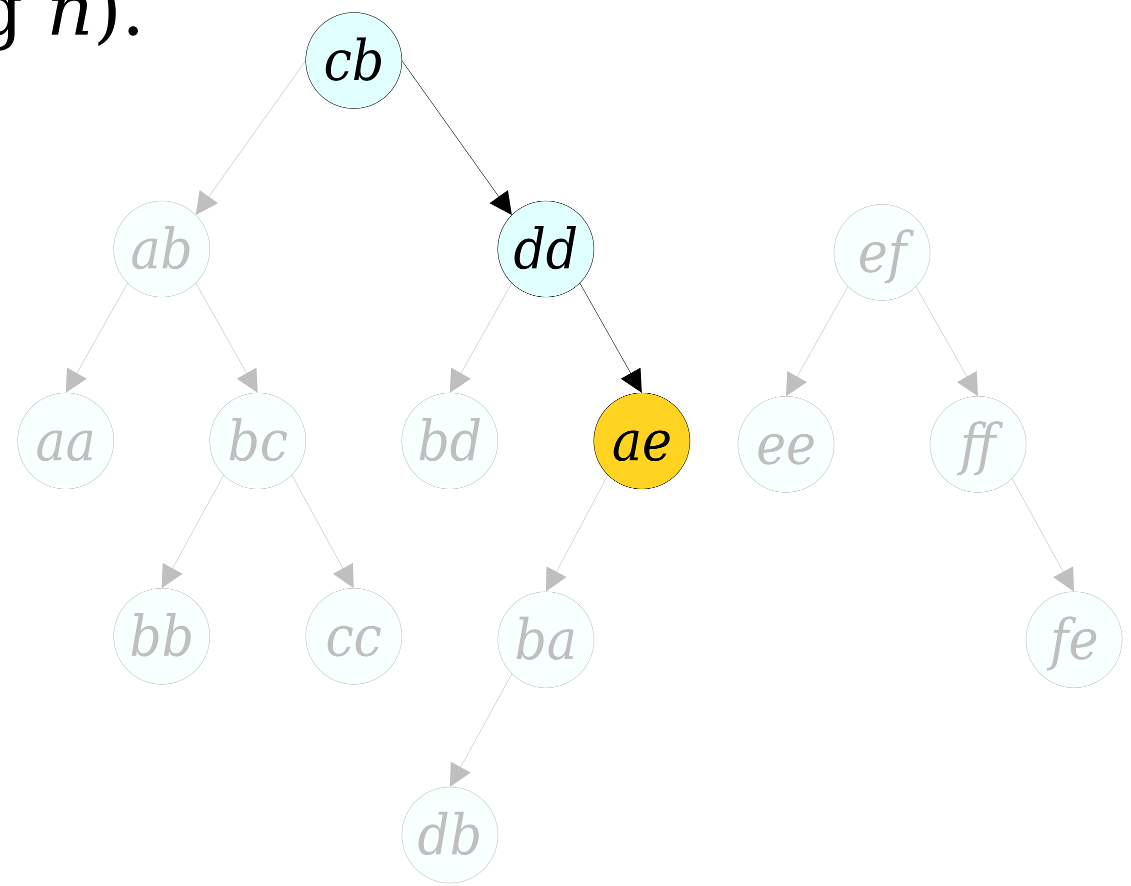

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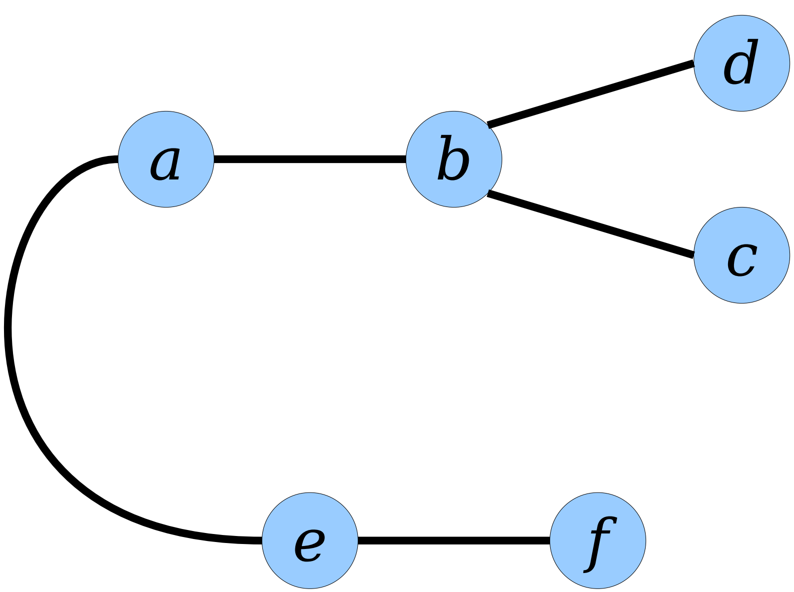

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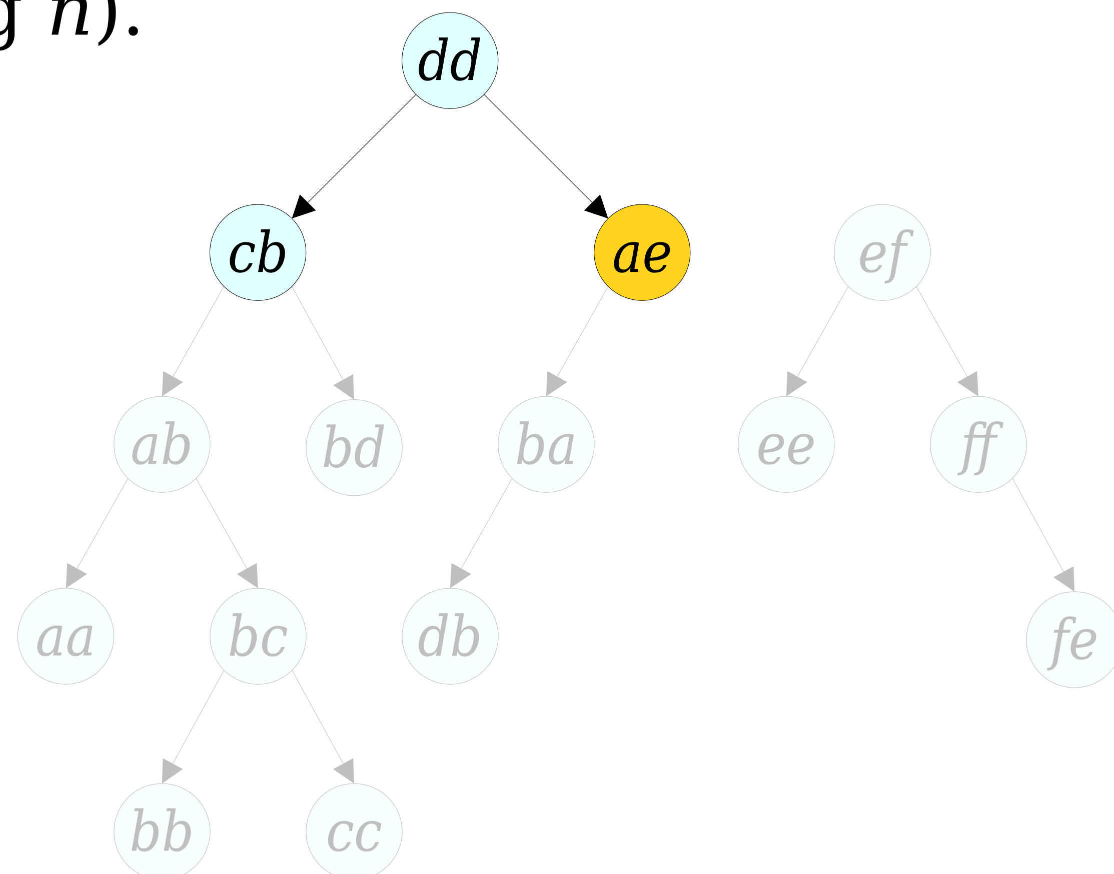

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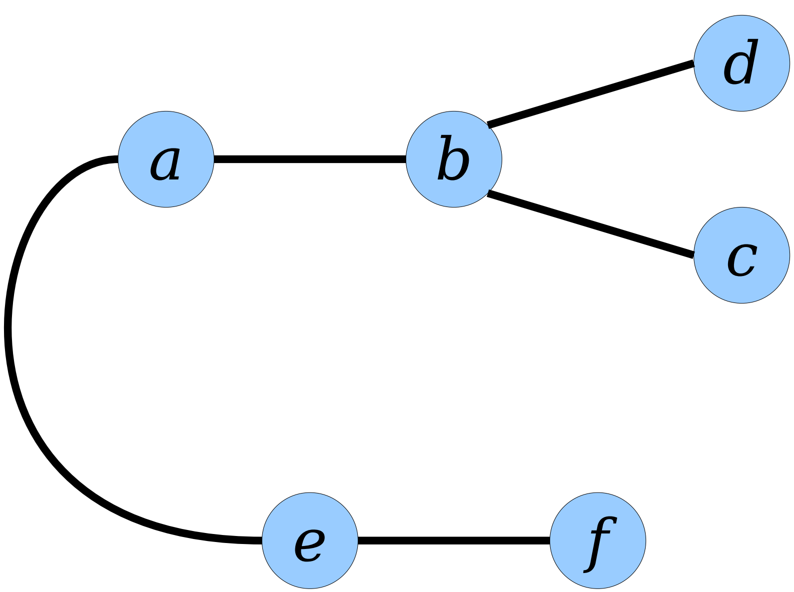

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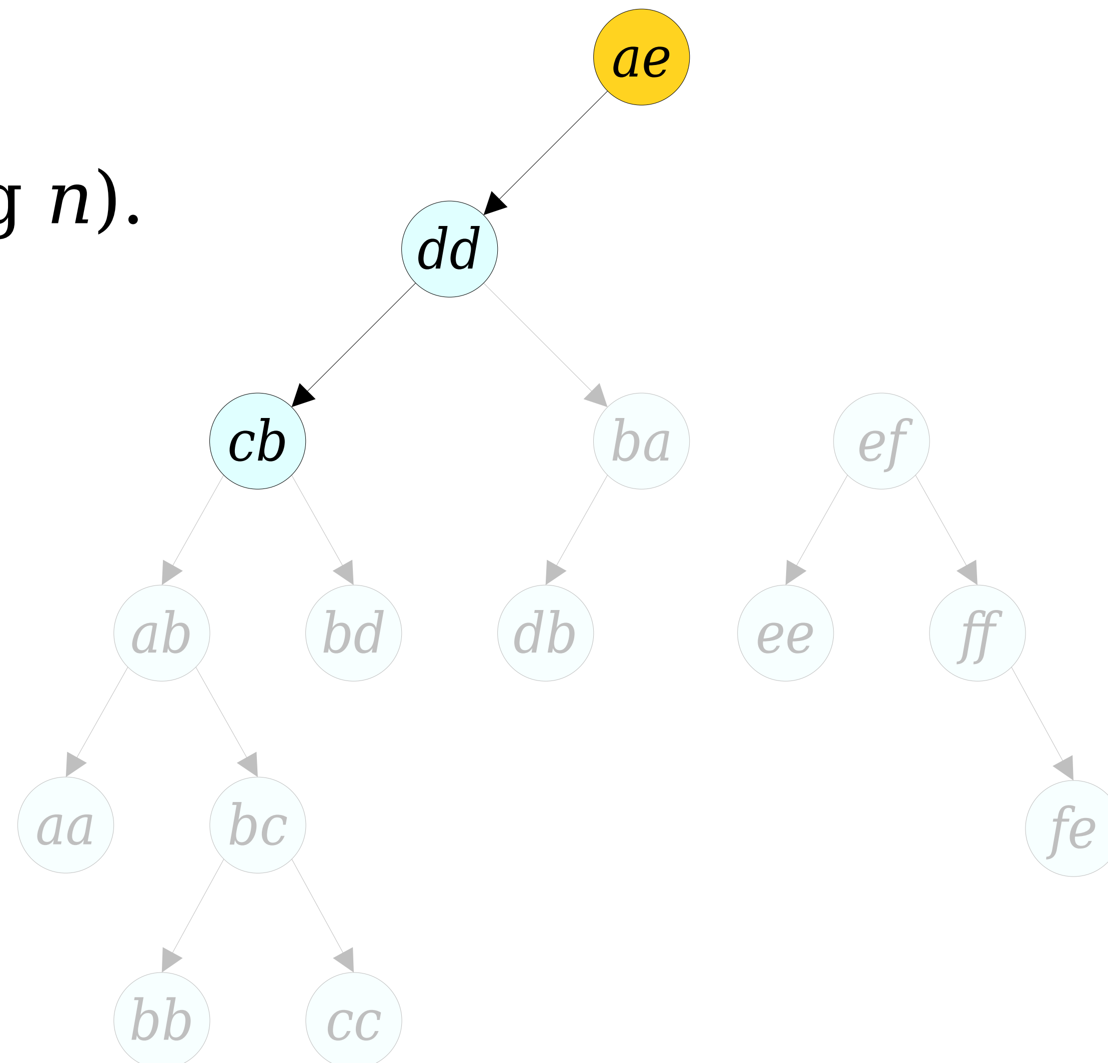

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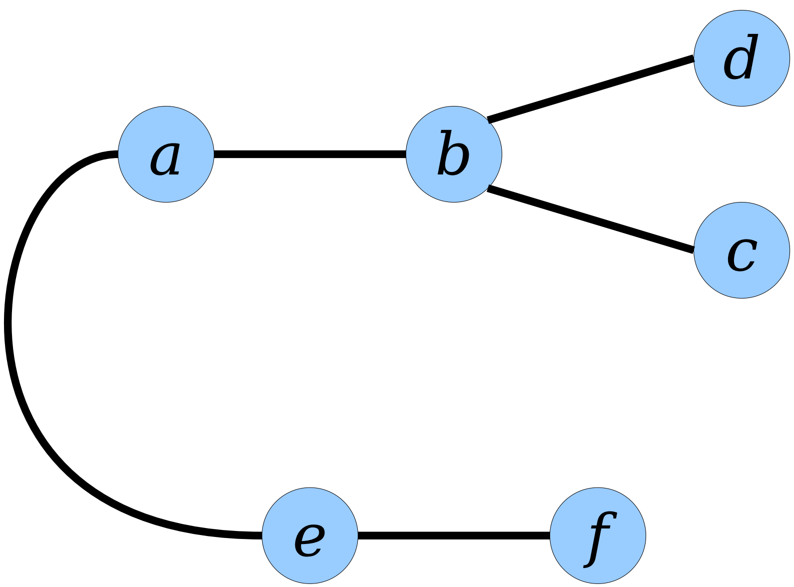

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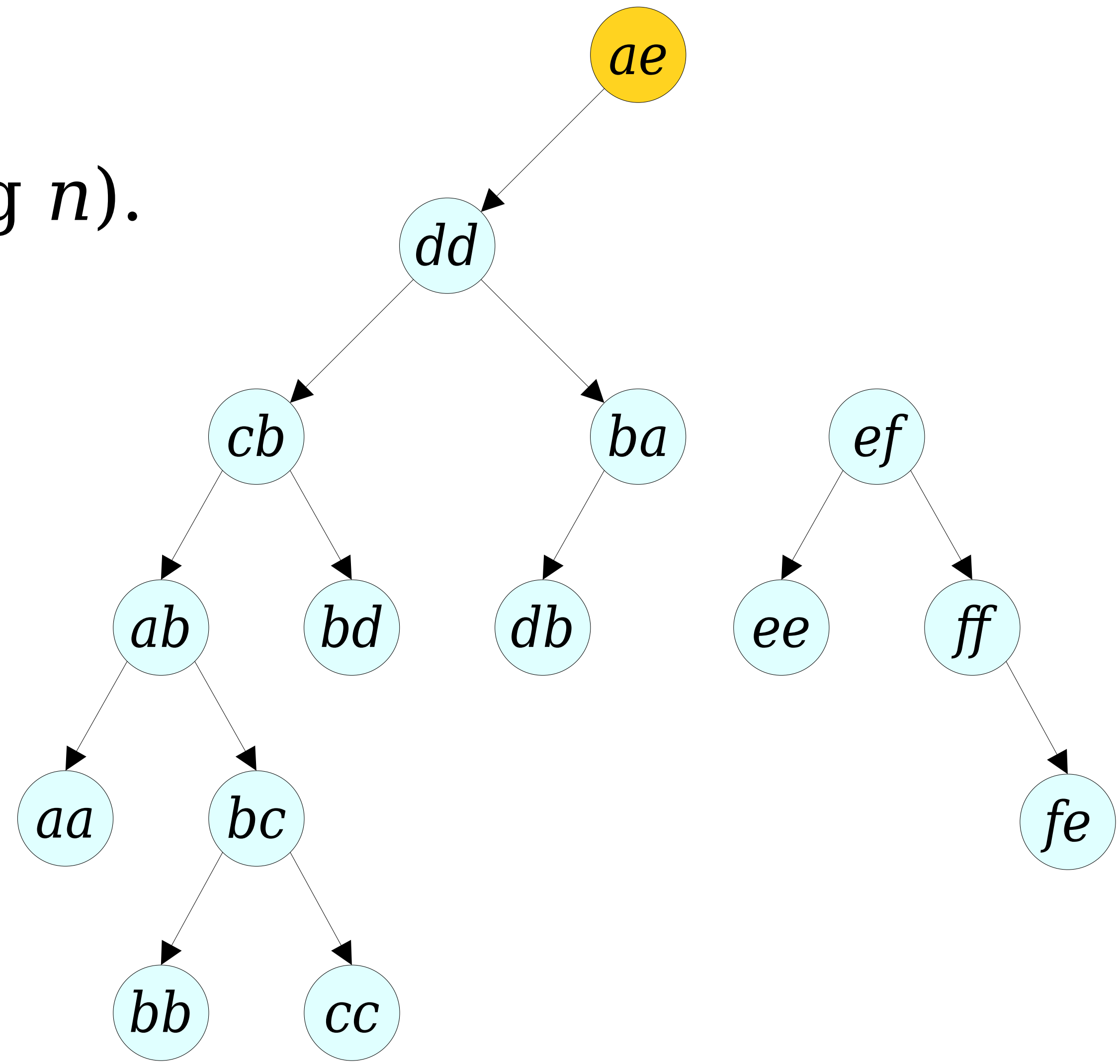

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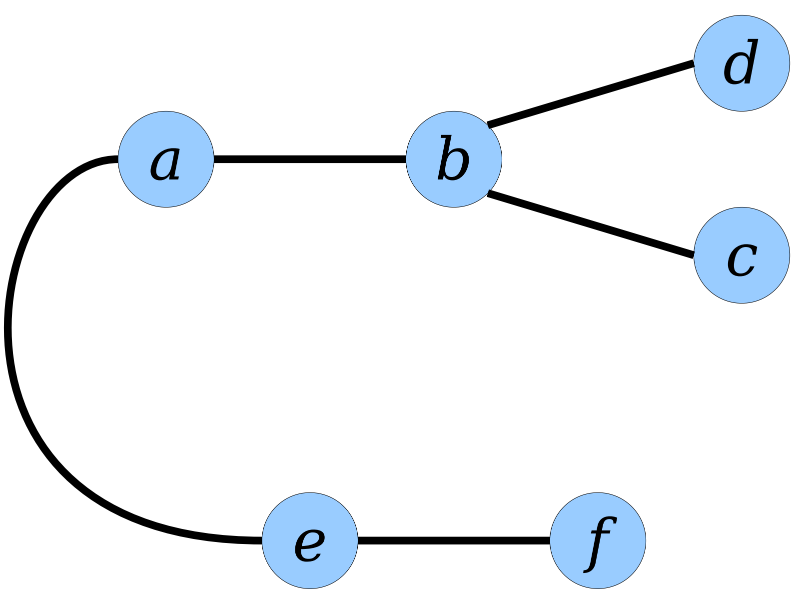

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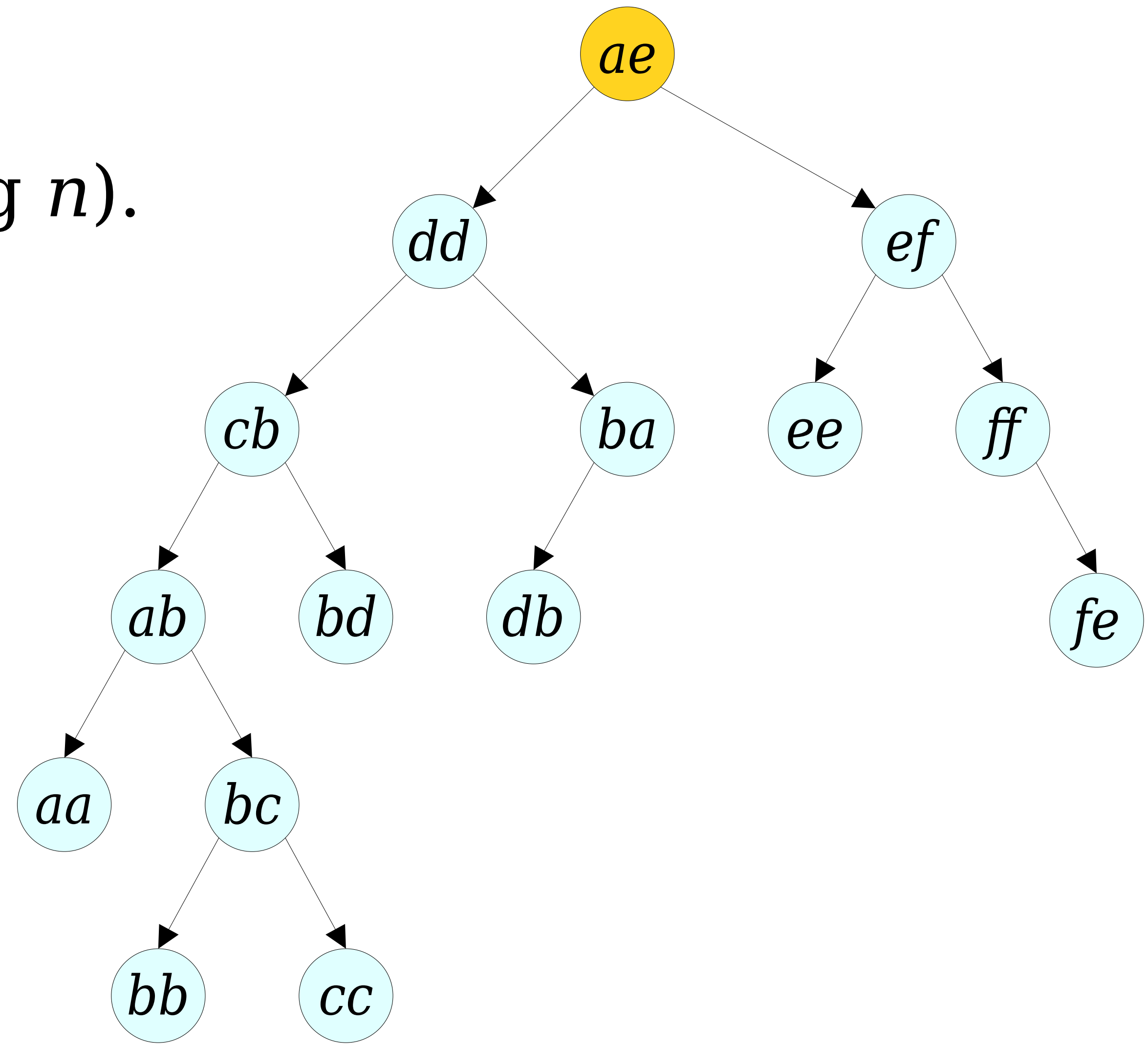

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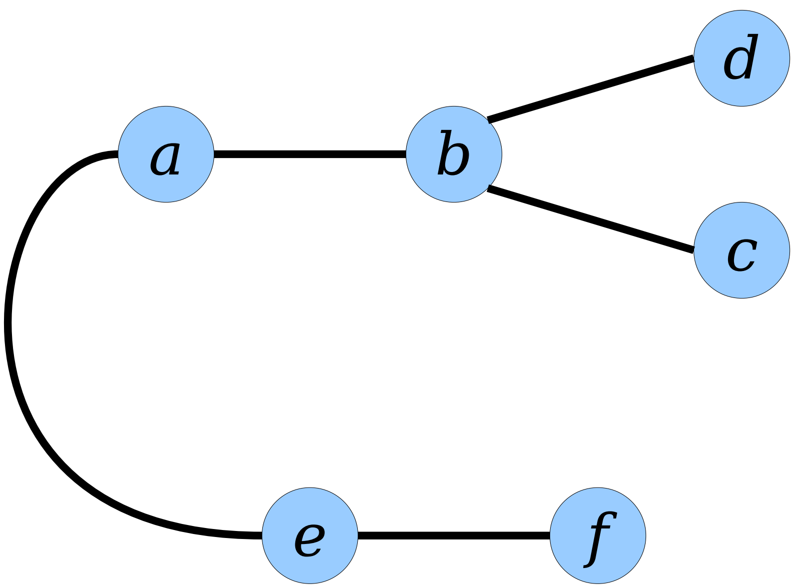

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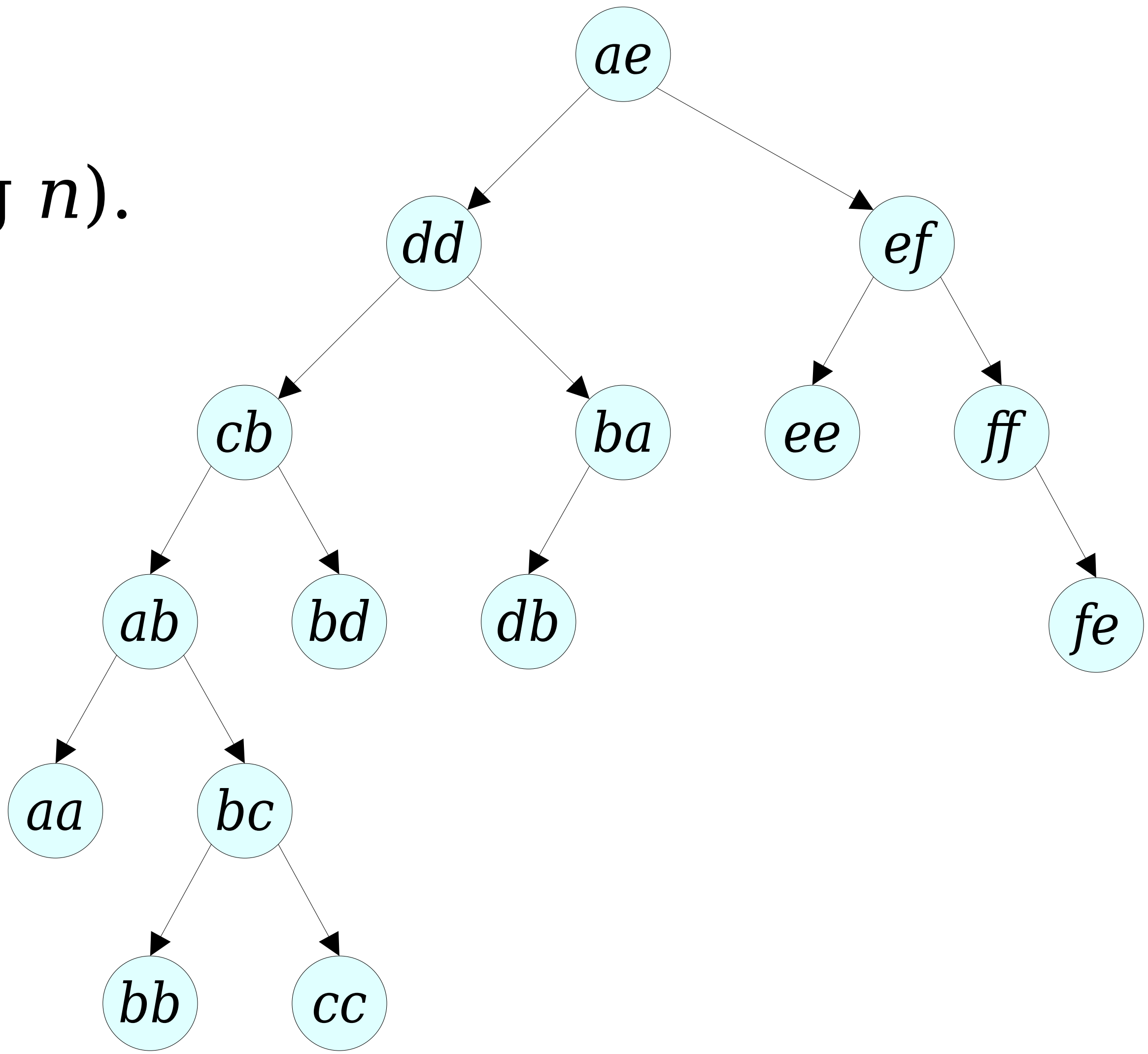

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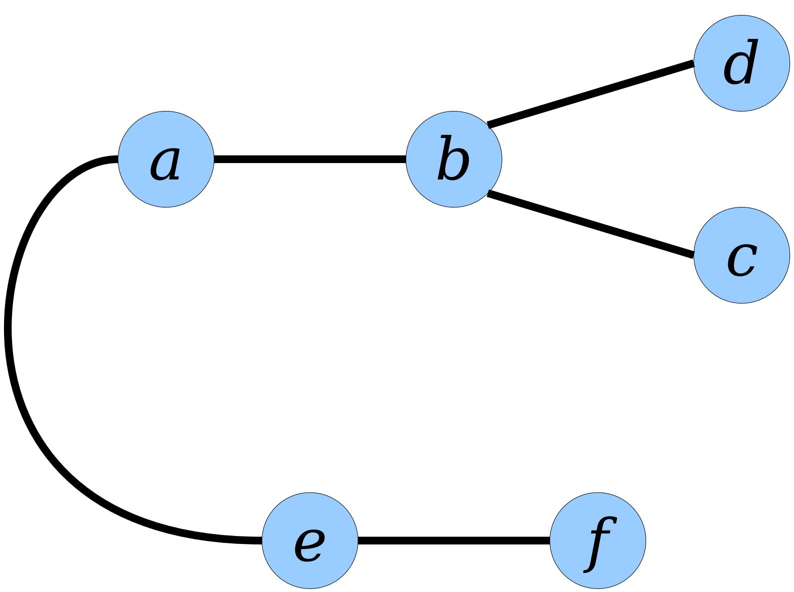

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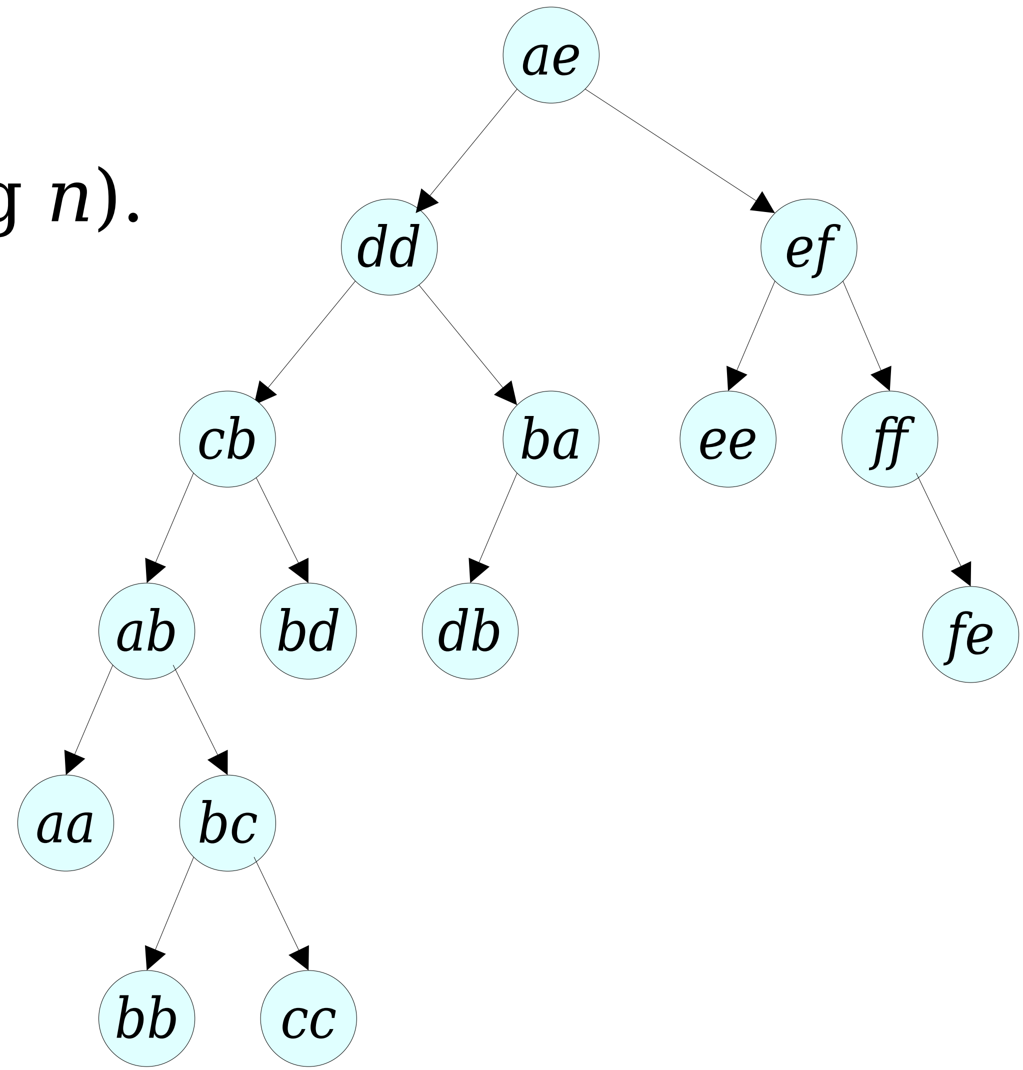

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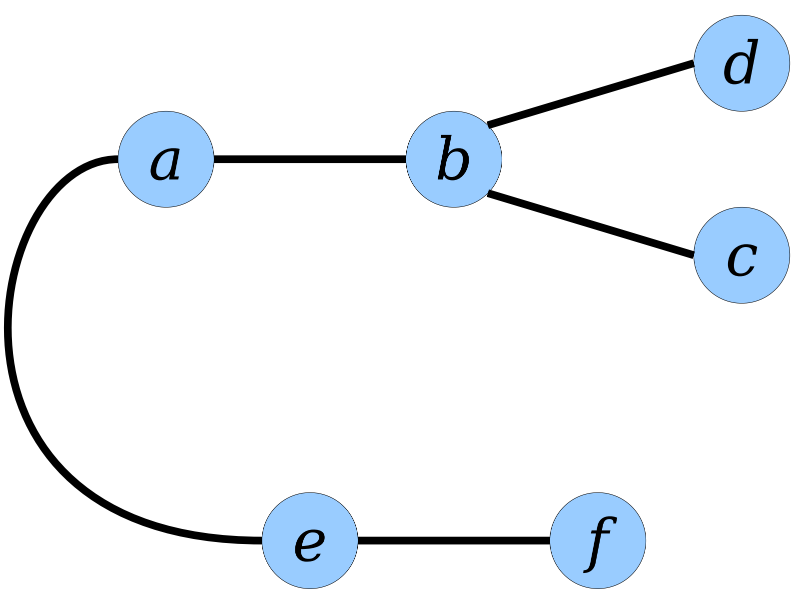

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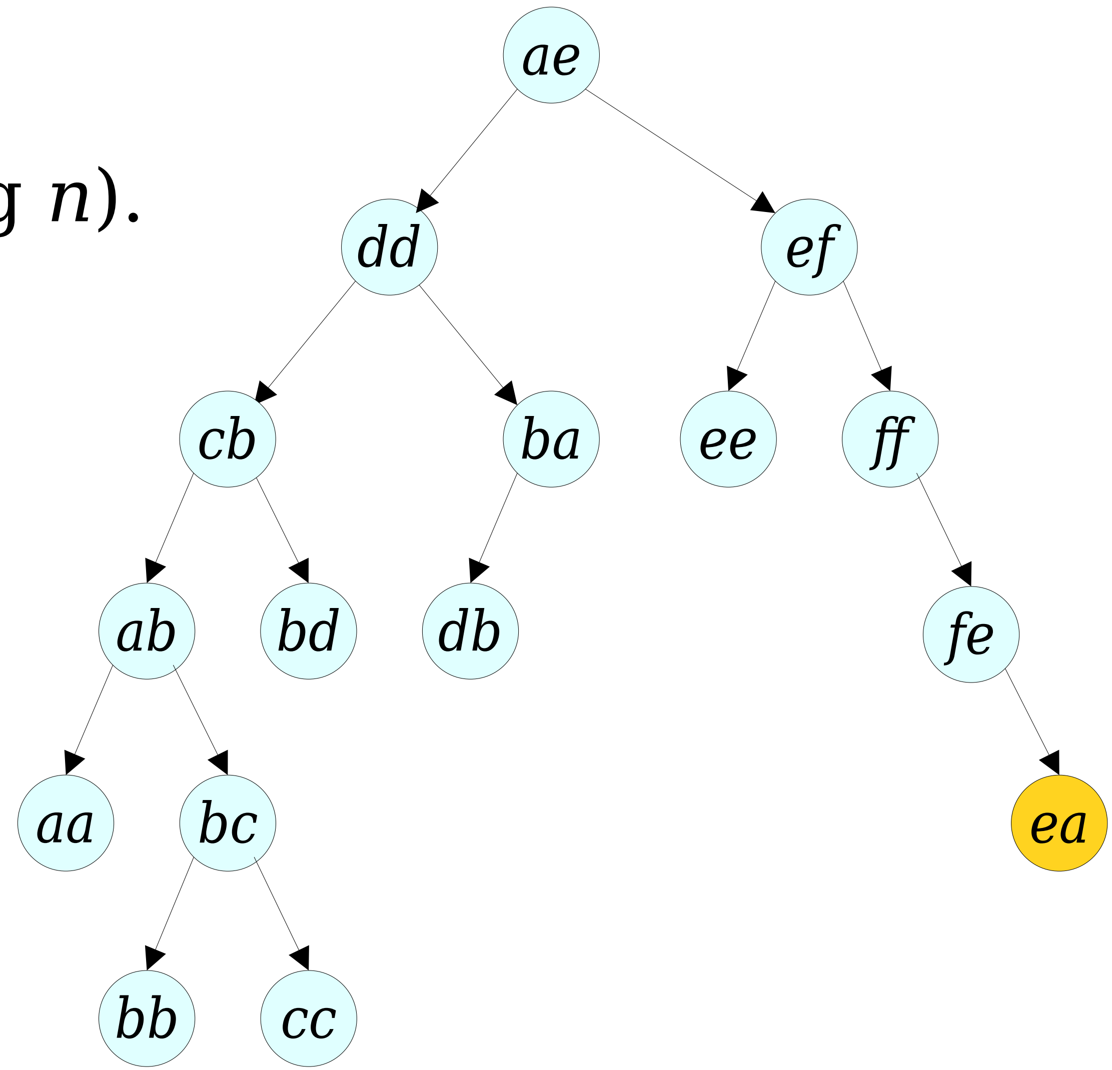

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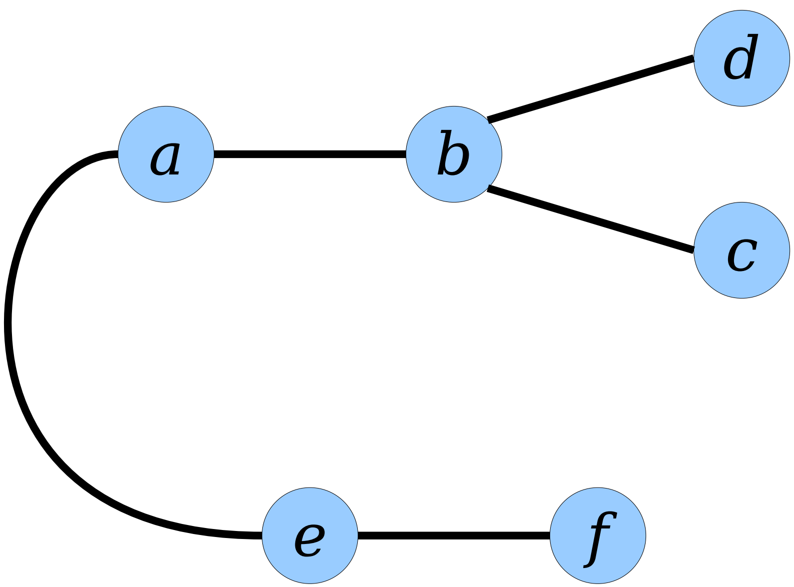

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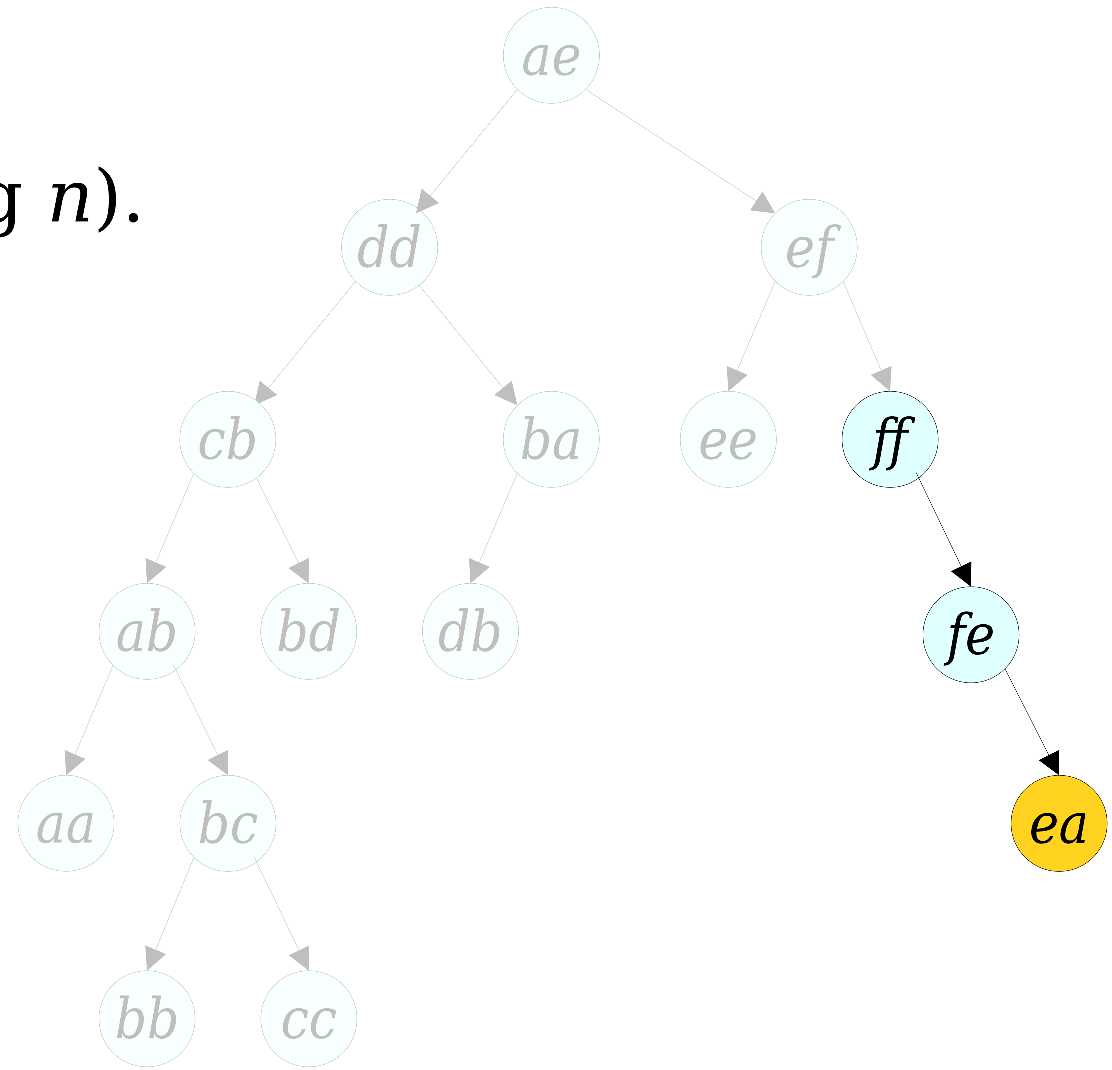

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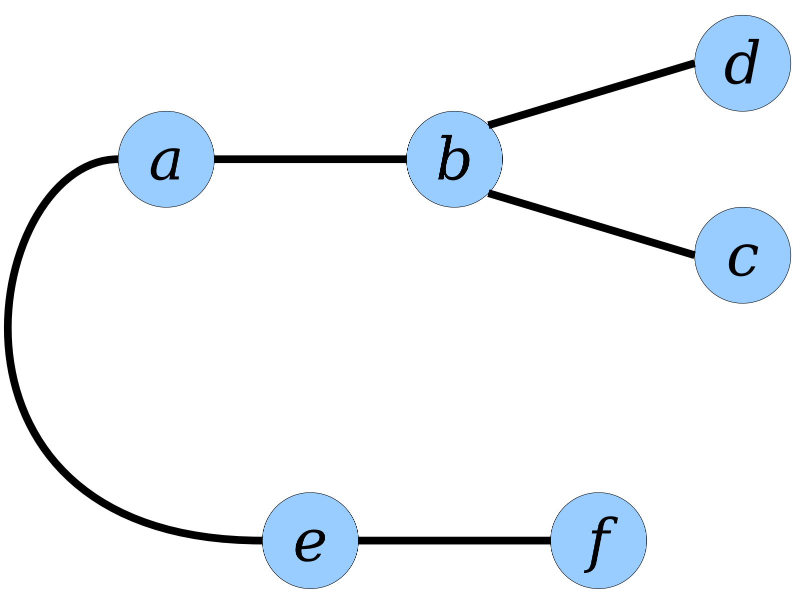

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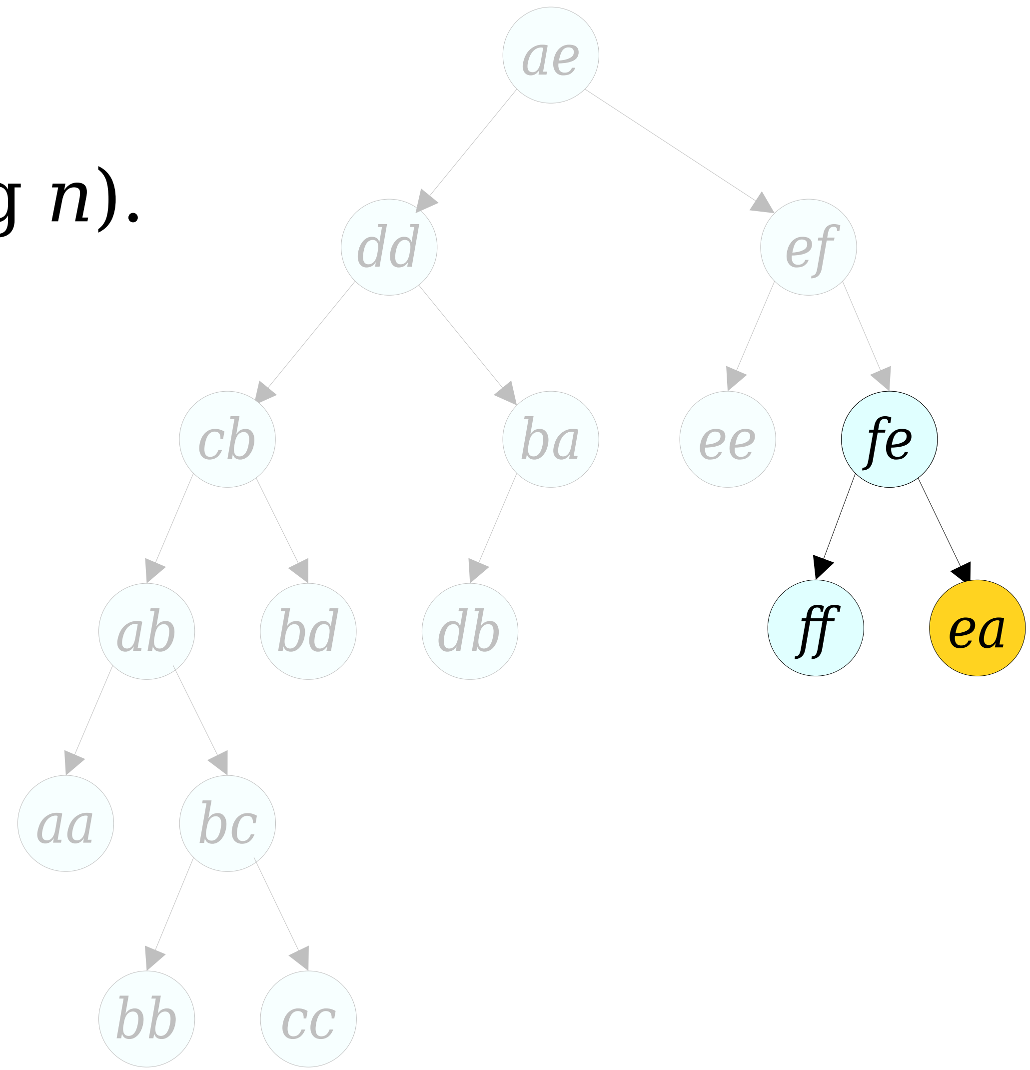

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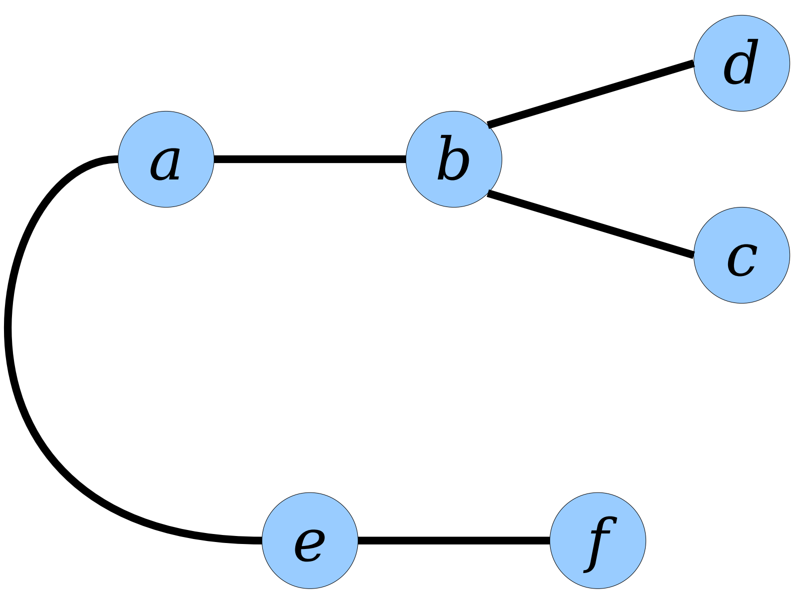

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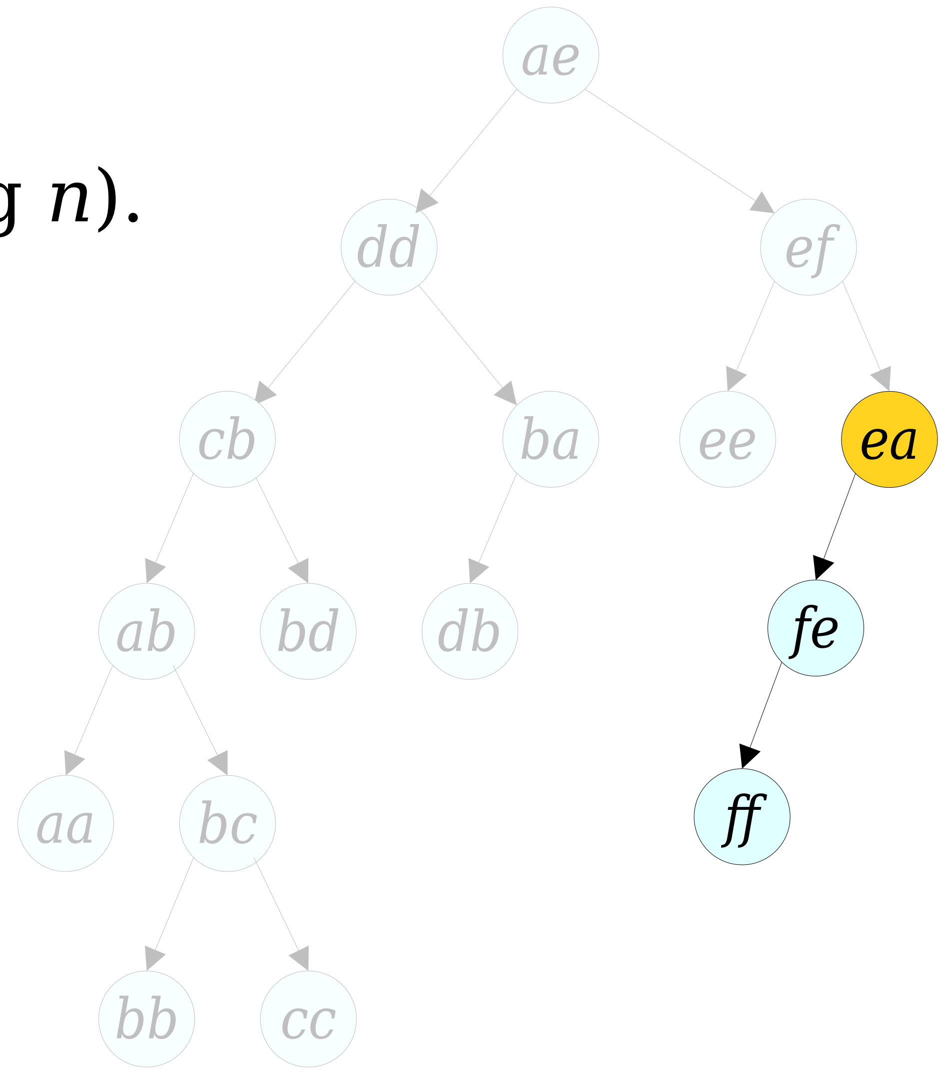

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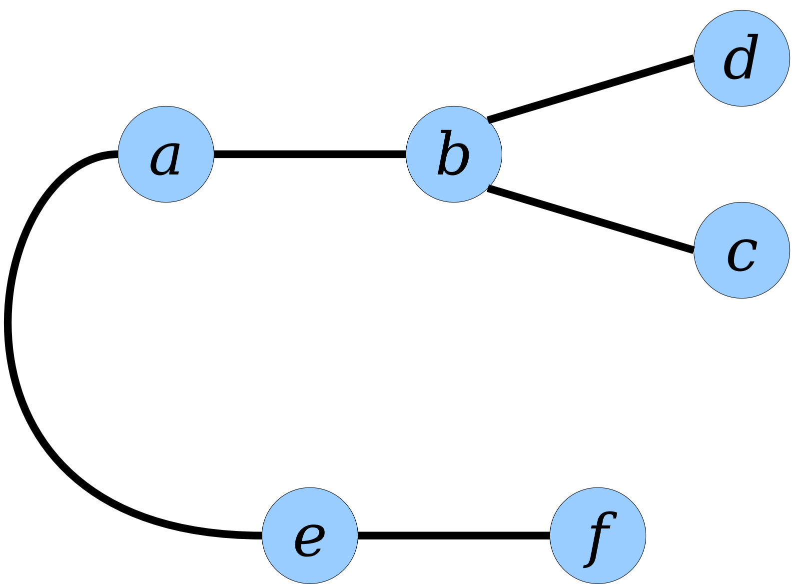

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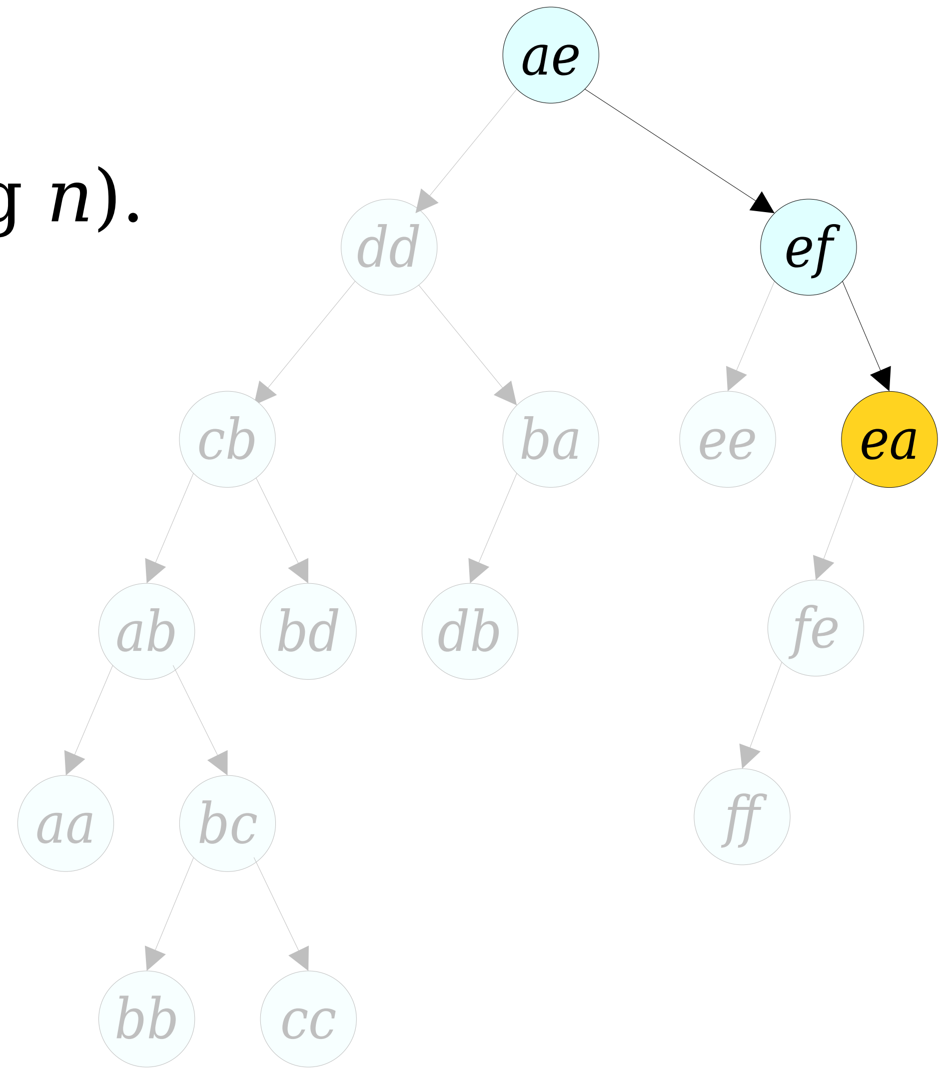

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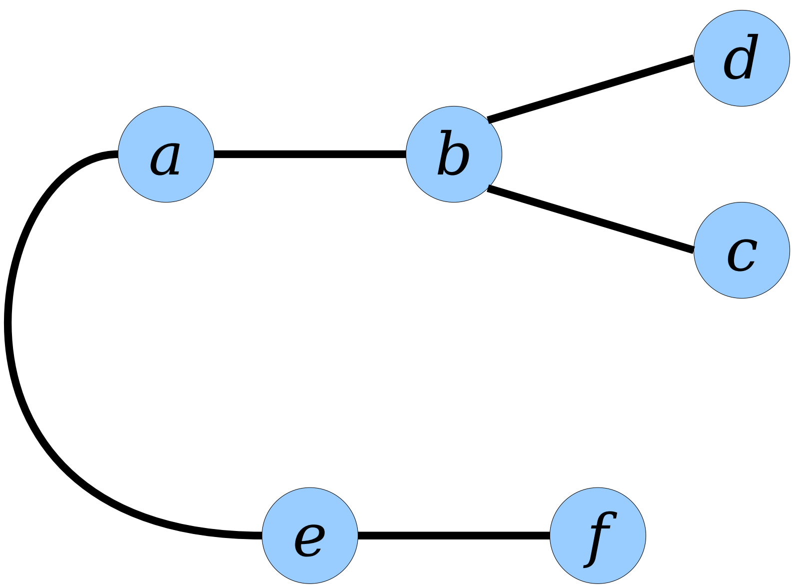

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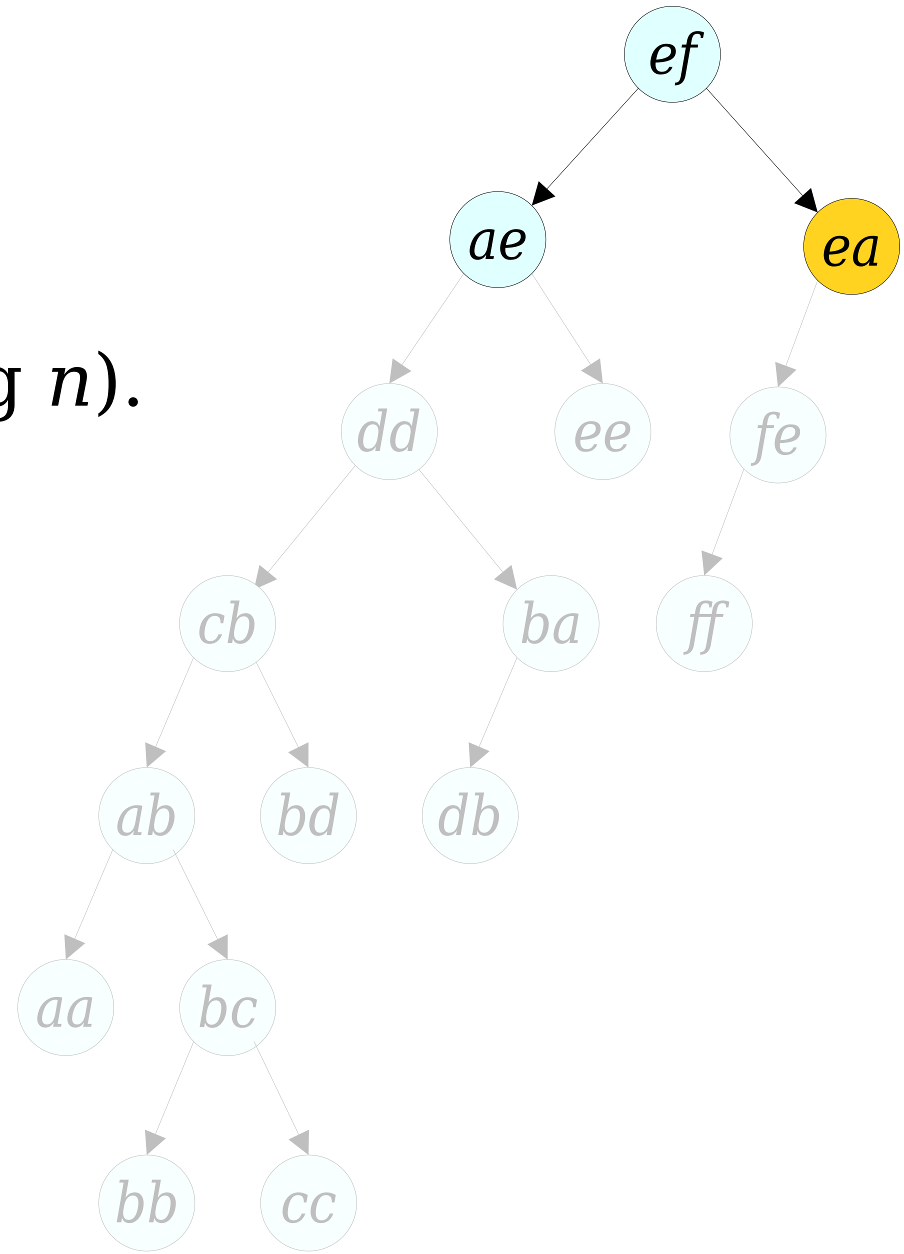

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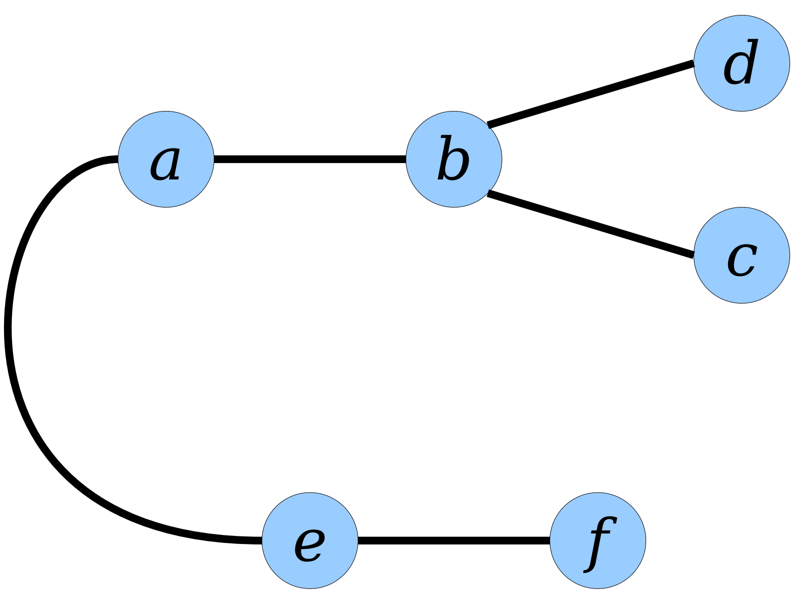

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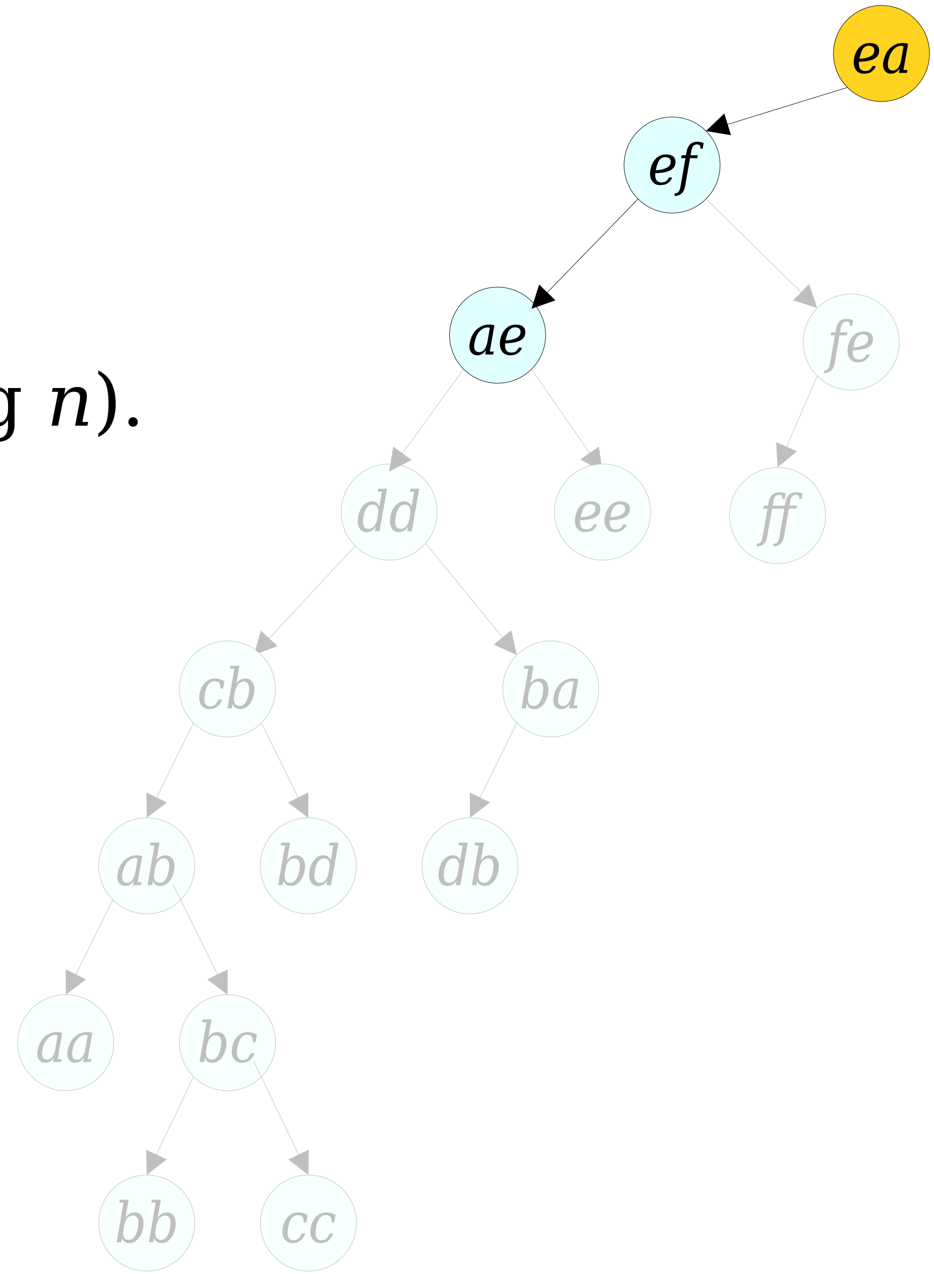

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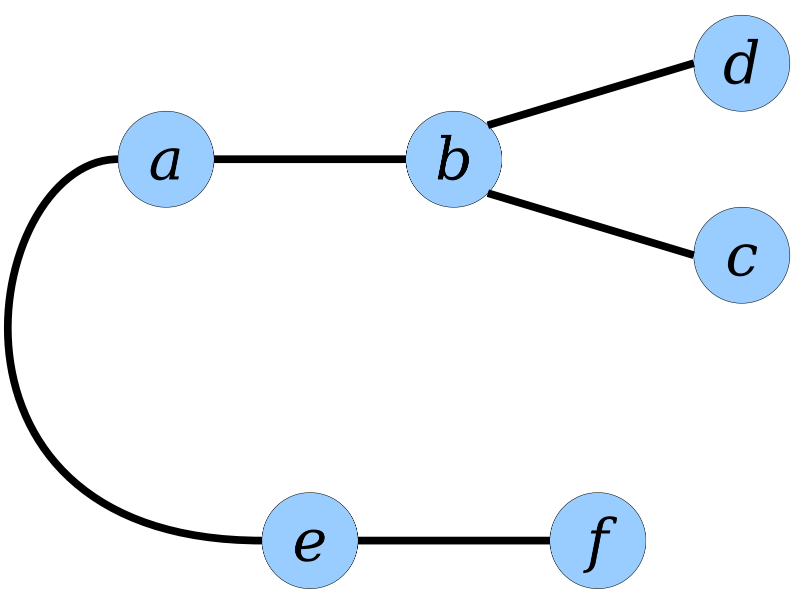

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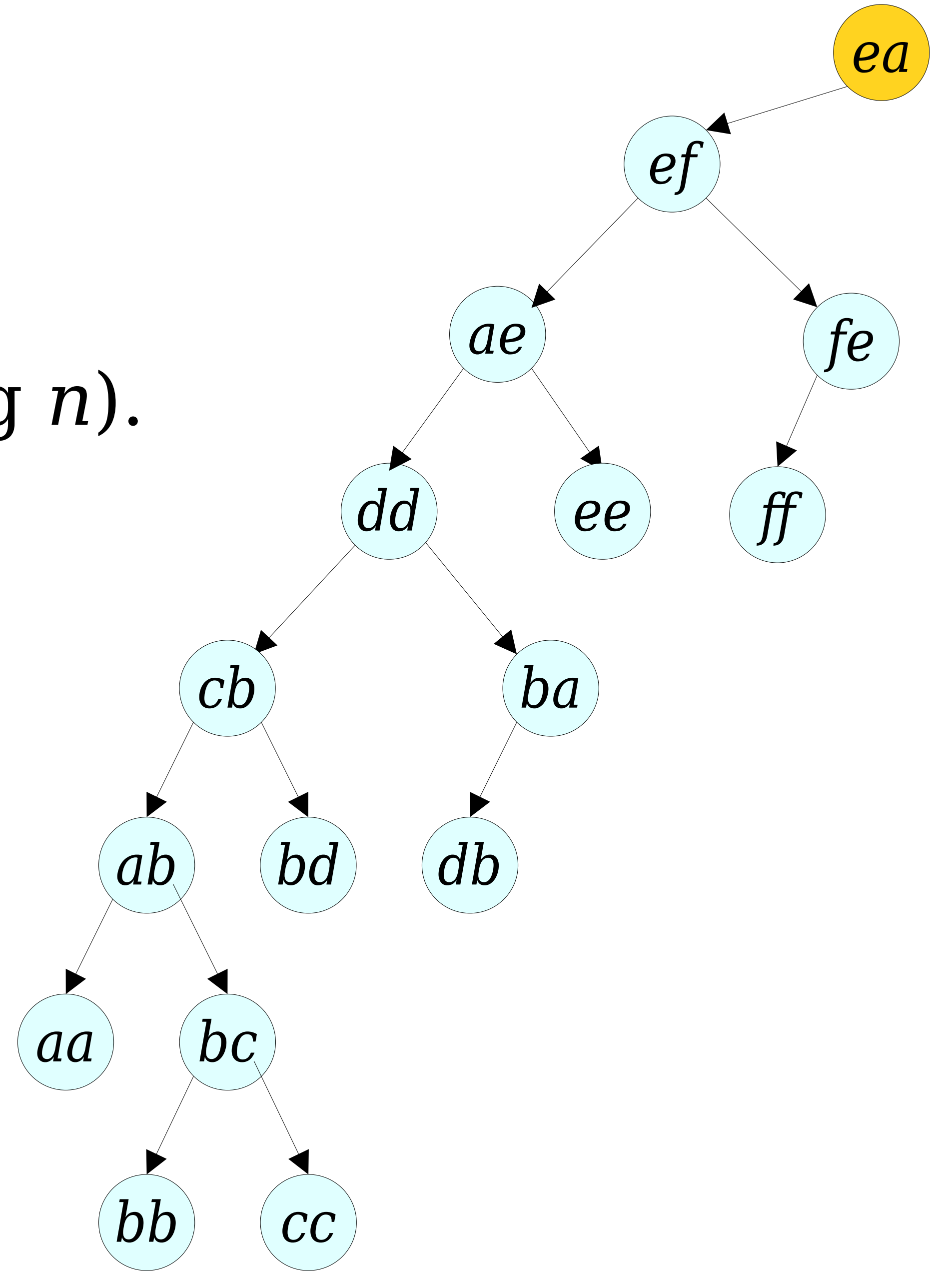

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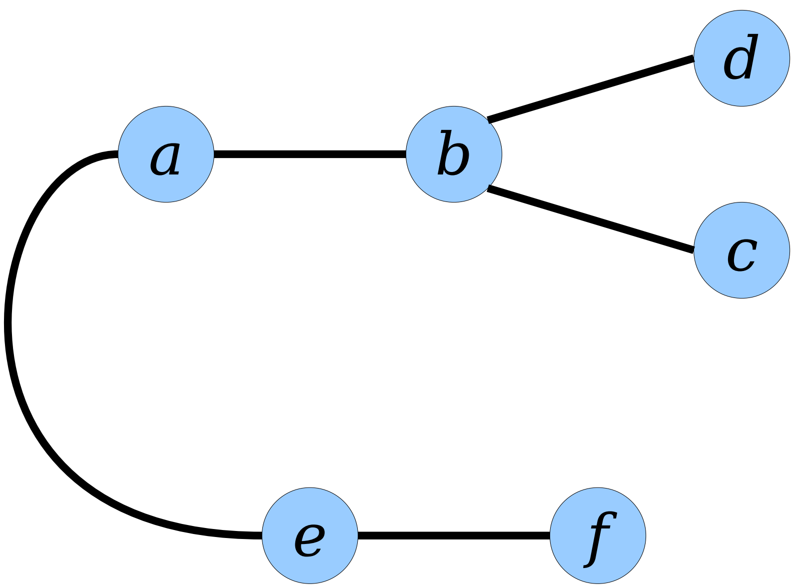

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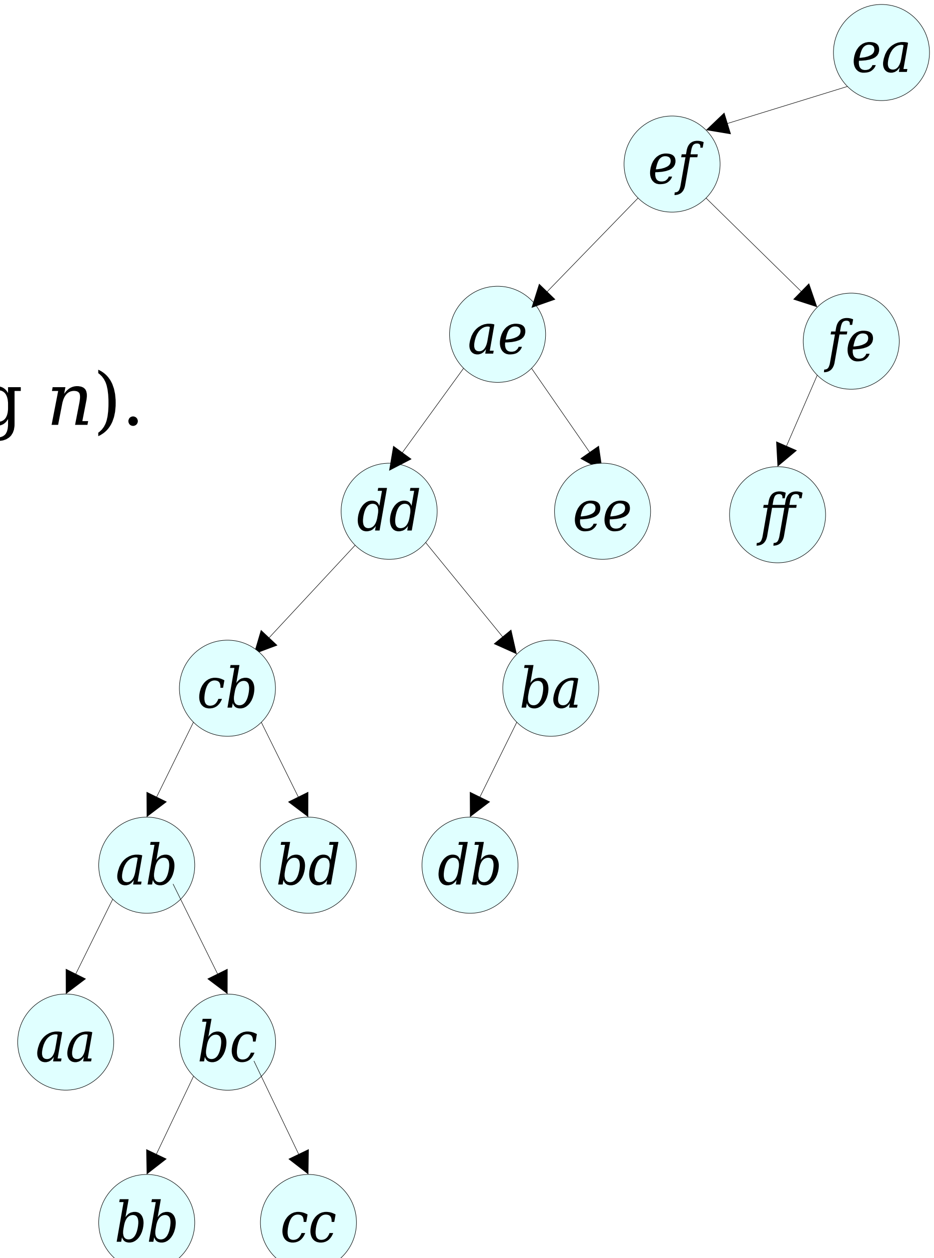

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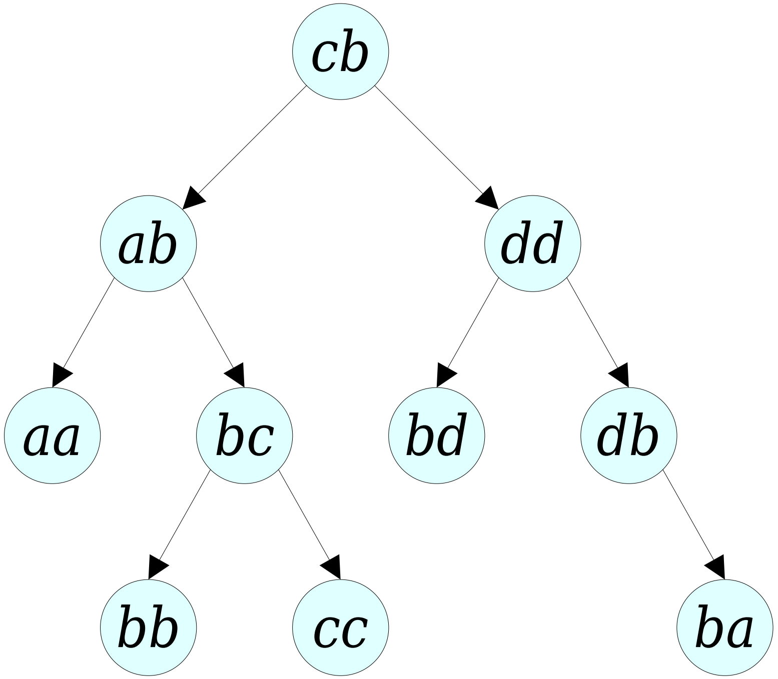

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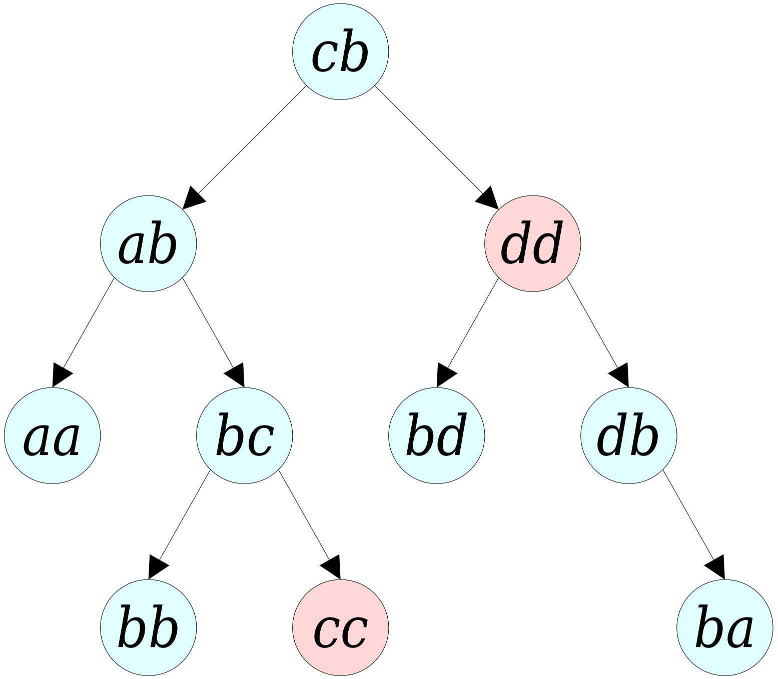

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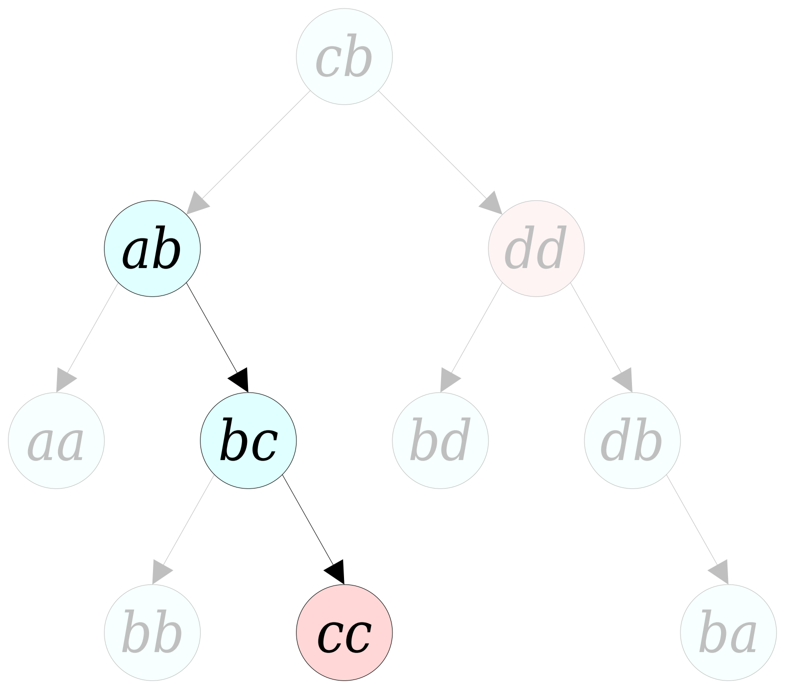

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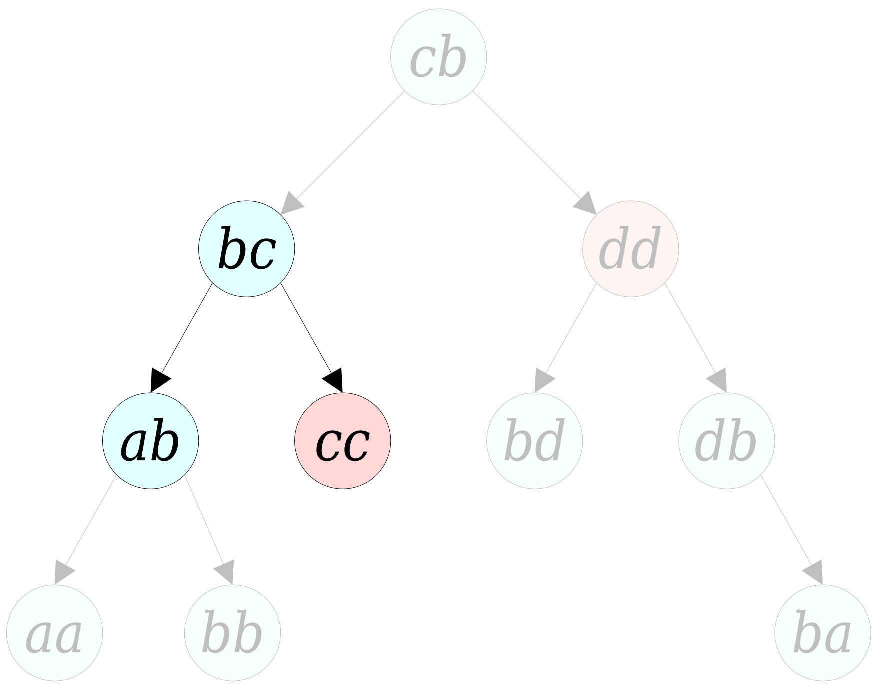

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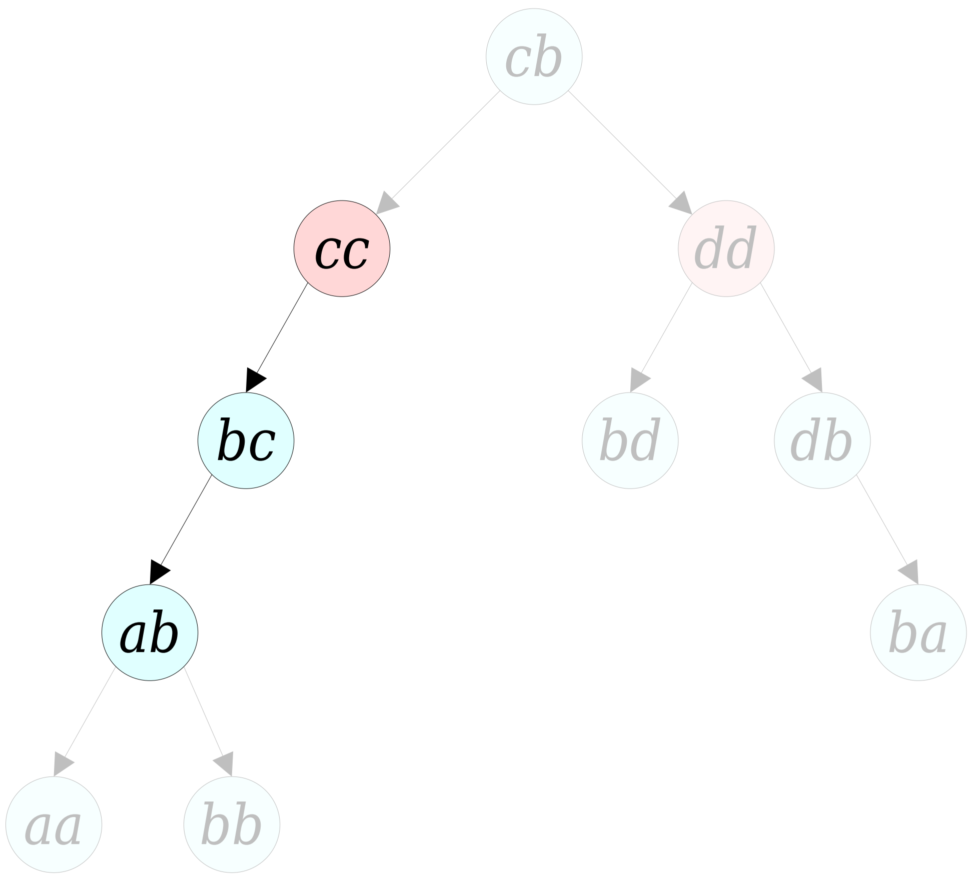

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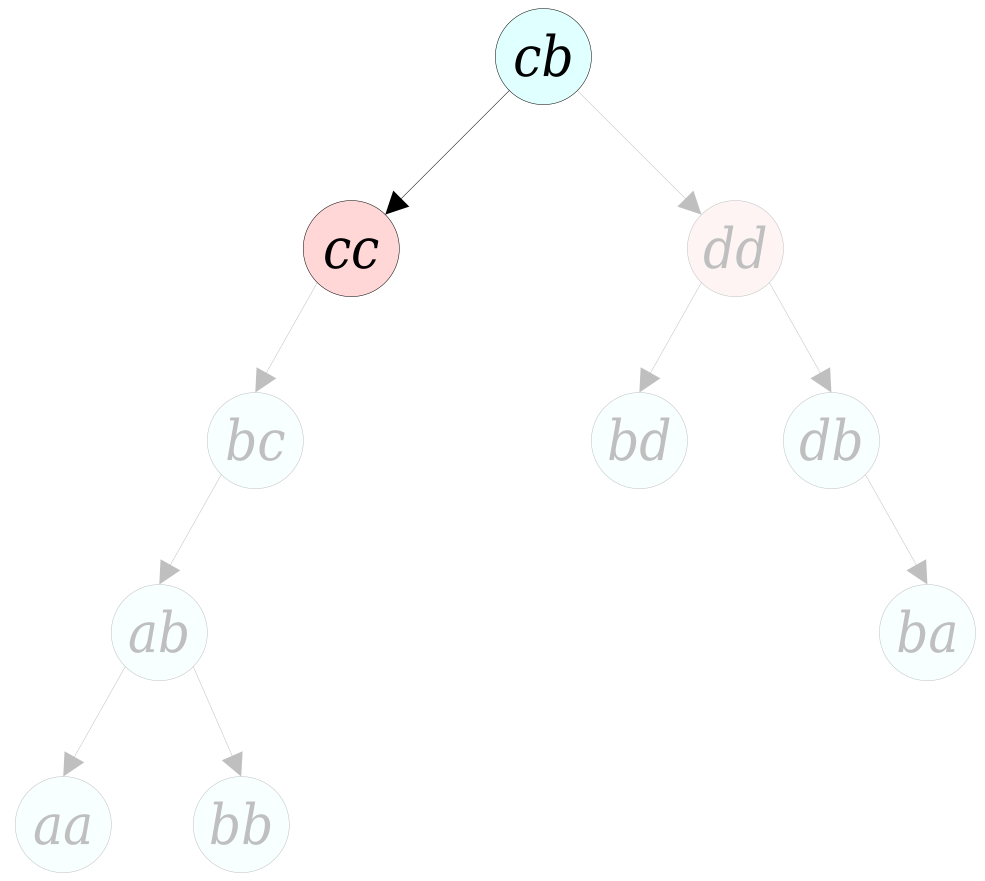

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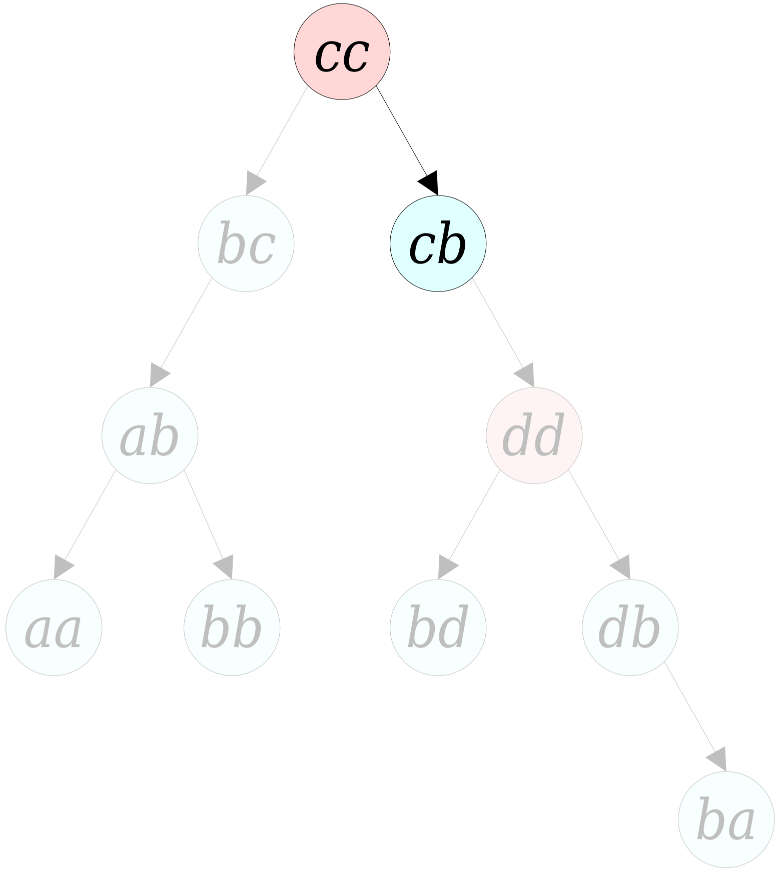

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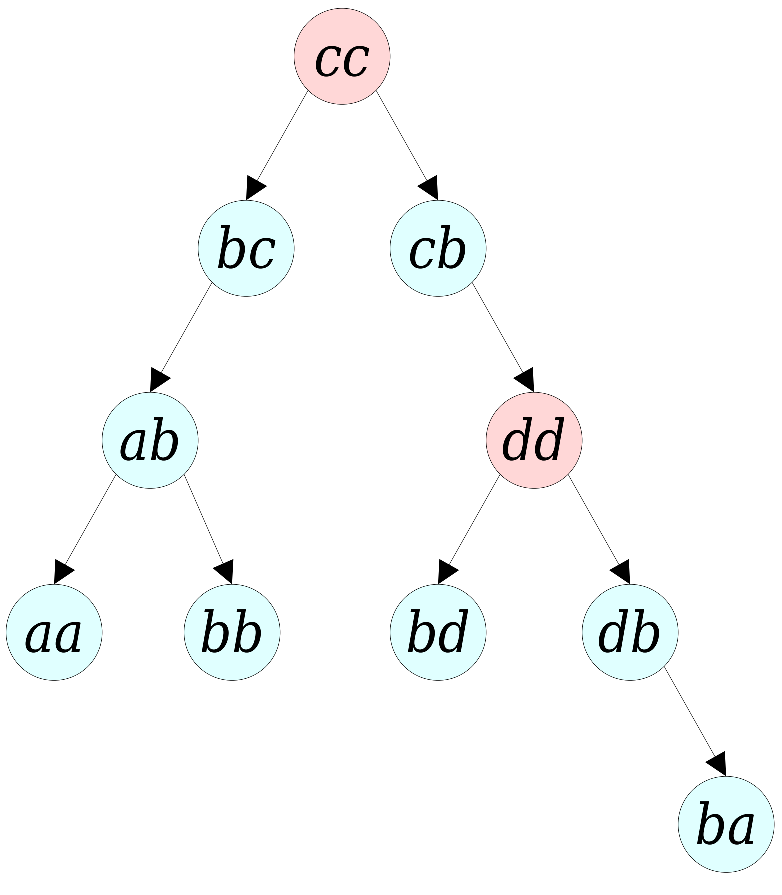

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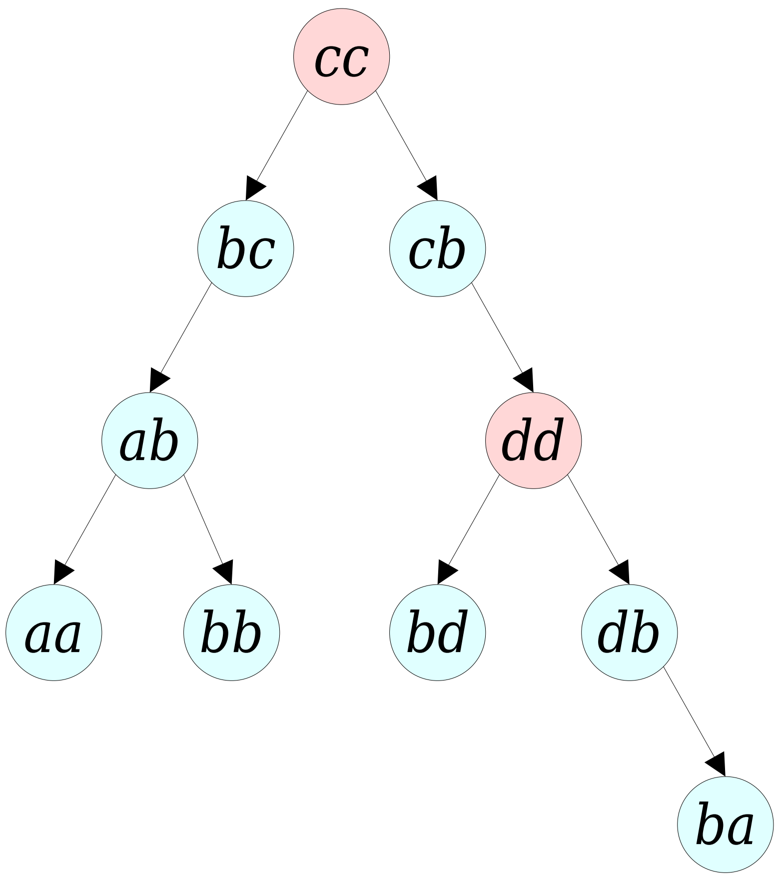

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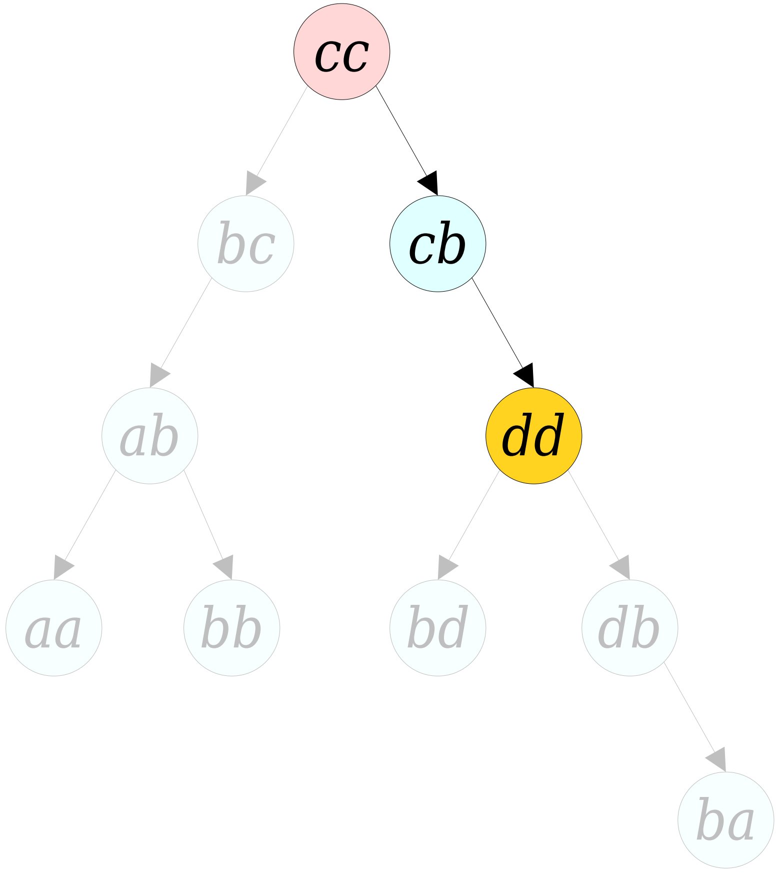

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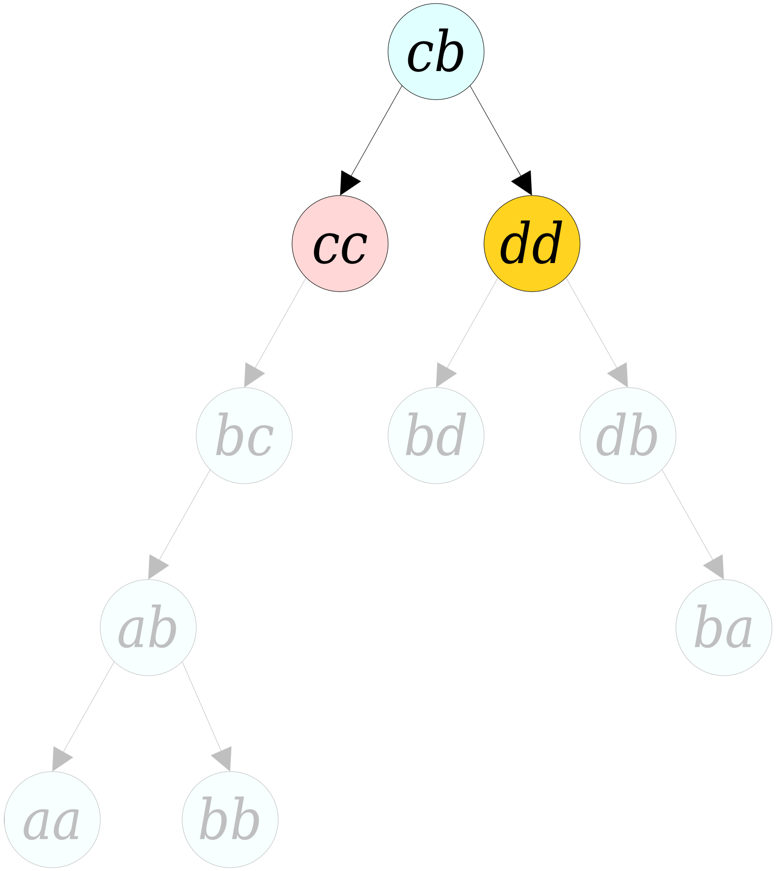

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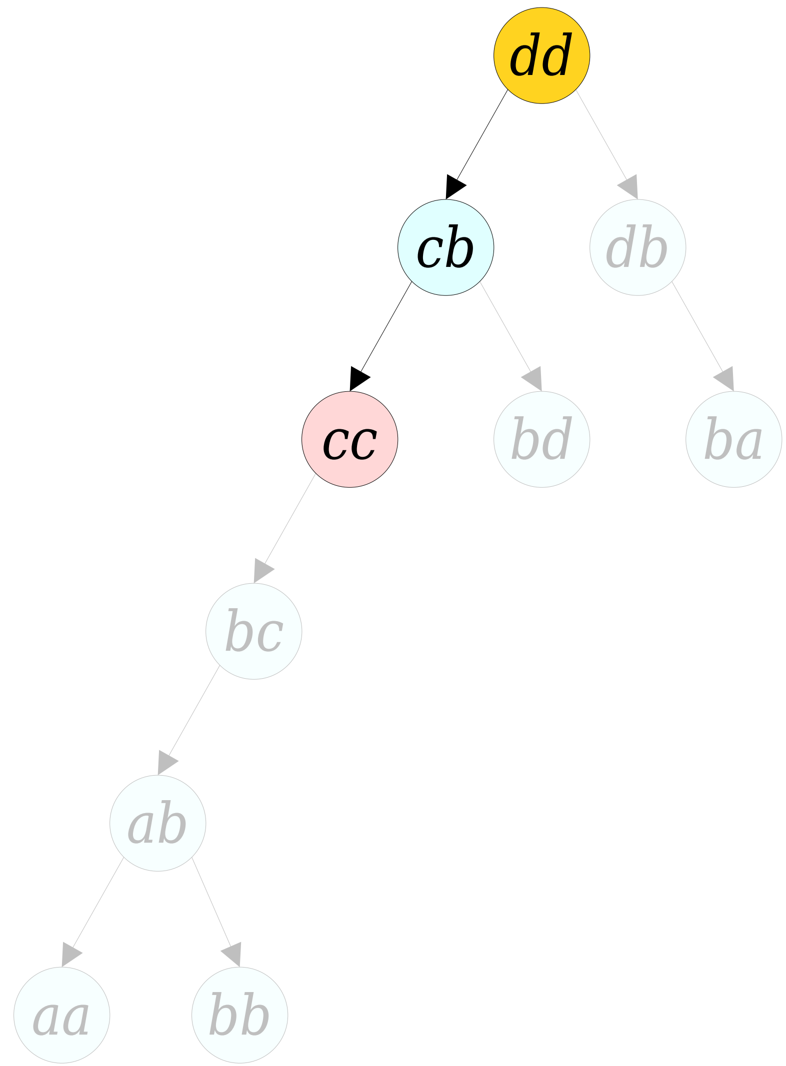
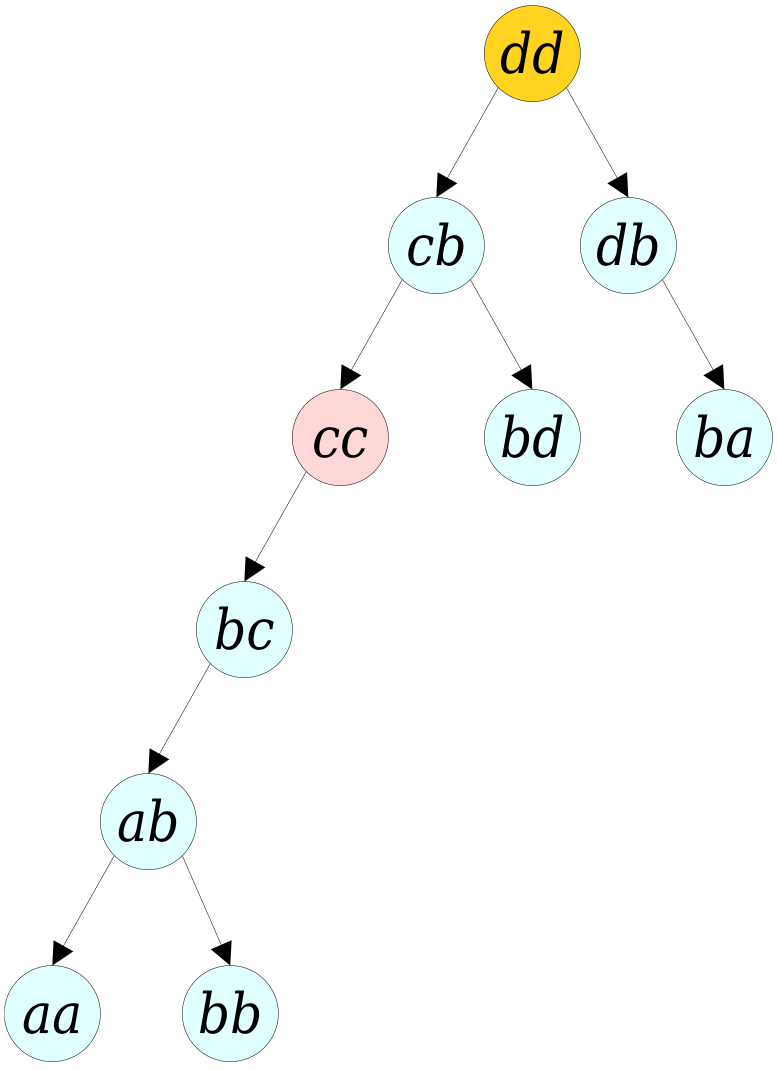
  - Splay xx.
  - Splay yy.



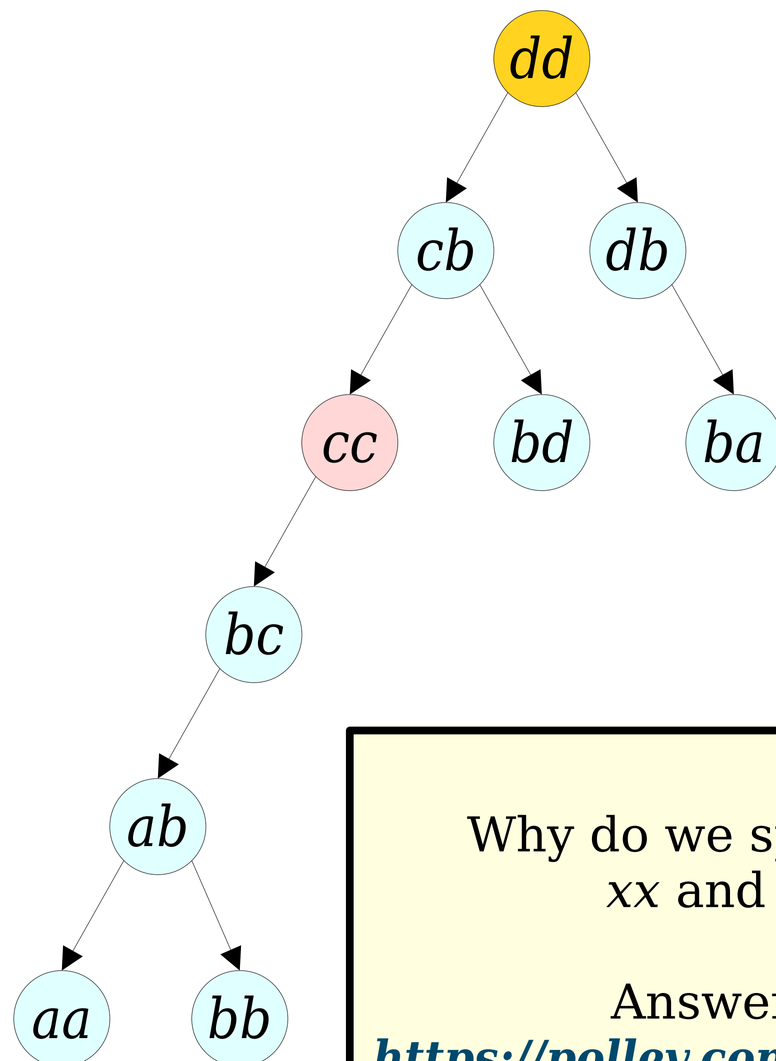
Euler Tour Trees

- To answer ***are-connected***(x, y):
  - Splay xx.
  - Splay yy.



Euler Tour Trees

- To answer ***are-connected***(x, y):
  - Splay *xx*.
  - Splay *yy*.
  - Return whether *xx* was encountered on the second splay.
- Amortized cost: **O(log *n*)**.

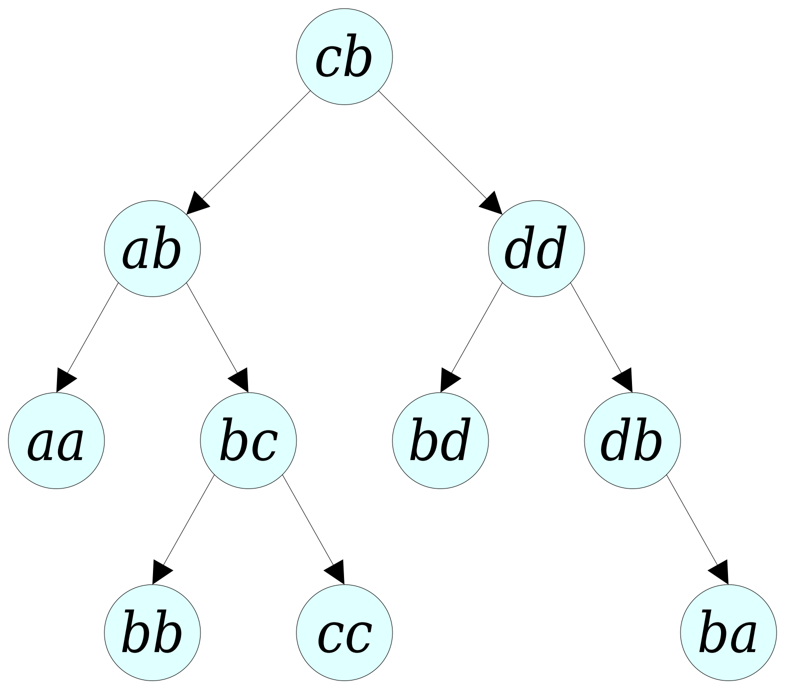


Why do we splay both *xx* and *yy*?

Answer at ***https://pollev.com/cs166spr23***

Euler Tour Trees

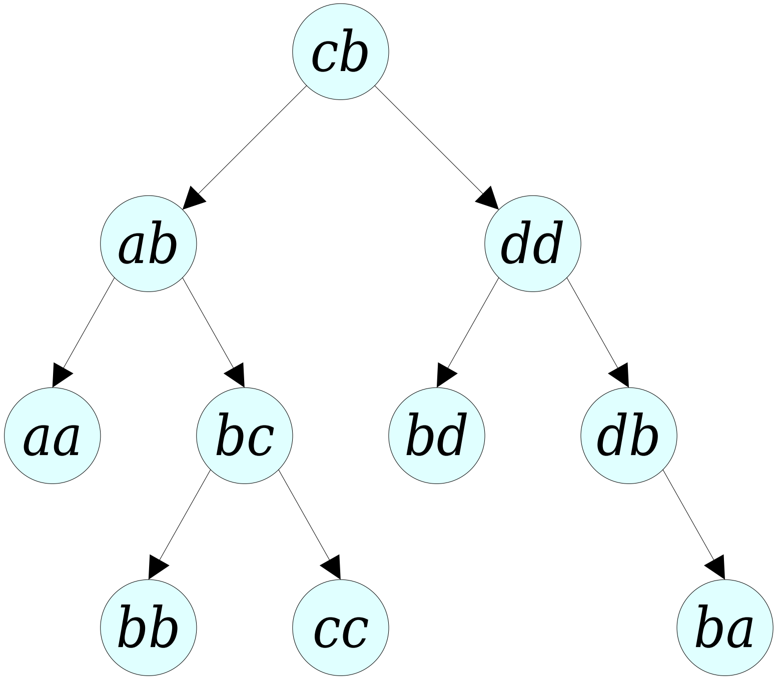
- To ***reroot***(x):



aa ab bb bc cc cb bd dd db ba

Euler Tour Trees

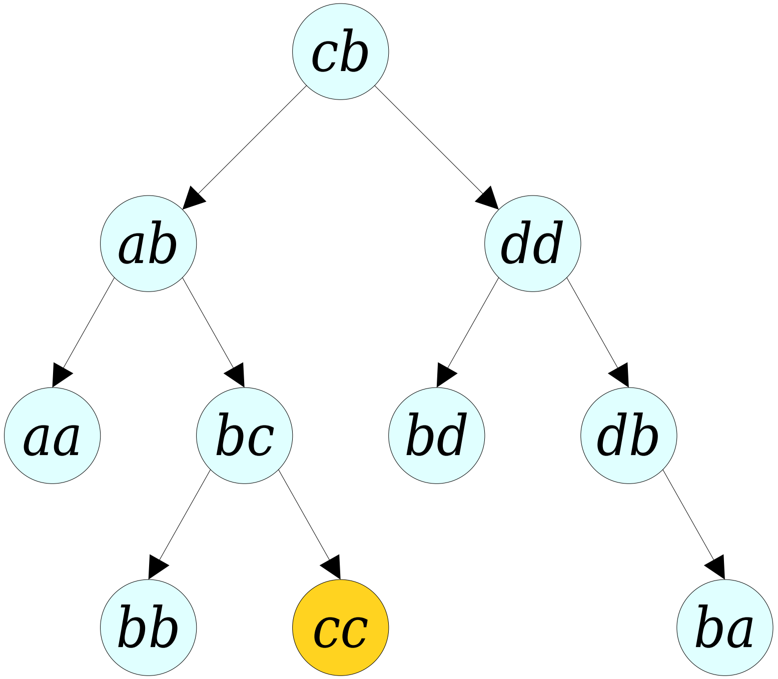
- To ***reroot***(x):
  - Splay xx.



aa ab bb bc cc cb bd dd db ba

Euler Tour Trees

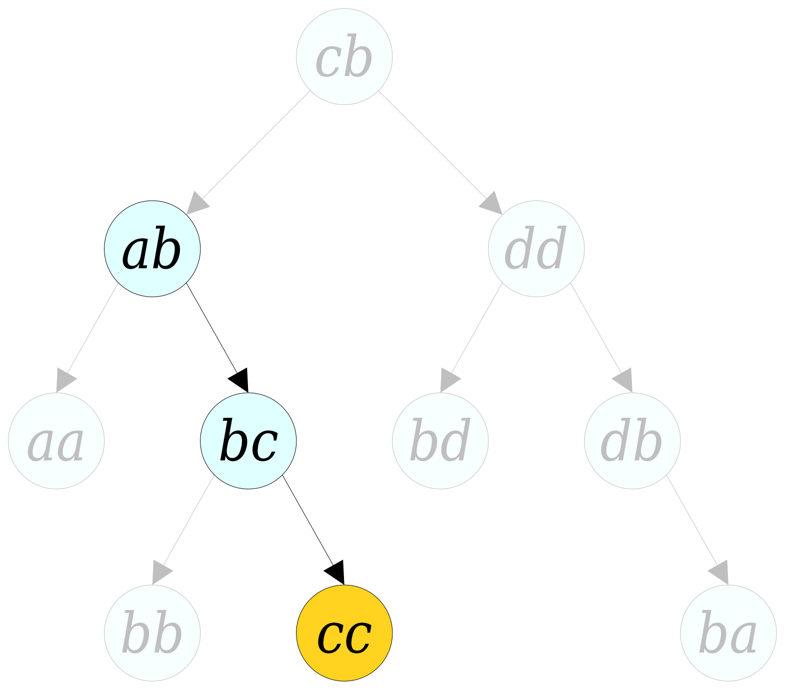
- To ***reroot***(x):
  - Splay xx.



aa ab bb bc cc cb bd dd db ba

Euler Tour Trees

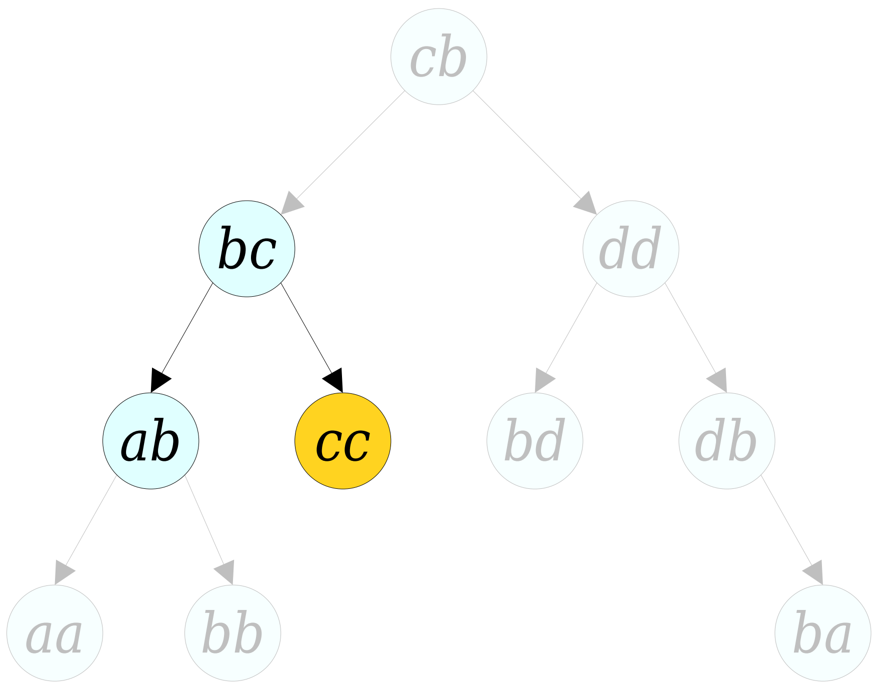
- To ***reroot***(x):
  - Splay xx.



aa ab bb bc cc cb bd dd db ba

Euler Tour Trees

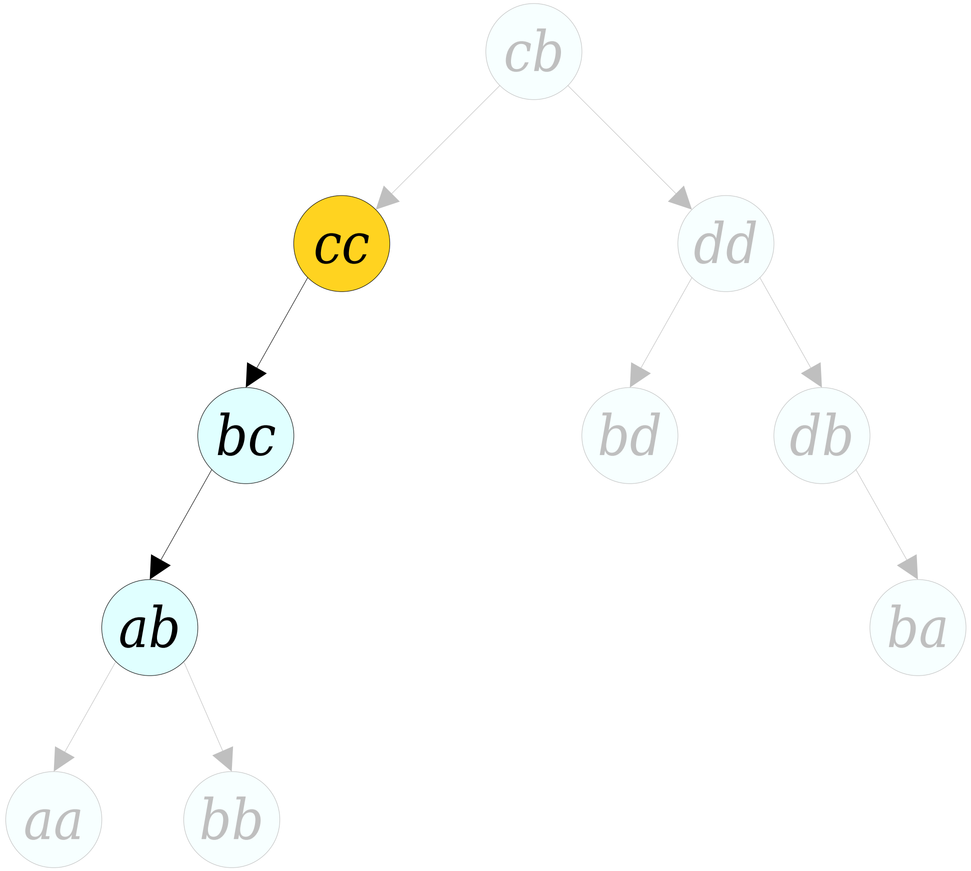
- To ***reroot***(x):
  - Splay xx.



aa ab bb bc cc cb bd dd db ba

Euler Tour Trees

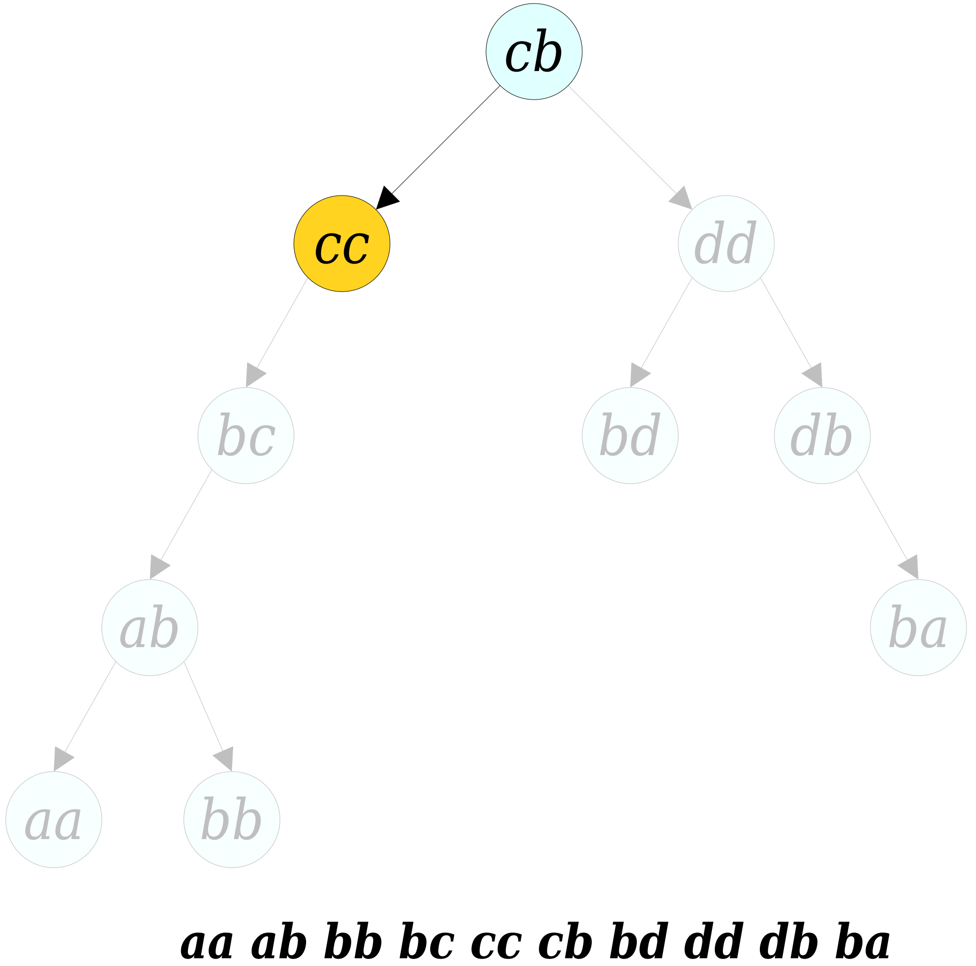
- To ***reroot***(x):
  - Splay xx.



aa ab bb bc cc cb bd dd db ba

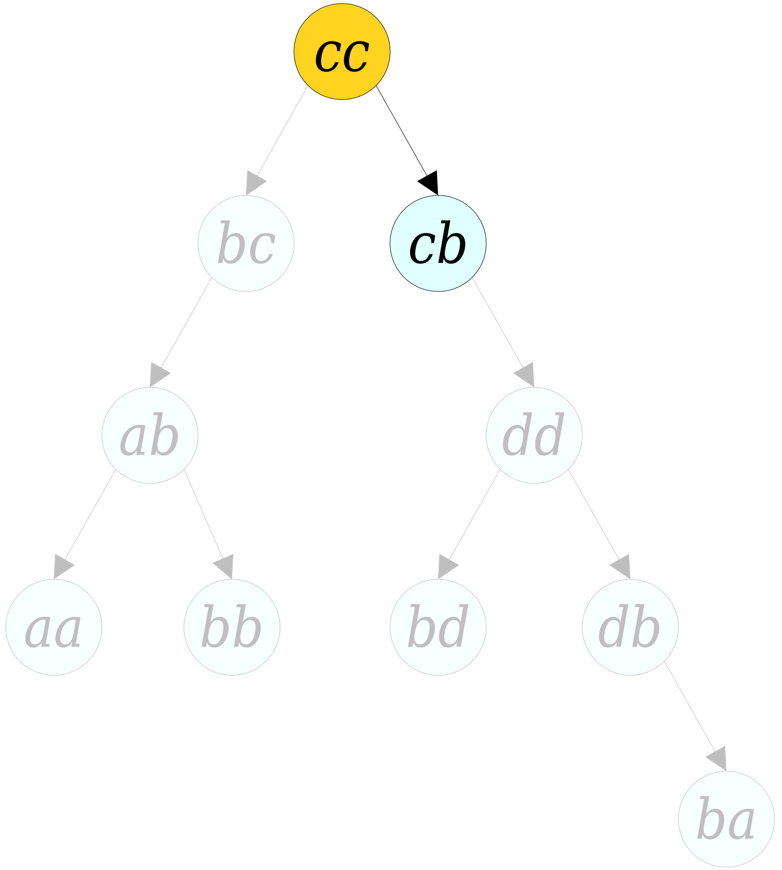
Euler Tour Trees

- To ***reroot***(x):
  - Splay xx.



Euler Tour Trees

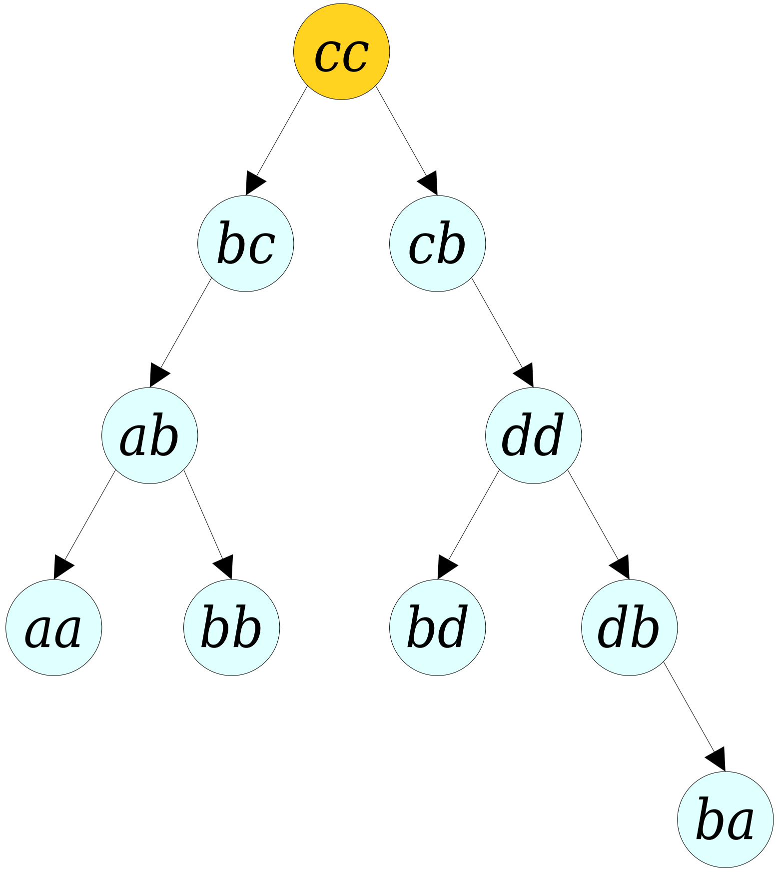
- To ***reroot***(x):
  - Splay xx.



aa ab bb bc cc cb bd dd db ba

Euler Tour Trees

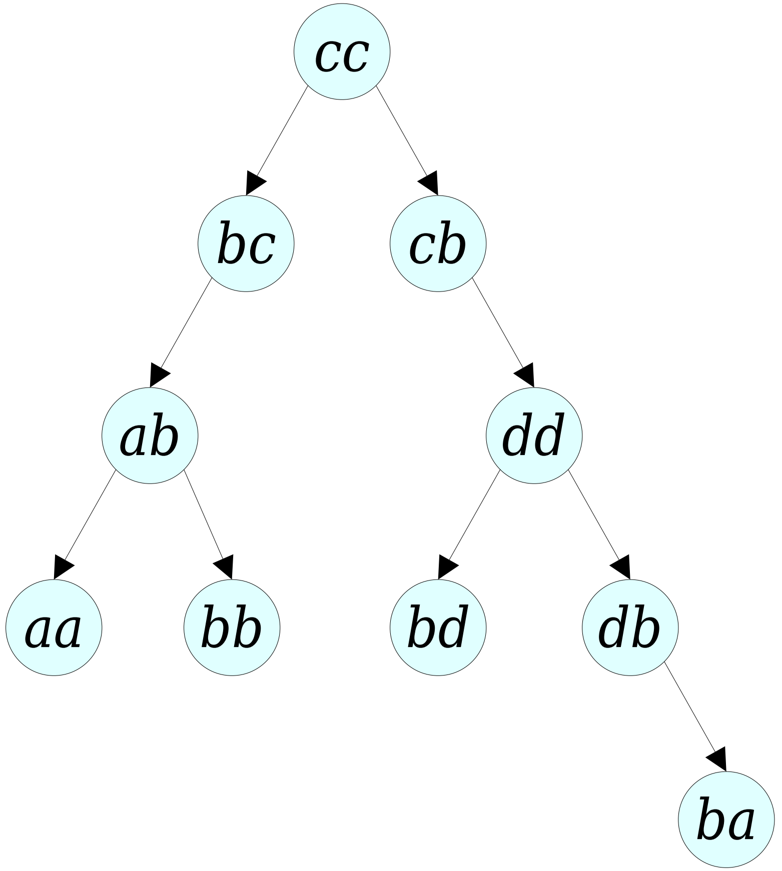
- To ***reroot***(x):
  - Splay xx.



aa ab bb bc cc cb bd dd db ba

Euler Tour Trees

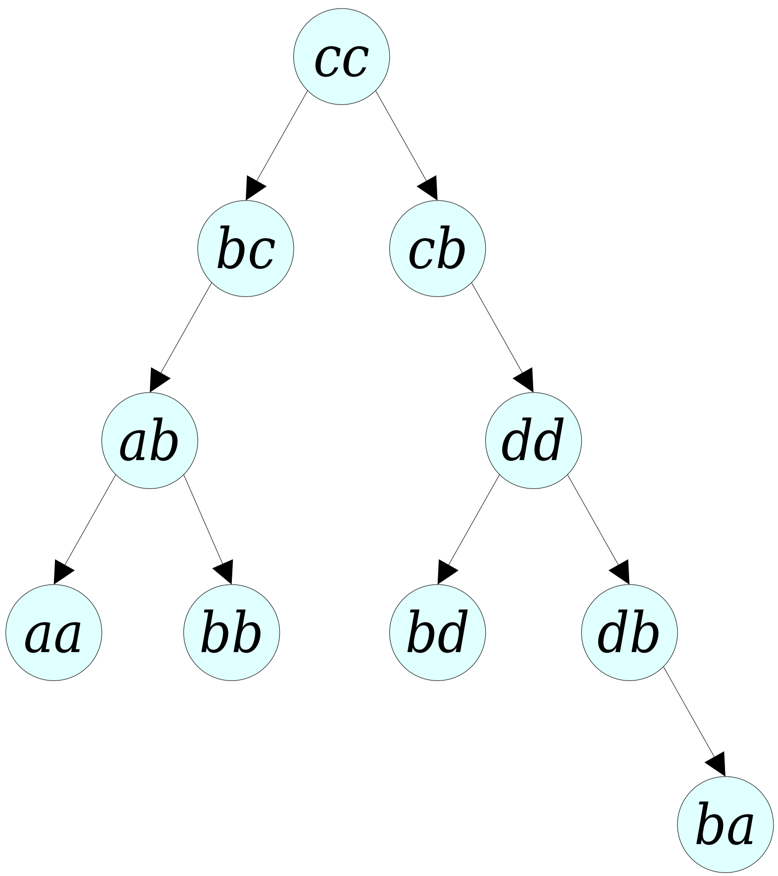
- To ***reroot***(x):
  - Splay xx.



aa ab bb bc cc cb bd dd db ba

Euler Tour Trees

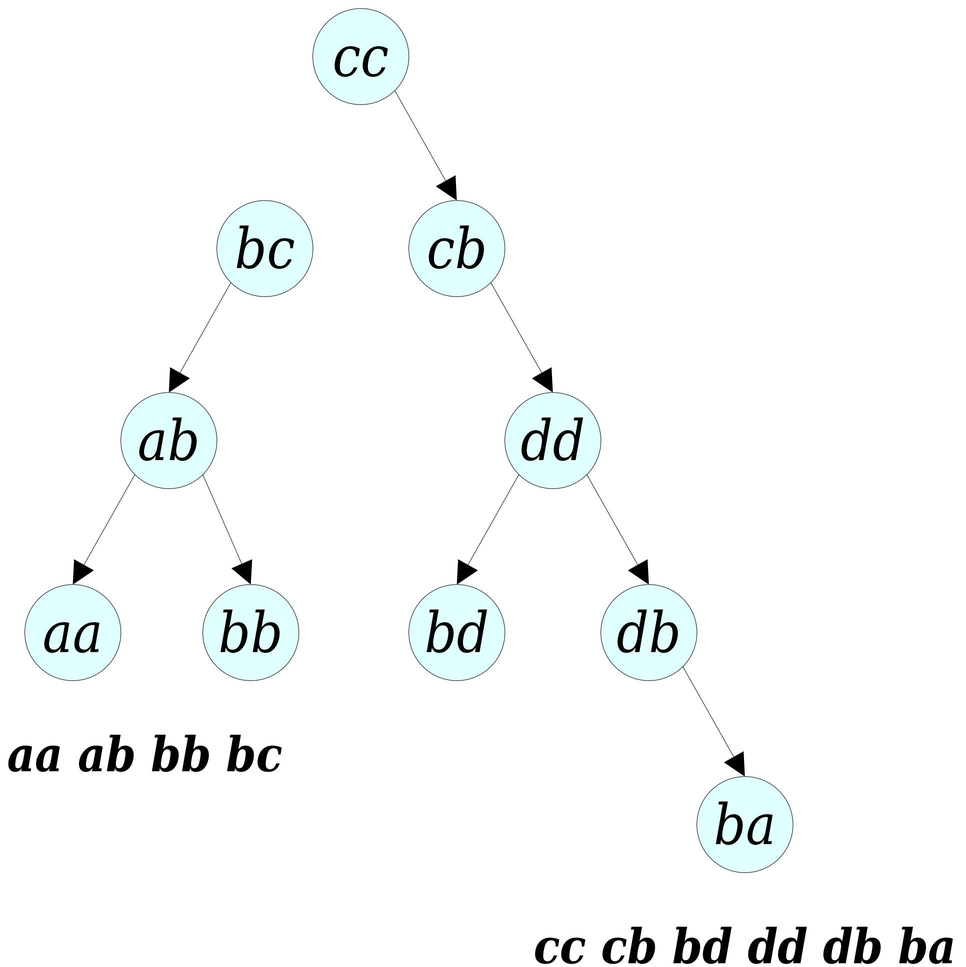
- To ***reroot***(x):
  - Splay xx.
  - Disconnect xx's left child tree T.



aa ab bb bc cc cb bd dd db ba

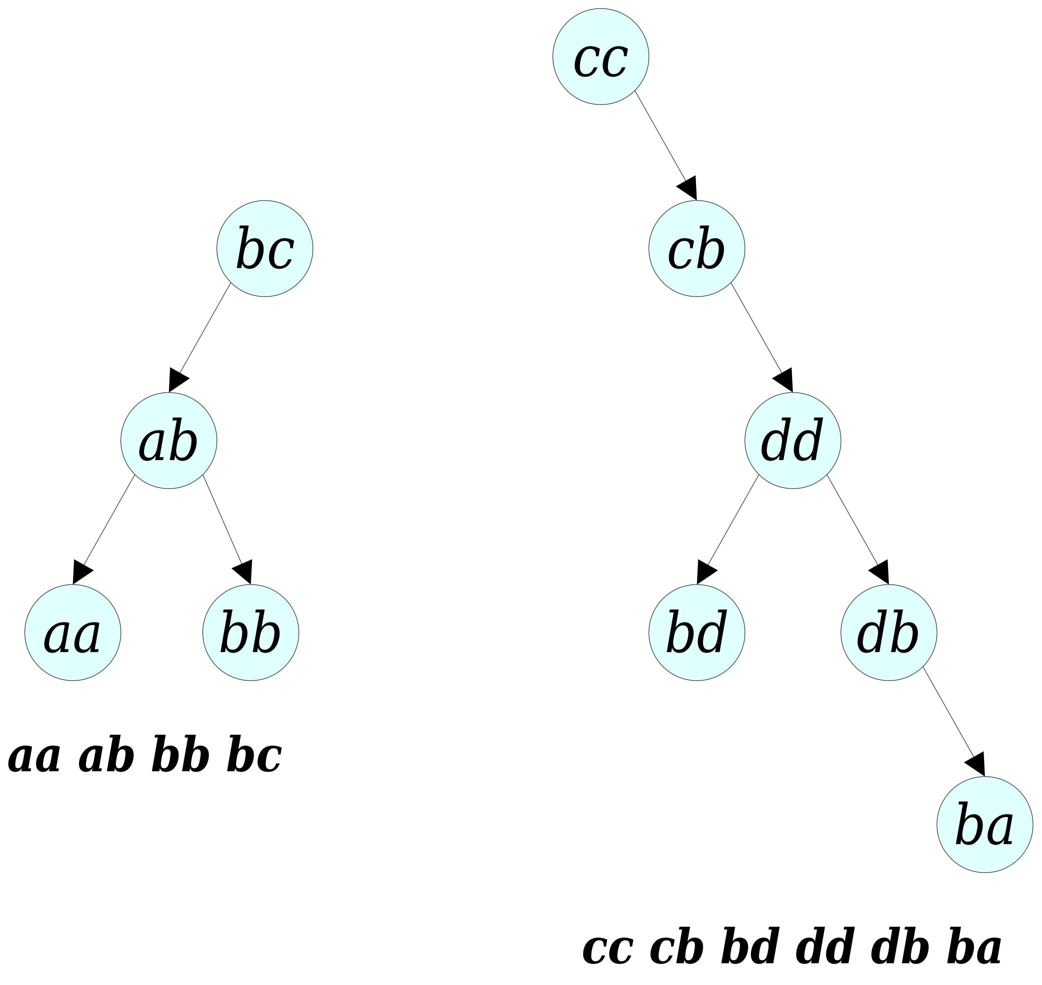
Euler Tour Trees

- To ***reroot***(x):
  - Splay xx.
  - Disconnect xx's left child tree T.



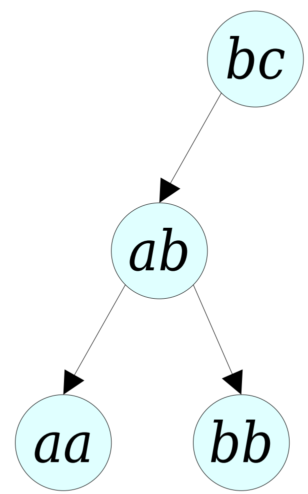
Euler Tour Trees

- To ***reroot***(x):
  - Splay xx.
  - Disconnect xx's left child tree T.

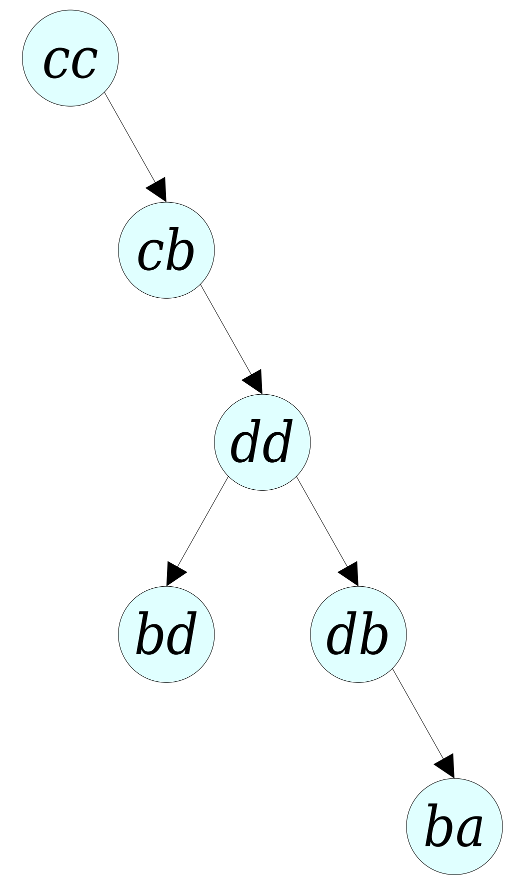


Euler Tour Trees

- To ***reroot***(x):
  - Splay xx.
  - Disconnect xx's left child tree T.
  - Splay the rightmost node in xx's subtree.



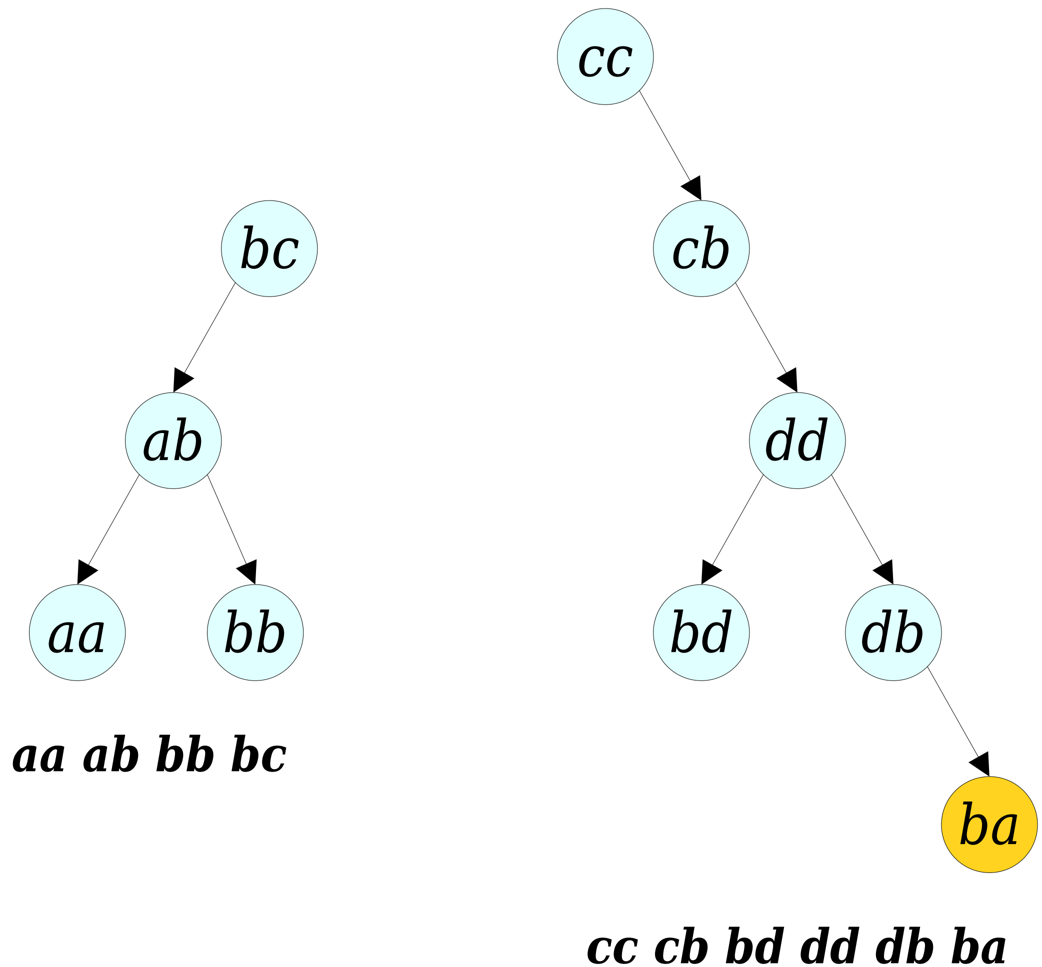
aa ab bb bc



cc cb bd dd db ba

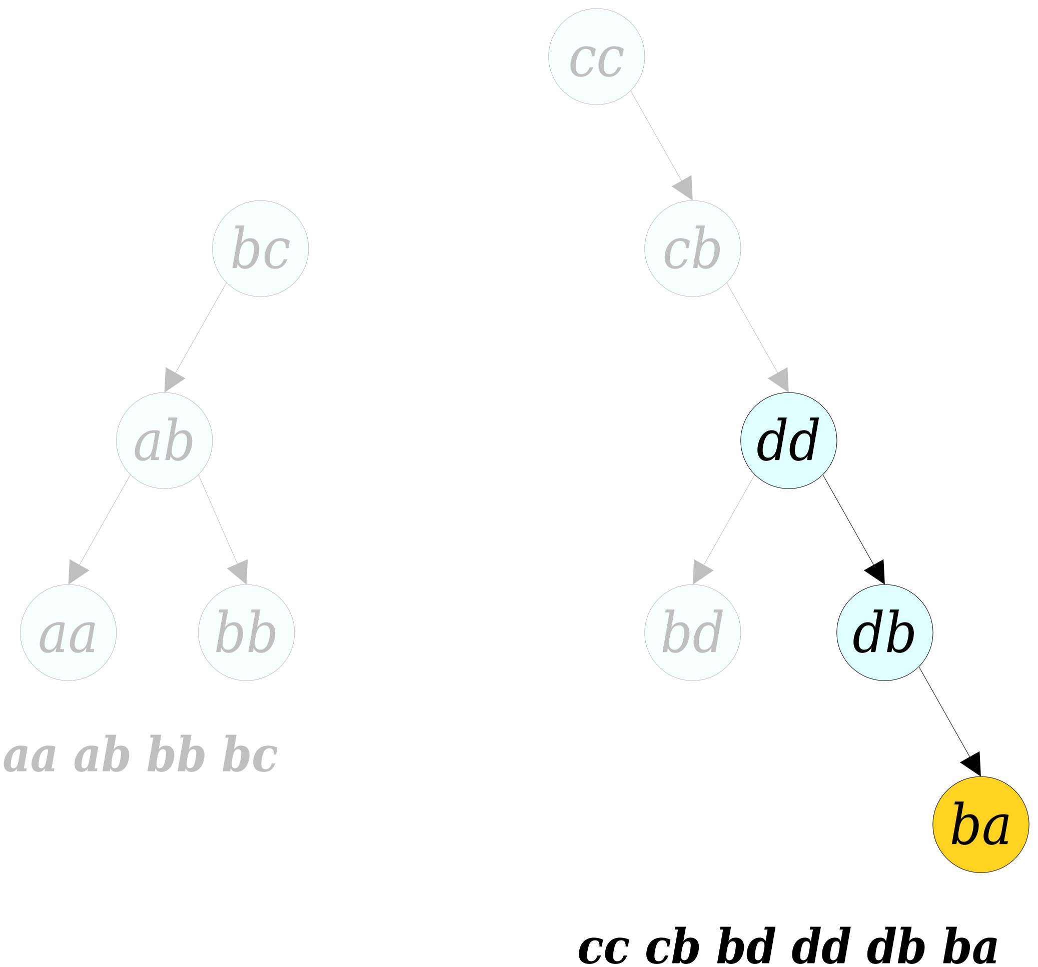
Euler Tour Trees

- To ***reroot***(x):
  - Splay *xx*.
  - Disconnect *xx*’s left child tree *T*.
  - Splay the rightmost node in *xx*’s subtree.



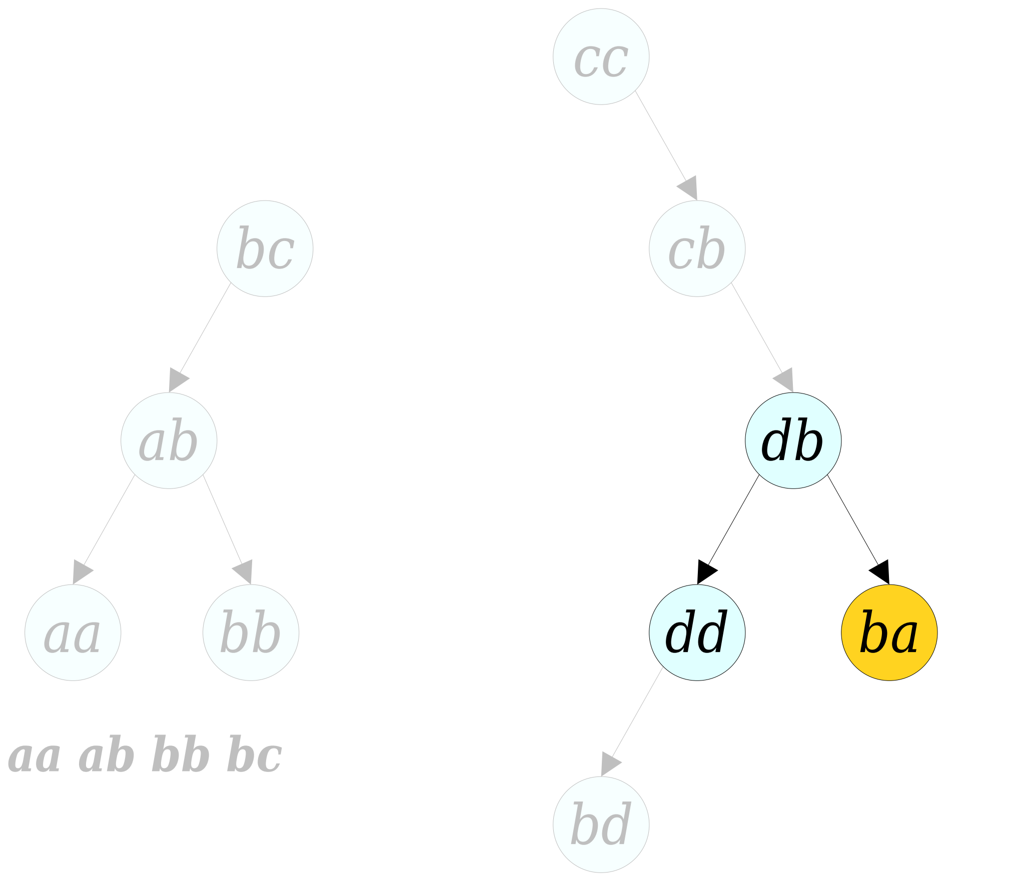
Euler Tour Trees

- To ***reroot***(x):
  - Splay xx.
  - Disconnect xx's left child tree T.
  - Splay the rightmost node in xx's subtree.



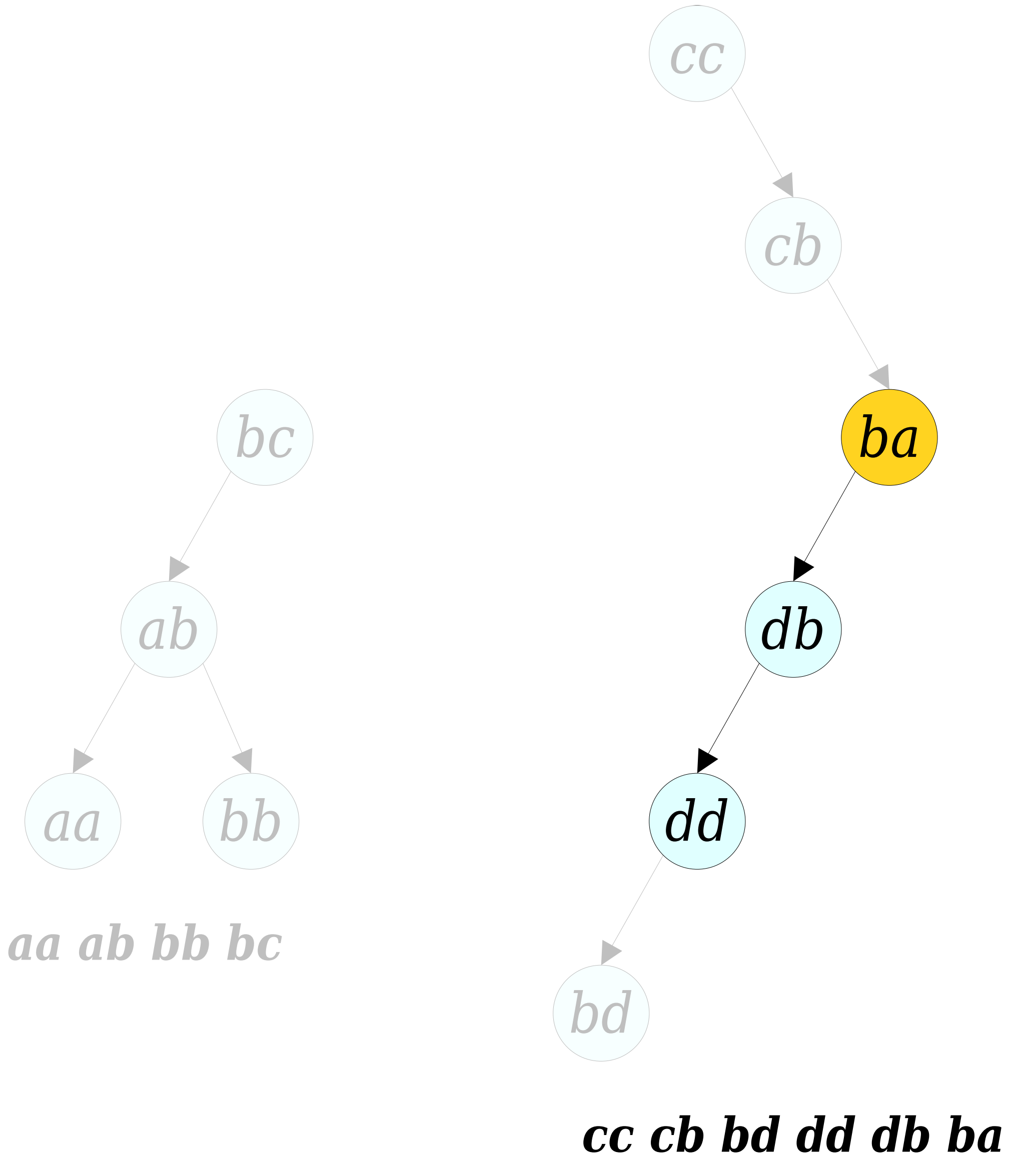
Euler Tour Trees

- To ***reroot***(x):
  - Splay xx.
  - Disconnect xx's left child tree T.
  - Splay the rightmost node in xx's subtree.



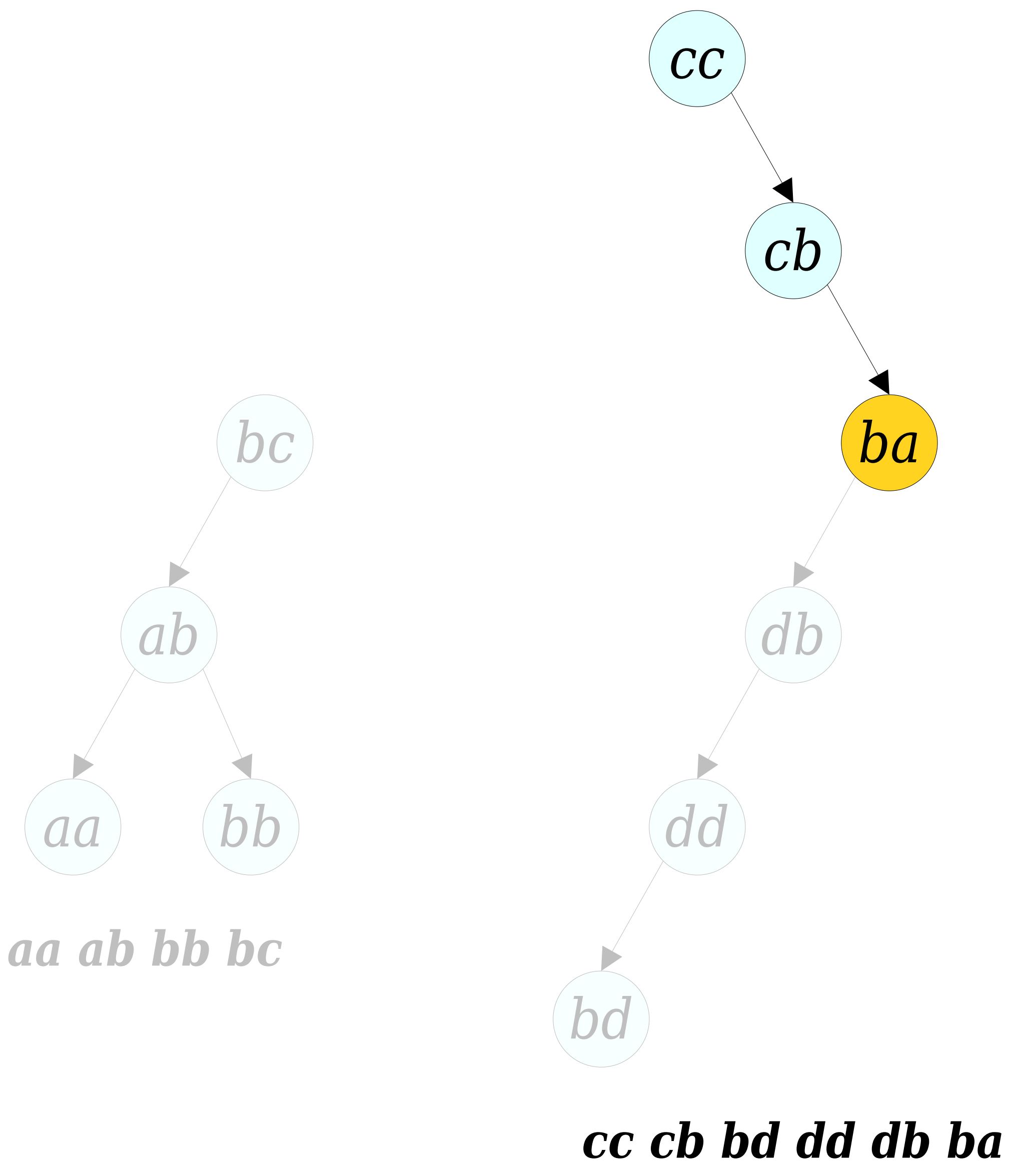
Euler Tour Trees

- To ***reroot***(x):
  - Splay xx.
  - Disconnect xx's left child tree T.
  - Splay the rightmost node in xx's subtree.



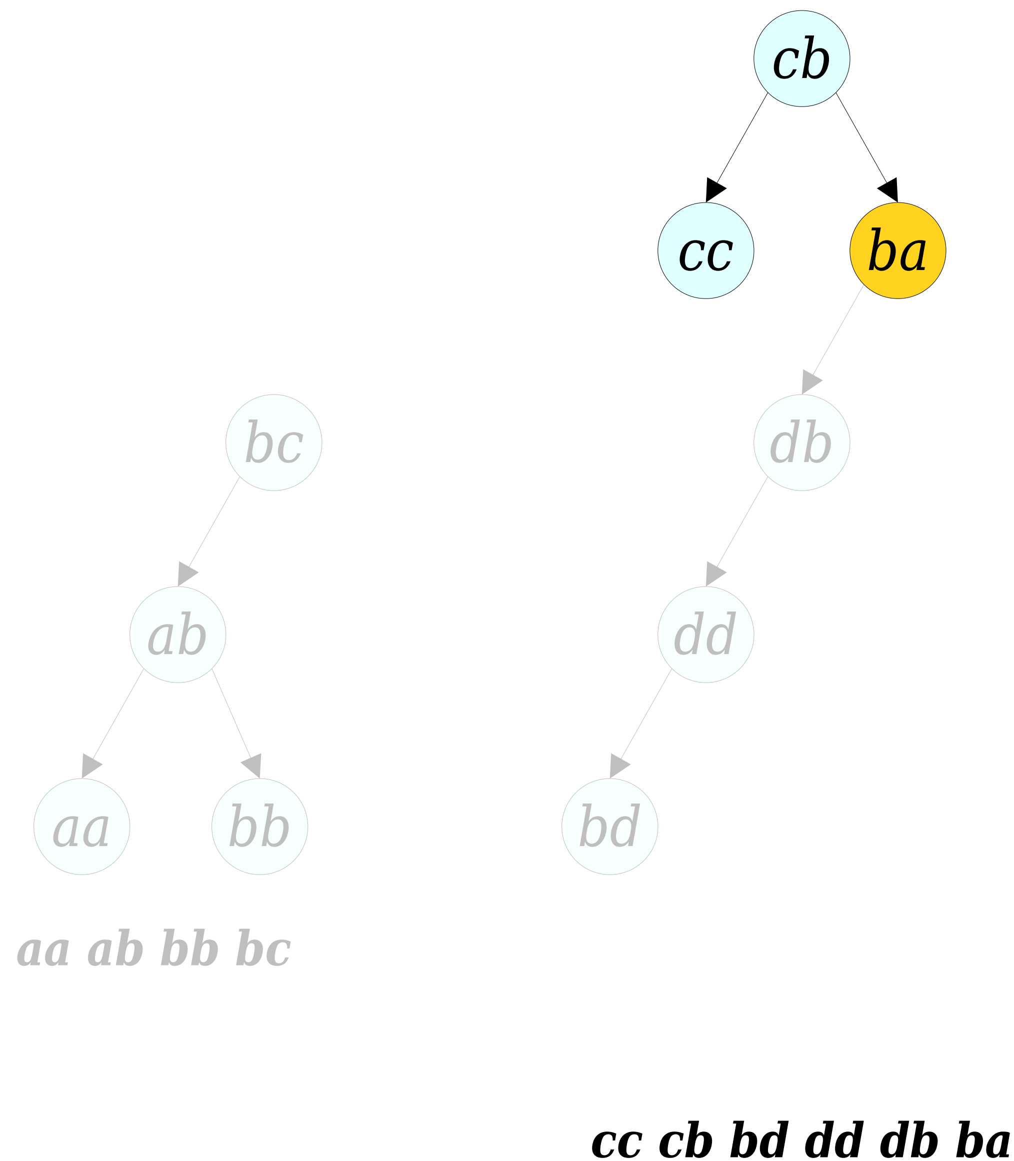
Euler Tour Trees

- To ***reroot***(x):
  - Splay xx.
  - Disconnect xx's left child tree T.
  - Splay the rightmost node in xx's subtree.



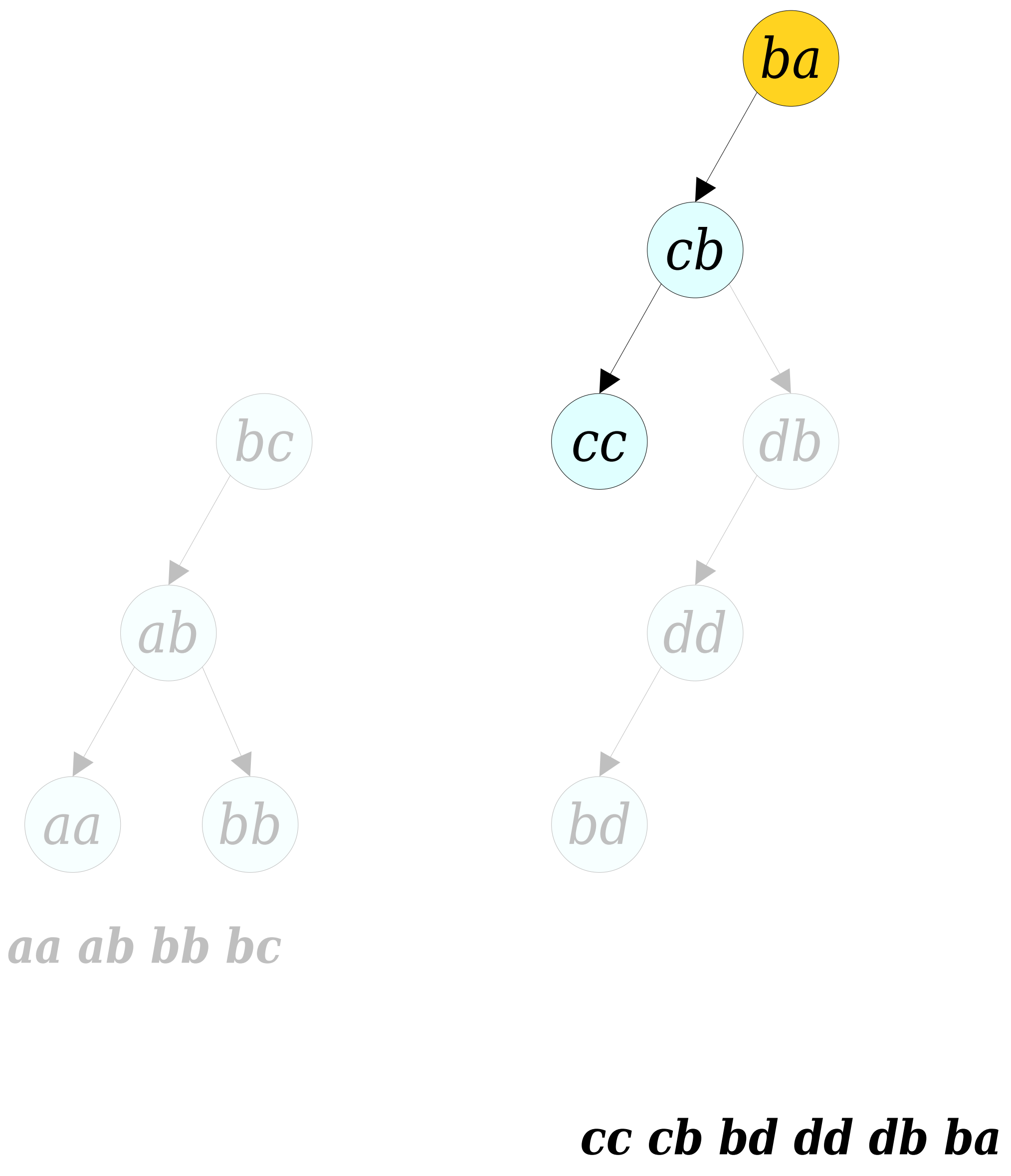
Euler Tour Trees

- To ***reroot***(x):
  - Splay xx.
  - Disconnect xx's left child tree T.
  - Splay the rightmost node in xx's subtree.



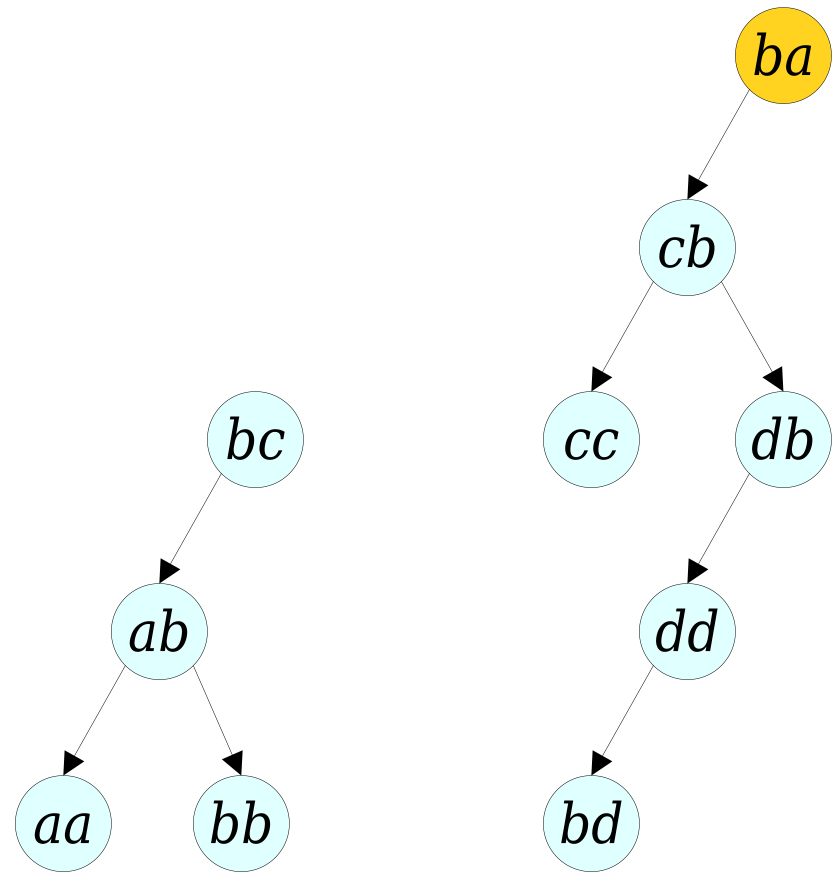
Euler Tour Trees

- To ***reroot***(x):
  - Splay xx.
  - Disconnect xx's left child tree T.
  - Splay the rightmost node in xx's subtree.



Euler Tour Trees

- To ***reroot***(x):
  - Splay xx.
  - Disconnect xx’s left child tree T.
  - Splay the rightmost node in xx’s subtree.

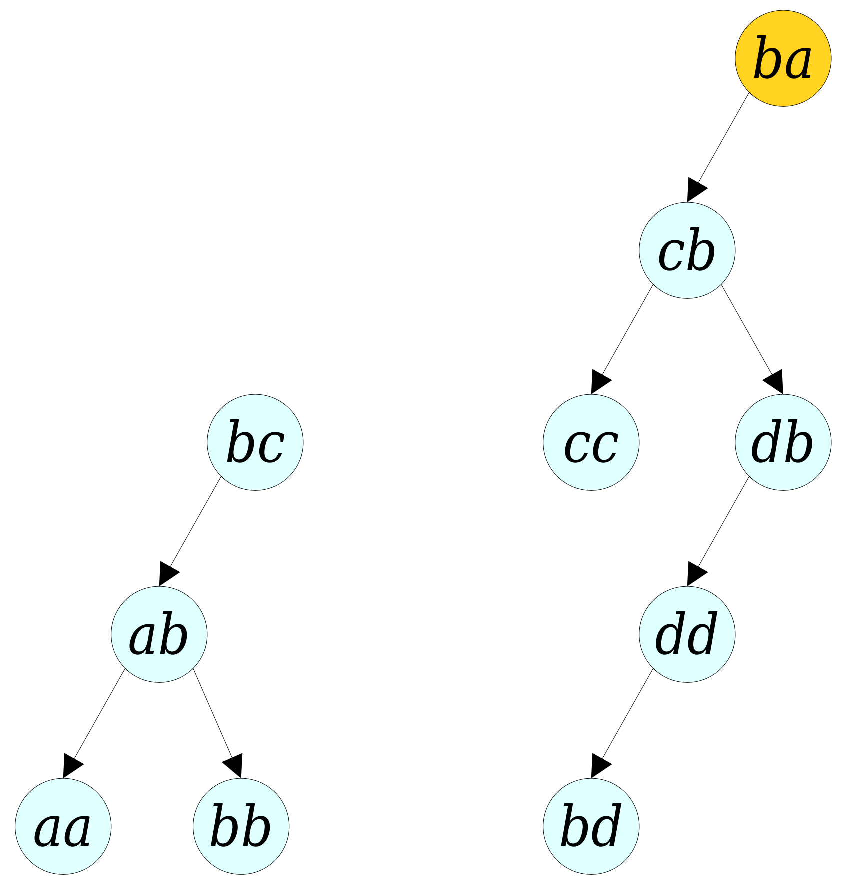


aa ab bb bc

cc cb bd dd db ba

Euler Tour Trees

- To ***reroot***(x):
  - Splay xx.
  - Disconnect xx's left child tree T.
  - Splay the rightmost node in xx's subtree.
  - Make T the right child of the root.

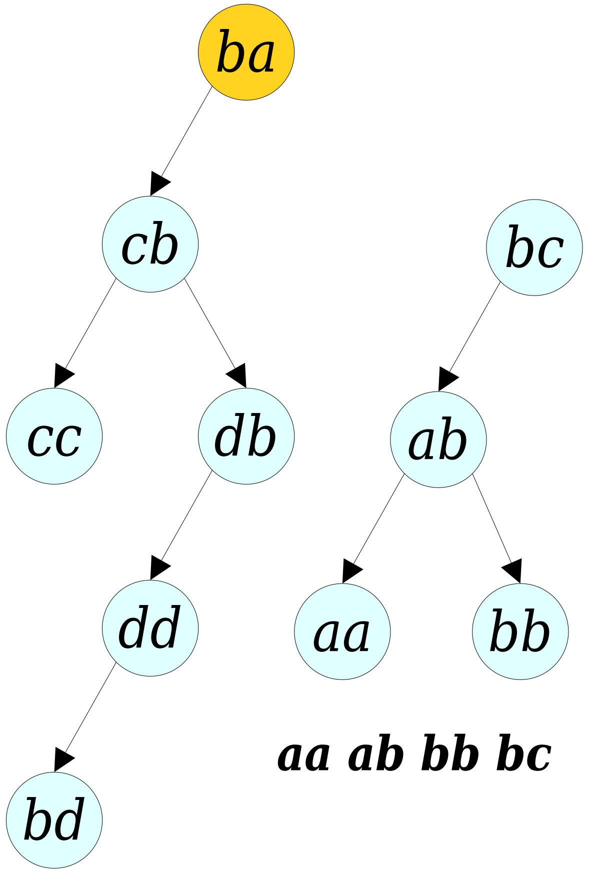


aa ab bb bc

cc cb bd dd db ba

Euler Tour Trees

- To ***reroot***(x):
  - Splay xx.
  - Disconnect xx's left child tree T.
  - Splay the rightmost node in xx's subtree.
  - Make T the right child of the root.

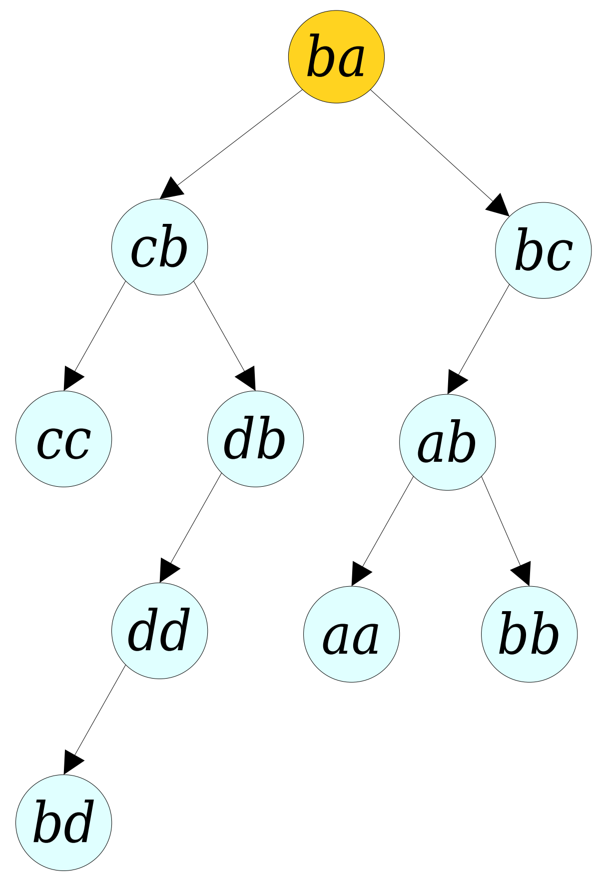


aa ab bb bc

cc cb bd dd db ba

Euler Tour Trees

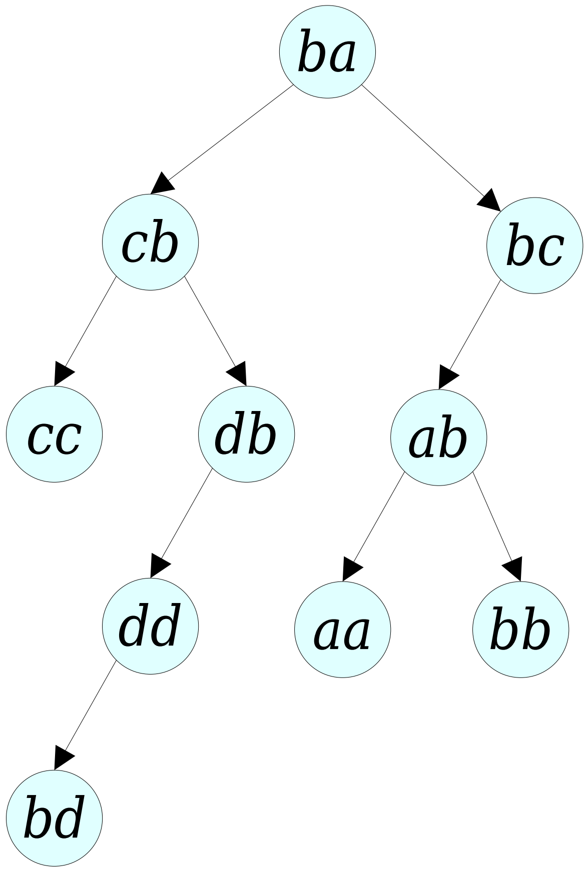
- To ***reroot***(x):
  - Splay xx.
  - Disconnect xx's left child tree T.
  - Splay the rightmost node in xx's subtree.
  - Make T the right child of the root.



cc cb bd dd db ba aa ab bb bc

Euler Tour Trees

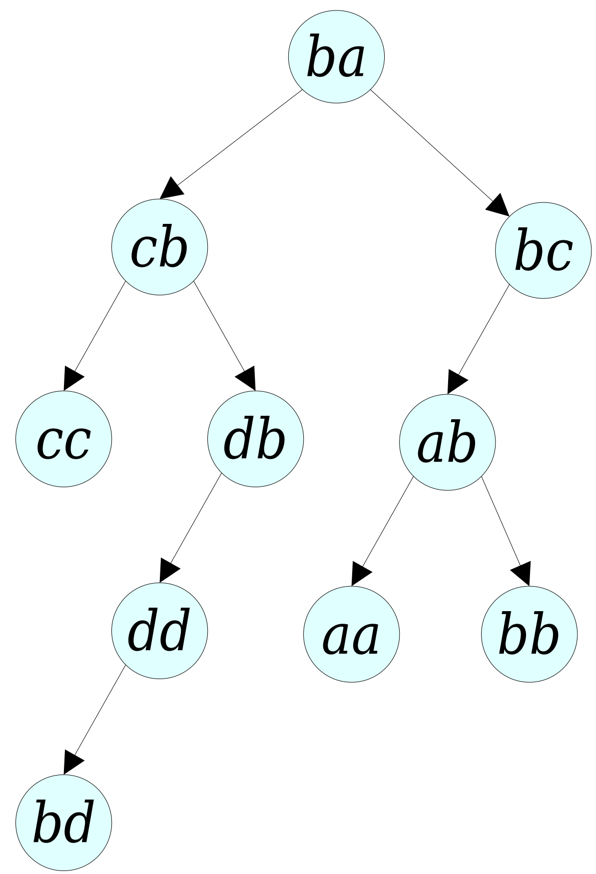
- To ***reroot***(x):
  - Splay xx.
  - Disconnect xx's left child tree T.
  - Splay the rightmost node in xx's subtree.
  - Make T the right child of the root.



cc cb bd dd db ba aa ab bb bc

Euler Tour Trees

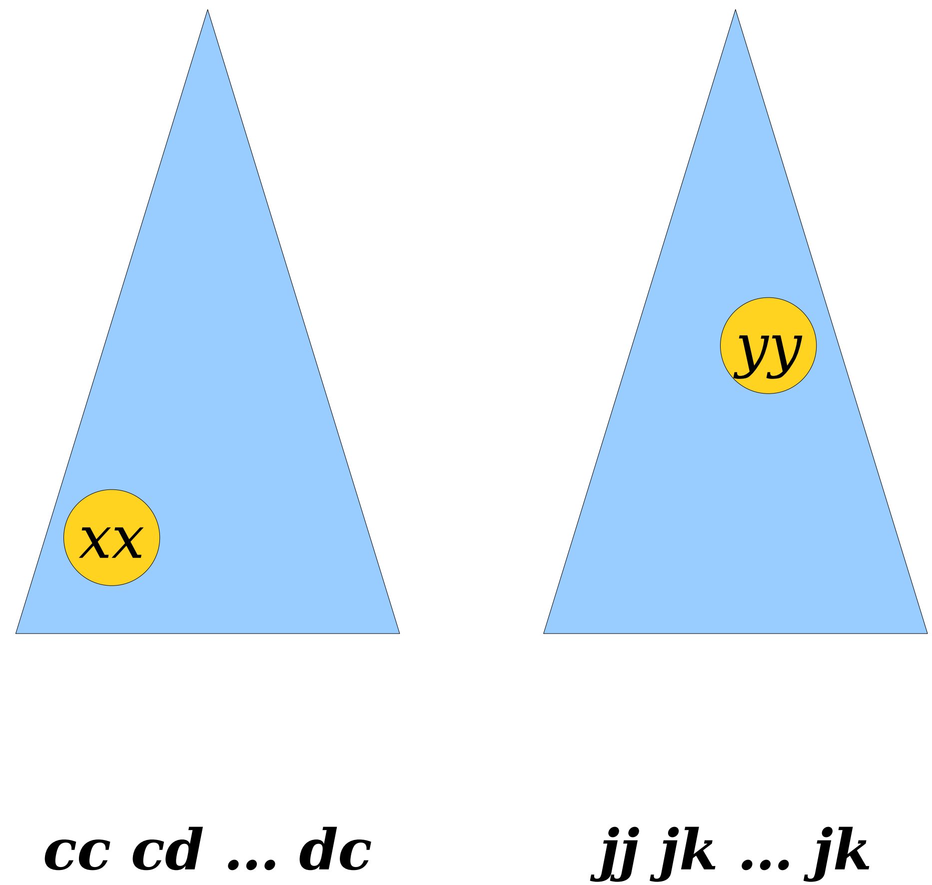
- To ***reroot***(x):
  - Splay xx.
  - Disconnect xx's left child tree T.
  - Splay the rightmost node in xx's subtree.
  - Make T the right child of the root.
- Amortized cost: **O(log *n*)**.



cc cb bd dd db ba aa ab bb bc

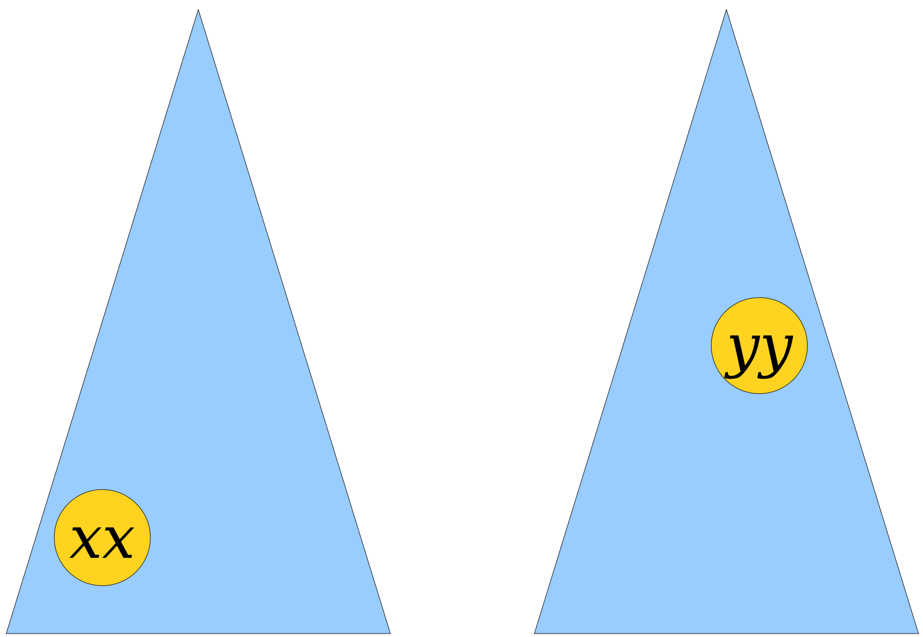
Euler Tour Trees

- To ***link***(x, y):



Euler Tour Trees

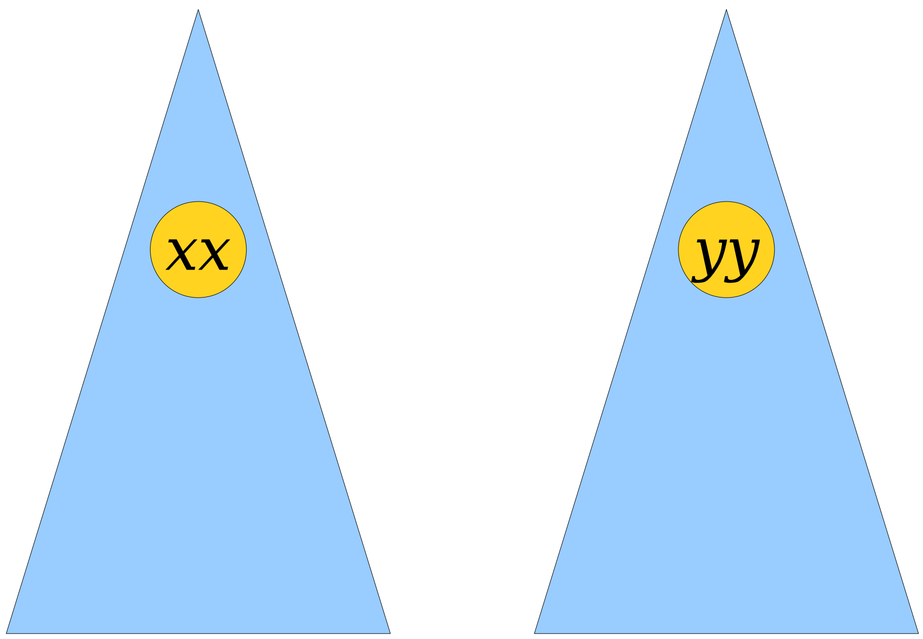
- To ***link***(x, y):
  - ***reroot***(x) and ***reroot***(y).



cc cd … dc ***jj jk … jk***

Euler Tour Trees

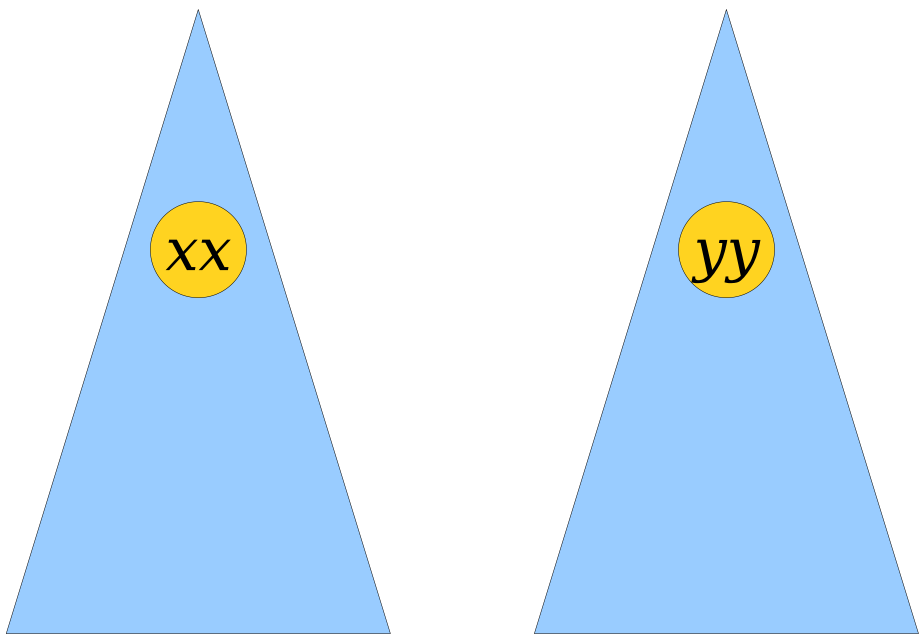
- To ***link***(x, y):
  - ***reroot***(x) and ***reroot***(y).



xx xa … ax ***yy yf … fy***

Euler Tour Trees

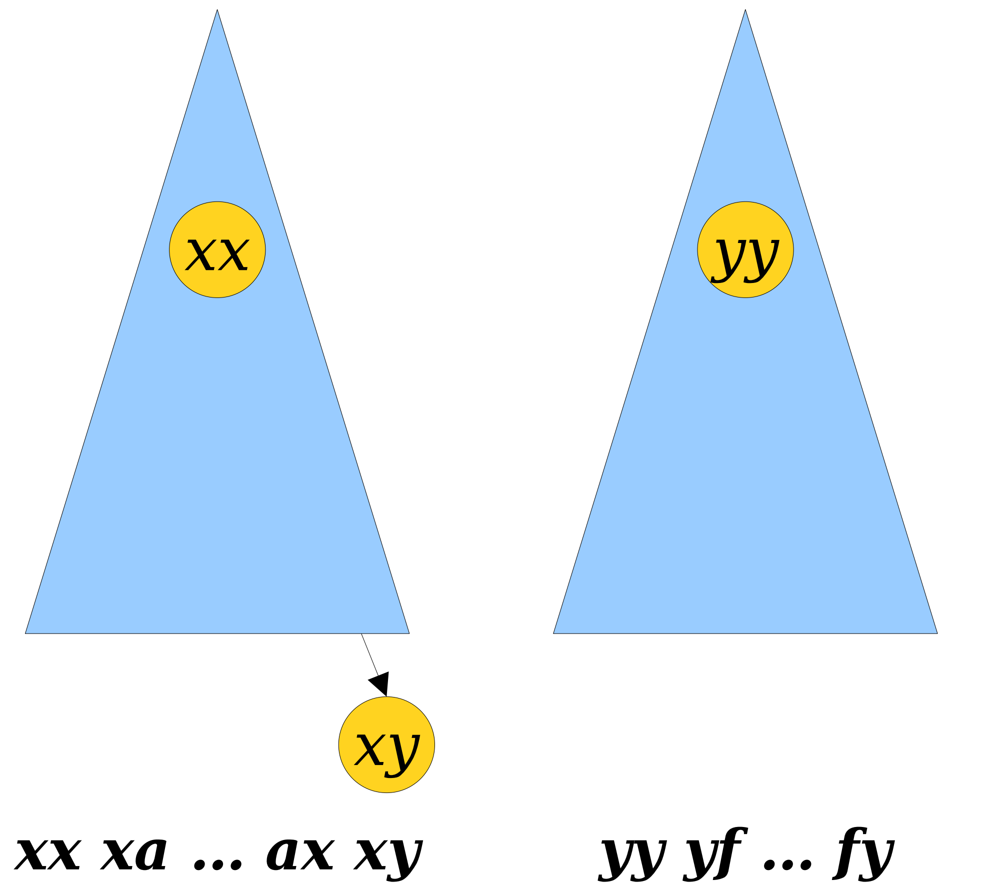
- To ***link***(x, y):
  - ***reroot***(x) and ***reroot***(y).
  - Add xy as the rightmost node of x's tree.



xx xa … ax ***yy yf … fy***

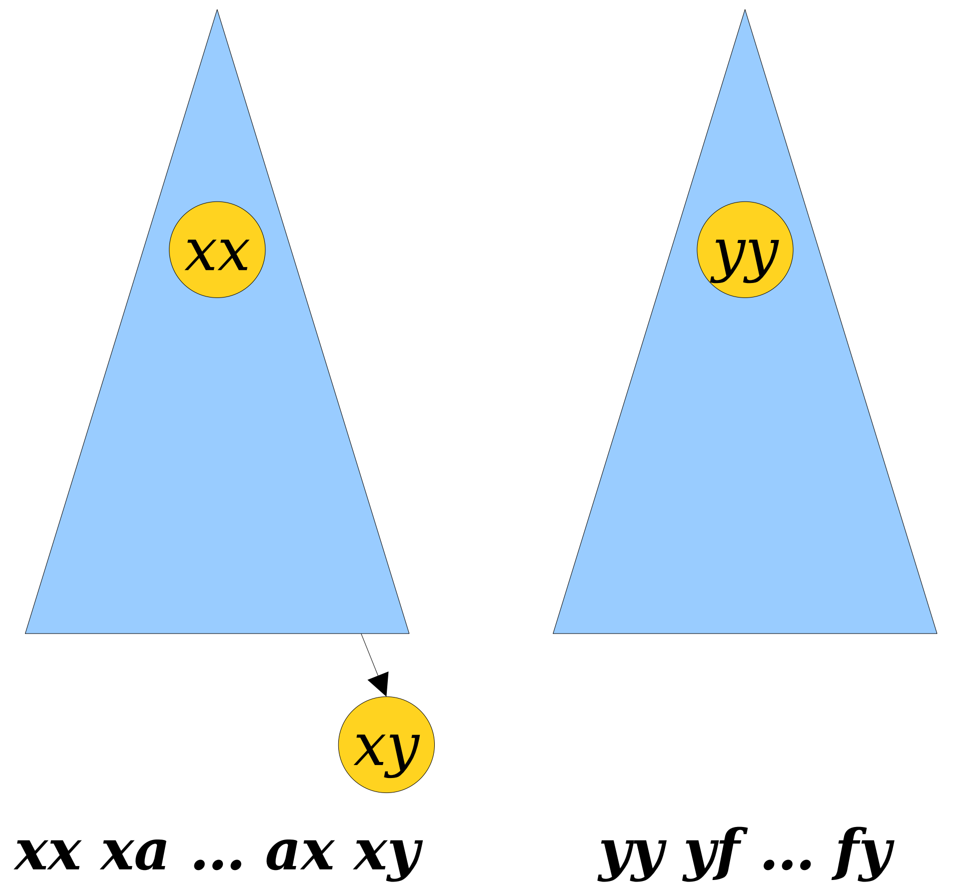
Euler Tour Trees

- To ***link***(x, y):
  - ***reroot***(x) and ***reroot***(y).
  - Add xy as the rightmost node of x's tree.



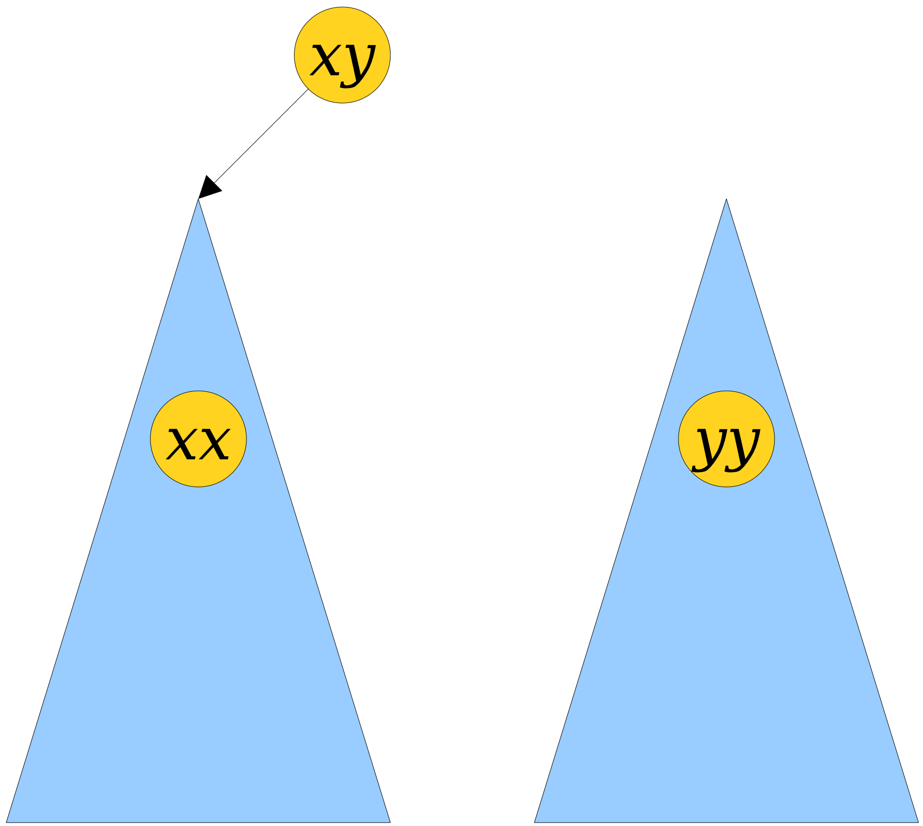
Euler Tour Trees

- To ***link***(x, y):
  - ***reroot***(x) and ***reroot***(y).
  - Add xy as the rightmost node of x's tree.
  - Splay xy.



Euler Tour Trees

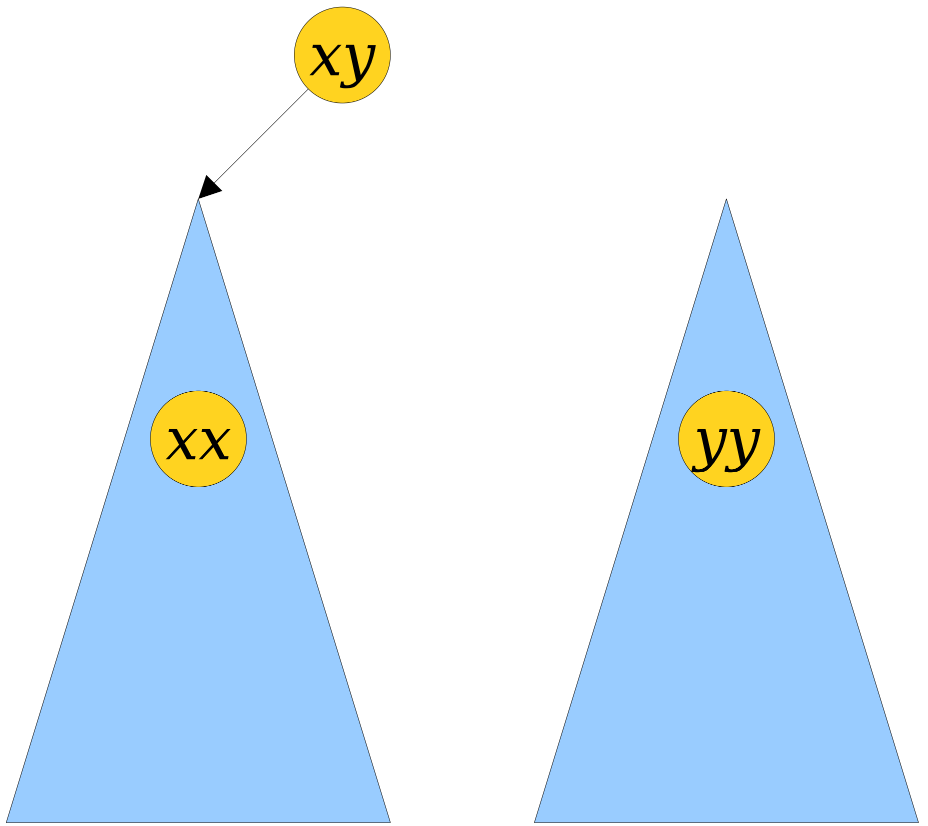
- To ***link***(x, y):
  - ***reroot***(x) and ***reroot***(y).
  - Add xy as the rightmost node of x's tree.
  - Splay xy.



xx xa … ax xy   ***yy yf … fy***

Euler Tour Trees

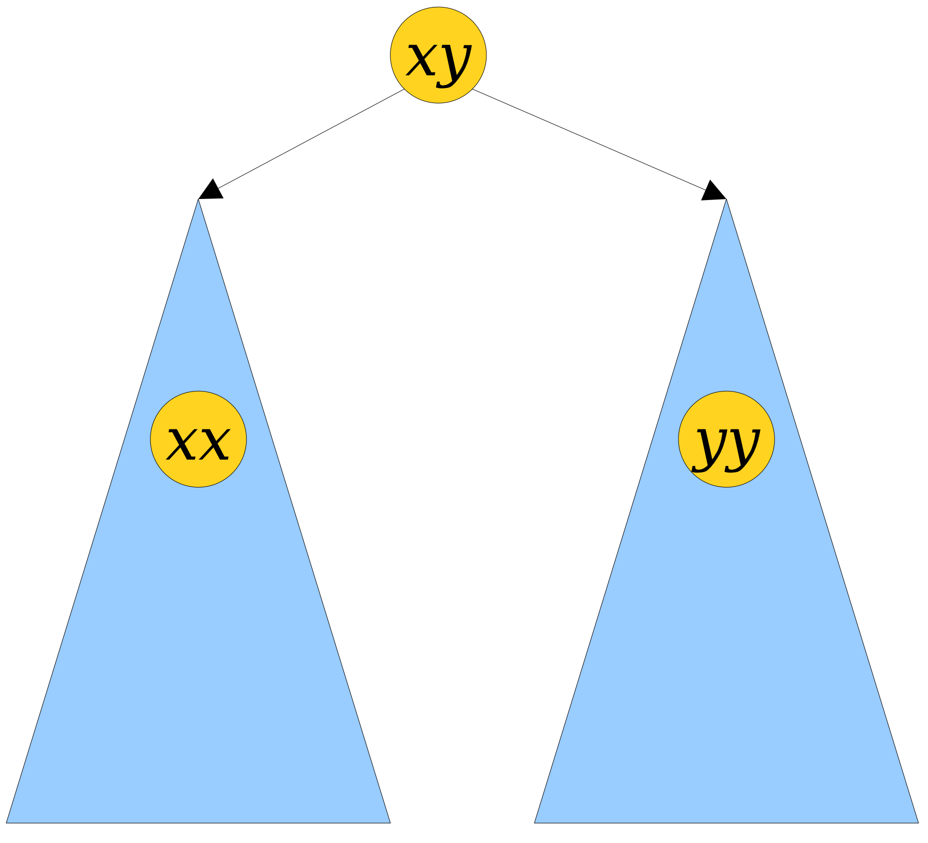
- To ***link***(x, y):
  - ***reroot***(x) and ***reroot***(y).
  - Add xy as the rightmost node of x's tree.
  - Splay xy.
  - Set yy's tree as xy's right child.



xx xa … ax xy ***yy yf … fy***

Euler Tour Trees

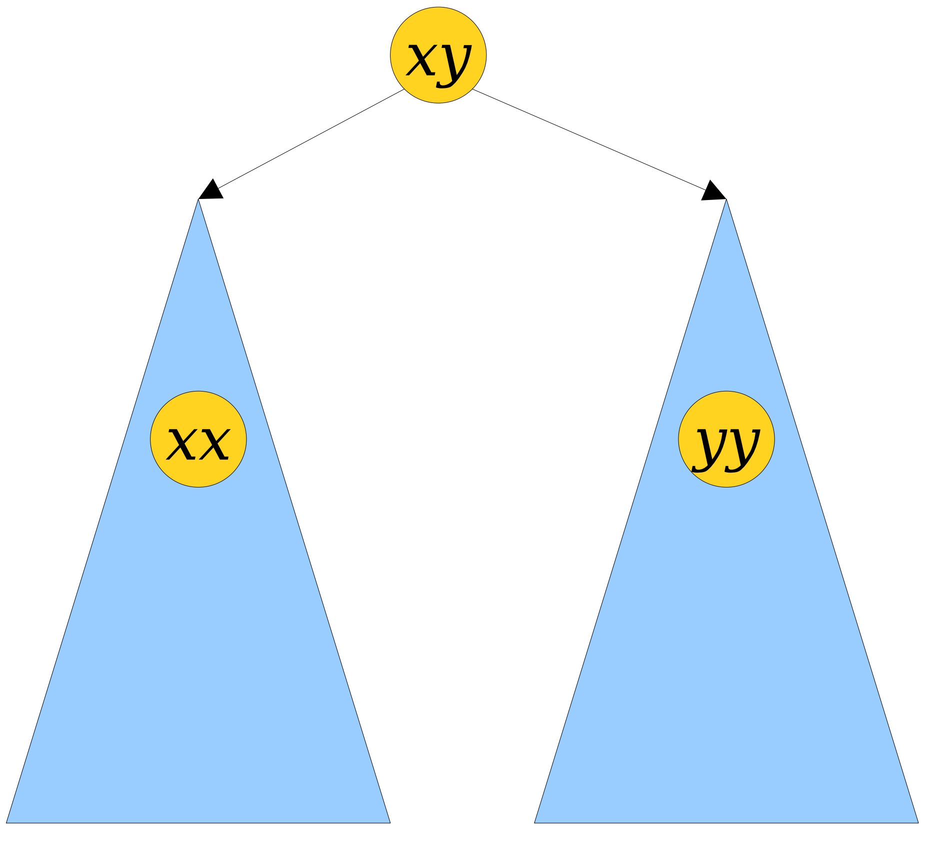
- To ***link***(x, y):
  - ***reroot***(x) and ***reroot***(y).
  - Add xy as the rightmost node of x's tree.
  - Splay xy.
  - Set yy's tree as xy's right child.



xx xa … ax xy yy yf … fy

Euler Tour Trees

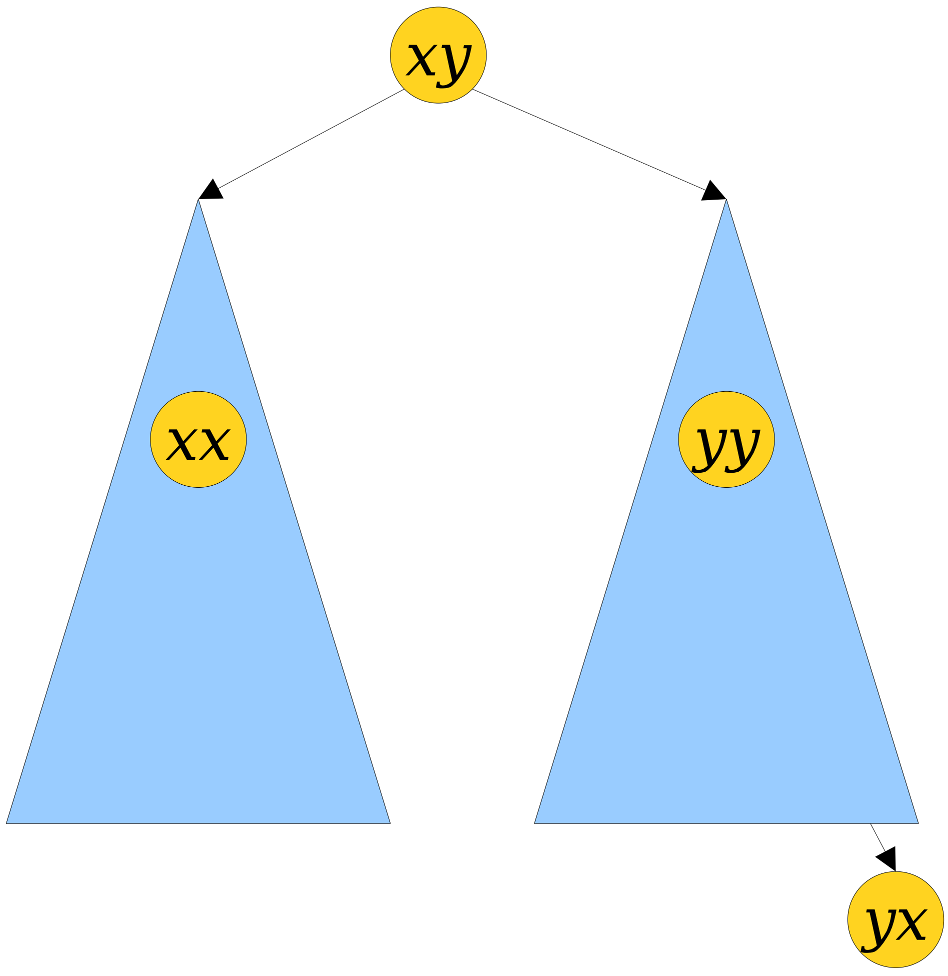
- To ***link***(*x*, *y*):
  - ***reroot***(*x*) and ***reroot***(*y*).
  - Add *xy* as the rightmost node of *x*'s tree.
  - Splay *xy*.
  - Set *yy*'s tree as *xy*'s right child.
  - Add *yx* as the rightmost node of the tree.



xx xa … ax xy yy yf … fy

Euler Tour Trees

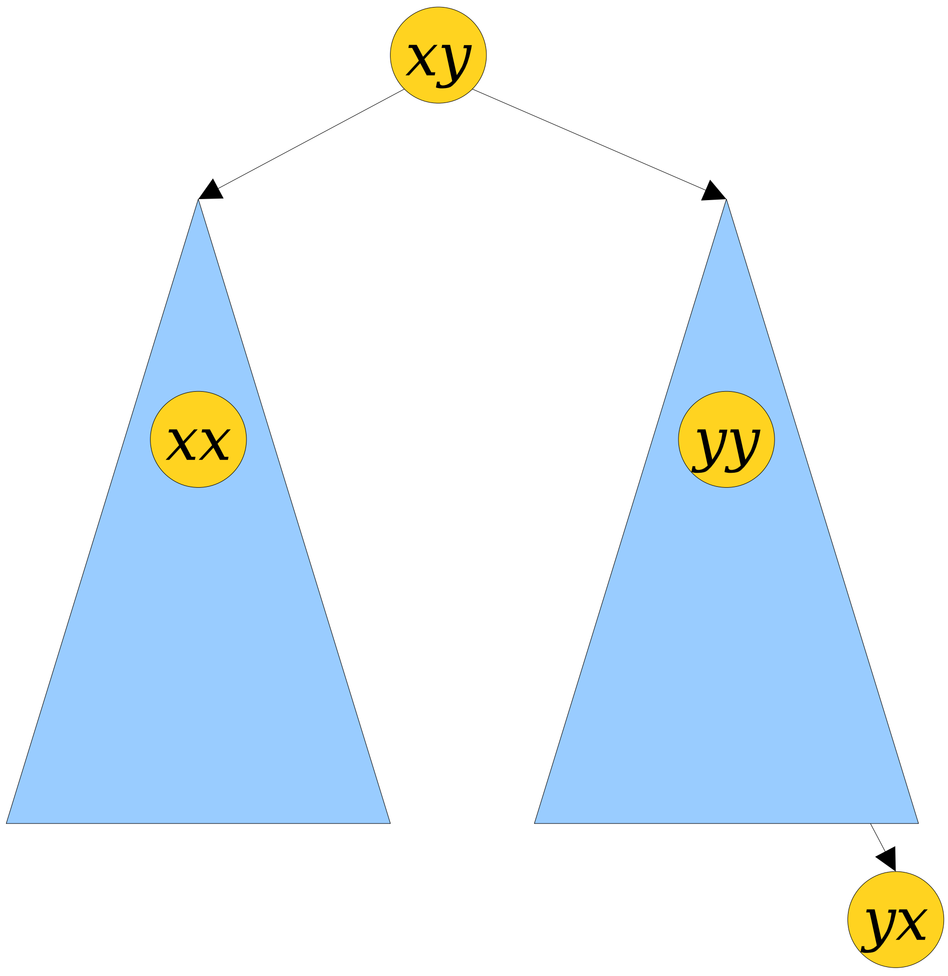
- To ***link***(x, y):
  - ***reroot***(x) and ***reroot***(y).
  - Add xy as the rightmost node of x's tree.
  - Splay xy.
  - Set yy's tree as xy's right child.
  - Add yx as the rightmost node of the tree.



xx xa … ax xy yy yf … fy yx

Euler Tour Trees

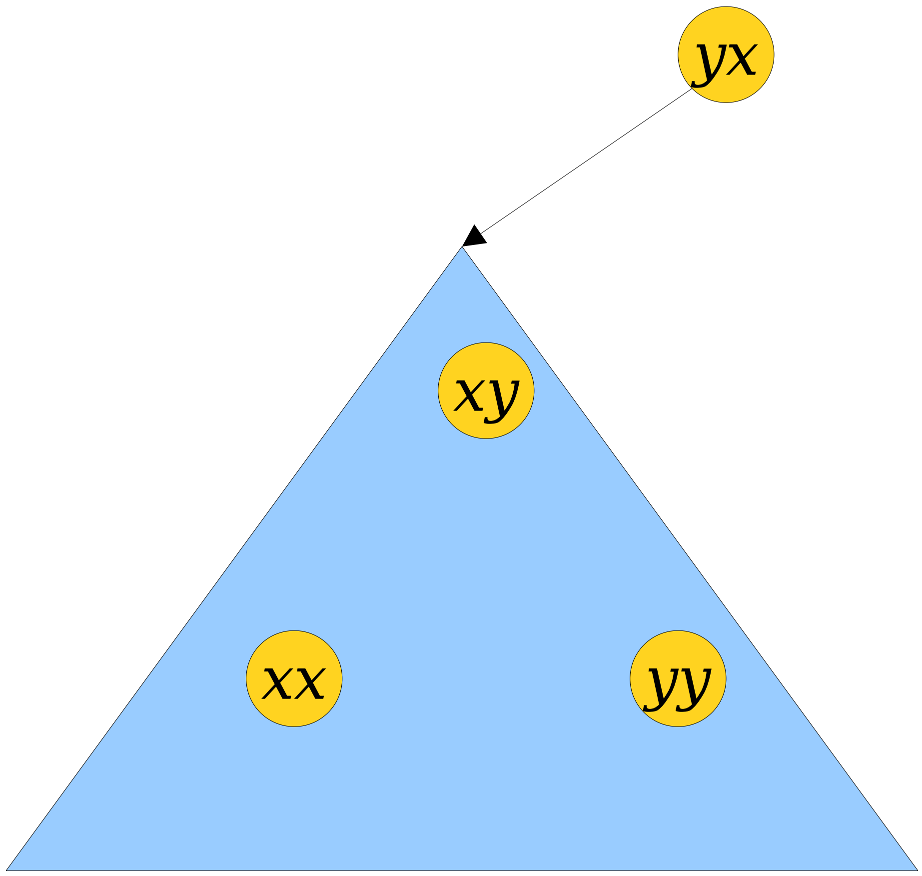
- To ***link***(x, y):
  - ***reroot***(x) and ***reroot***(y).
  - Add xy as the rightmost node of x's tree.
  - Splay xy.
  - Set yy's tree as xy's right child.
  - Add yx as the rightmost node of the tree.
  - Splay yx.



xx xa … ax xy yy yf … fy yx

Euler Tour Trees

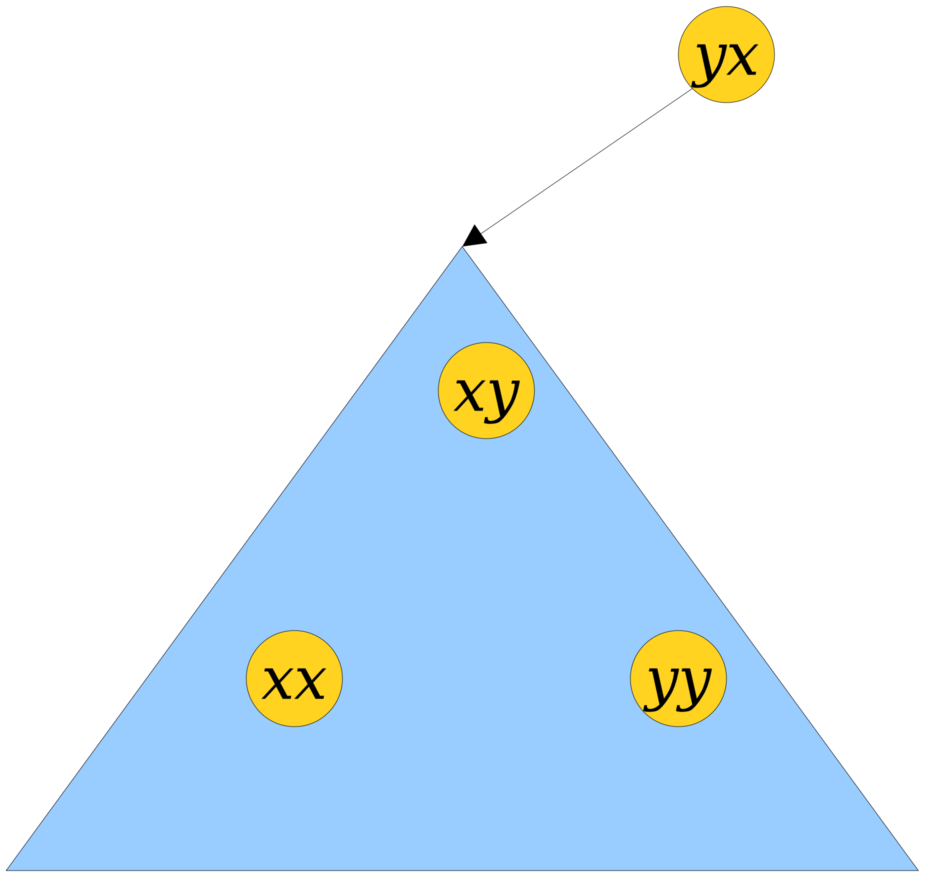
- To ***link***(*x*, *y*):
  - ***reroot***(*x*) and ***reroot***(*y*).
  - Add *xy* as the rightmost node of *x*'s tree.
  - Splay *xy*.
  - Set *yy*'s tree as *xy*'s right child.
  - Add *yx* as the rightmost node of the tree.
  - Splay *yx*.



xx xa … ax xy yy yf … fy yx

Euler Tour Trees

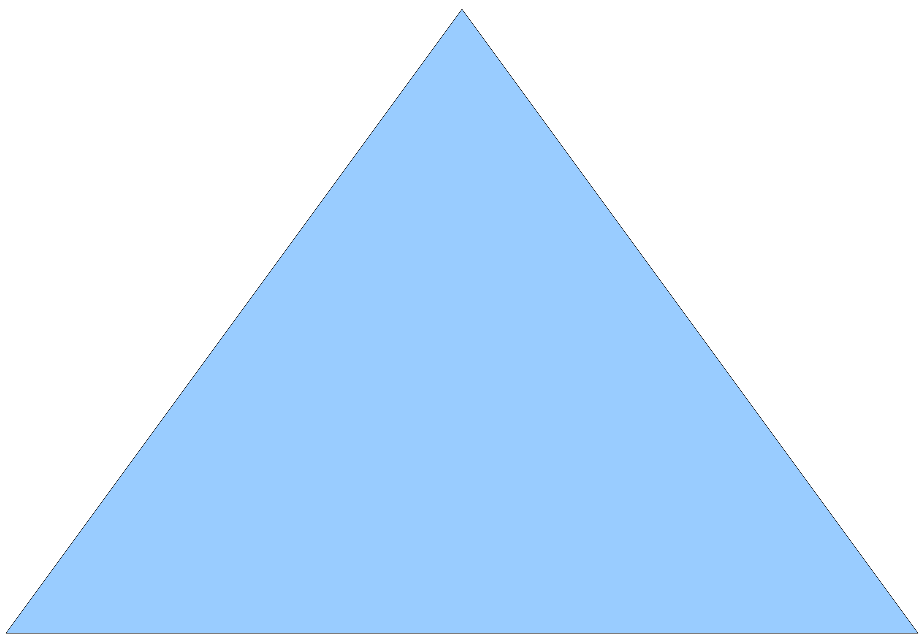
- To ***link***(*x*, *y*):
  - ***reroot***(*x*) and ***reroot***(*y*).
  - Add *xy* as the rightmost node of *x*'s tree.
  - Splay *xy*.
  - Set *yy*'s tree as *xy*'s right child.
  - Add *yx* as the rightmost node of the tree.
  - Splay *yx*.
- Amortized cost: **O(log *n*)**.



xx xa … ax xy yy yf … fy yx

Euler Tour Trees

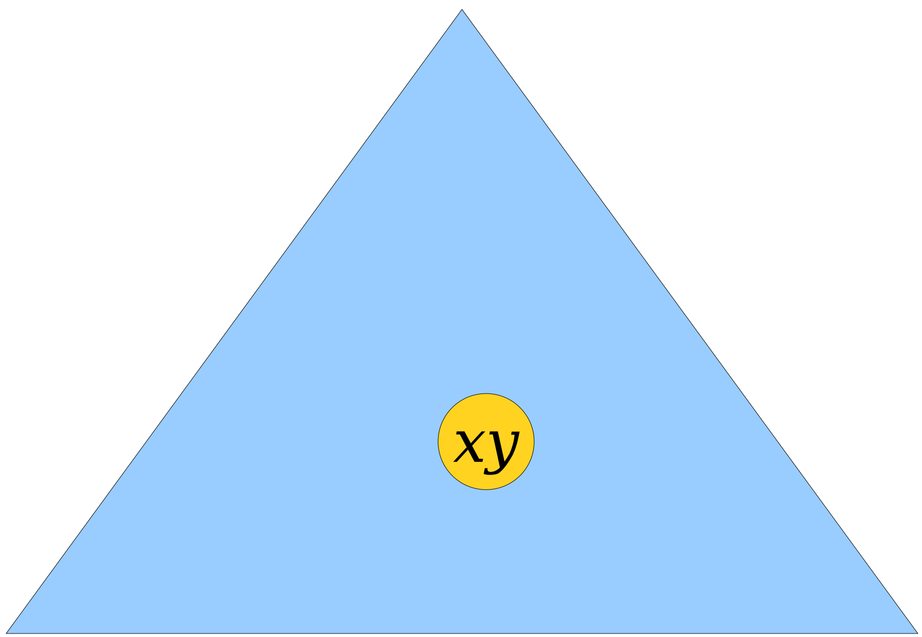
- To ***cut***(x, y):



aa ab … cx xy yy yf … fy yx xt … ba

Euler Tour Trees

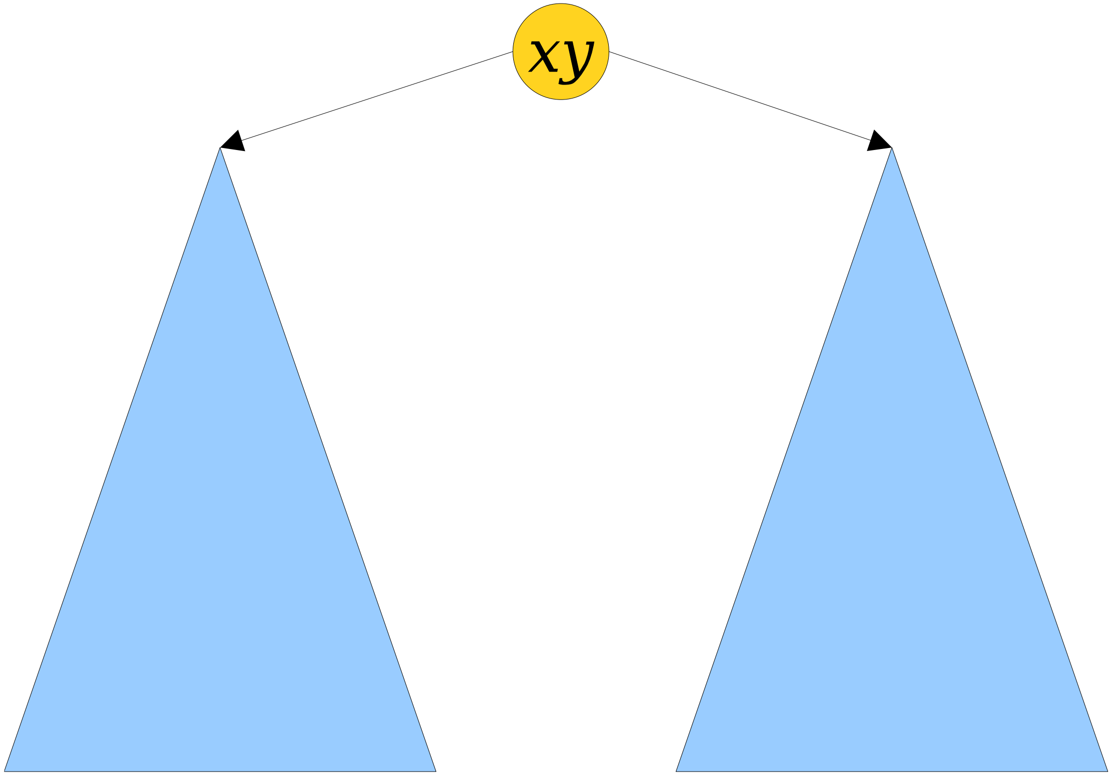
- To ***cut***(x, y):
  - Splay xy.



aa ab … cx xy yy yf … fy yx xt … ba

Euler Tour Trees

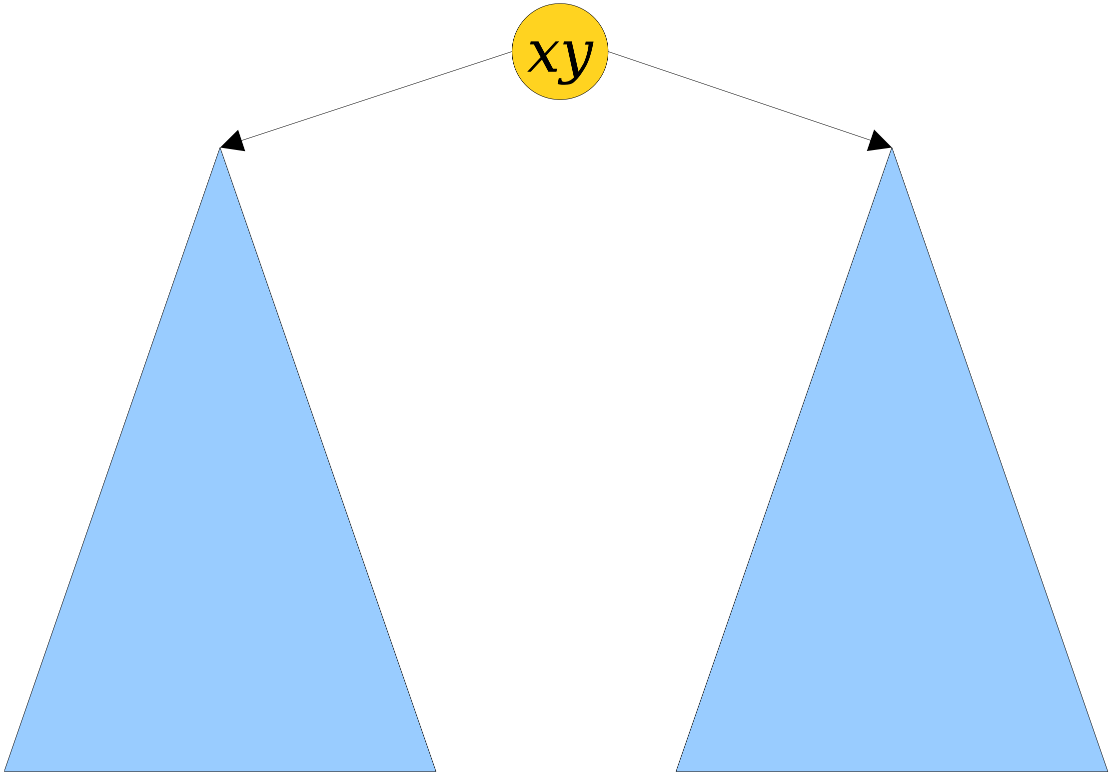
- To ***cut***(x, y):
  - Splay xy.



aa ab … cx xy yy yf … fy yx xt … ba

Euler Tour Trees

- To ***cut***(x, y):
  - Splay xy.
  - Delete xy.



aa ab … cx xy yy yf … fy yx xt … ba

Euler Tour Trees

- To ***cut***(x, y):
  - Splay xy.
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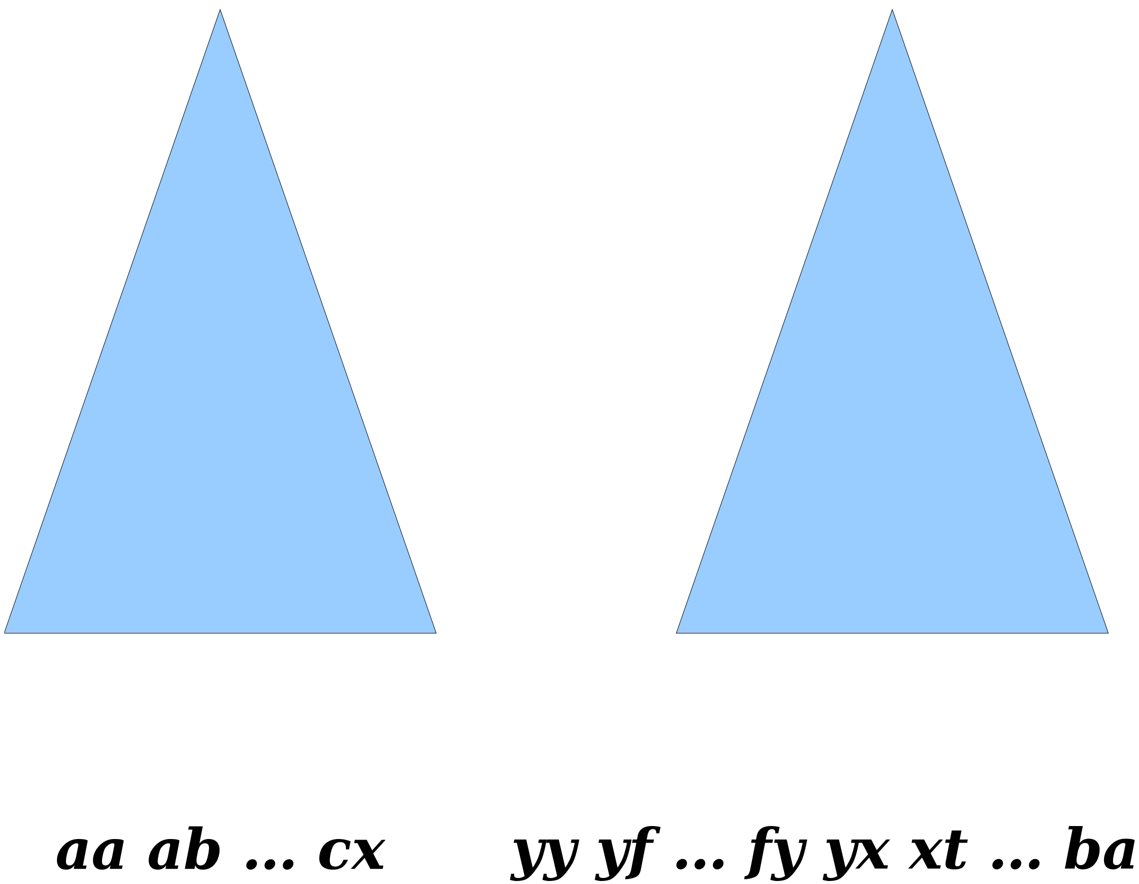

Euler Tour Trees

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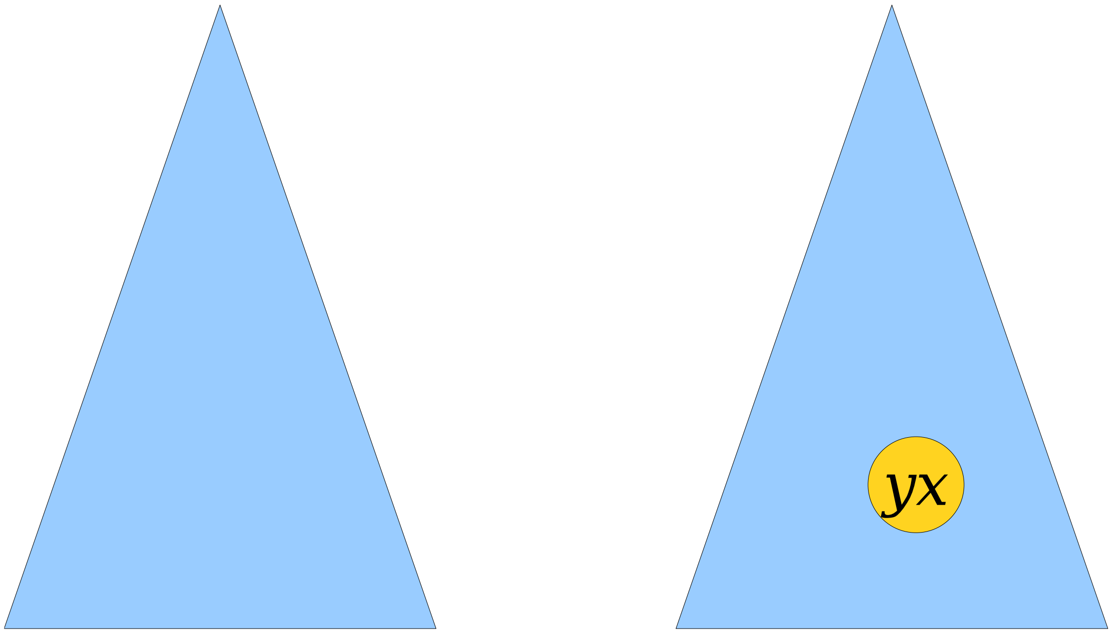

aa ab … cx ***yy yf … fy yx xt … ba***

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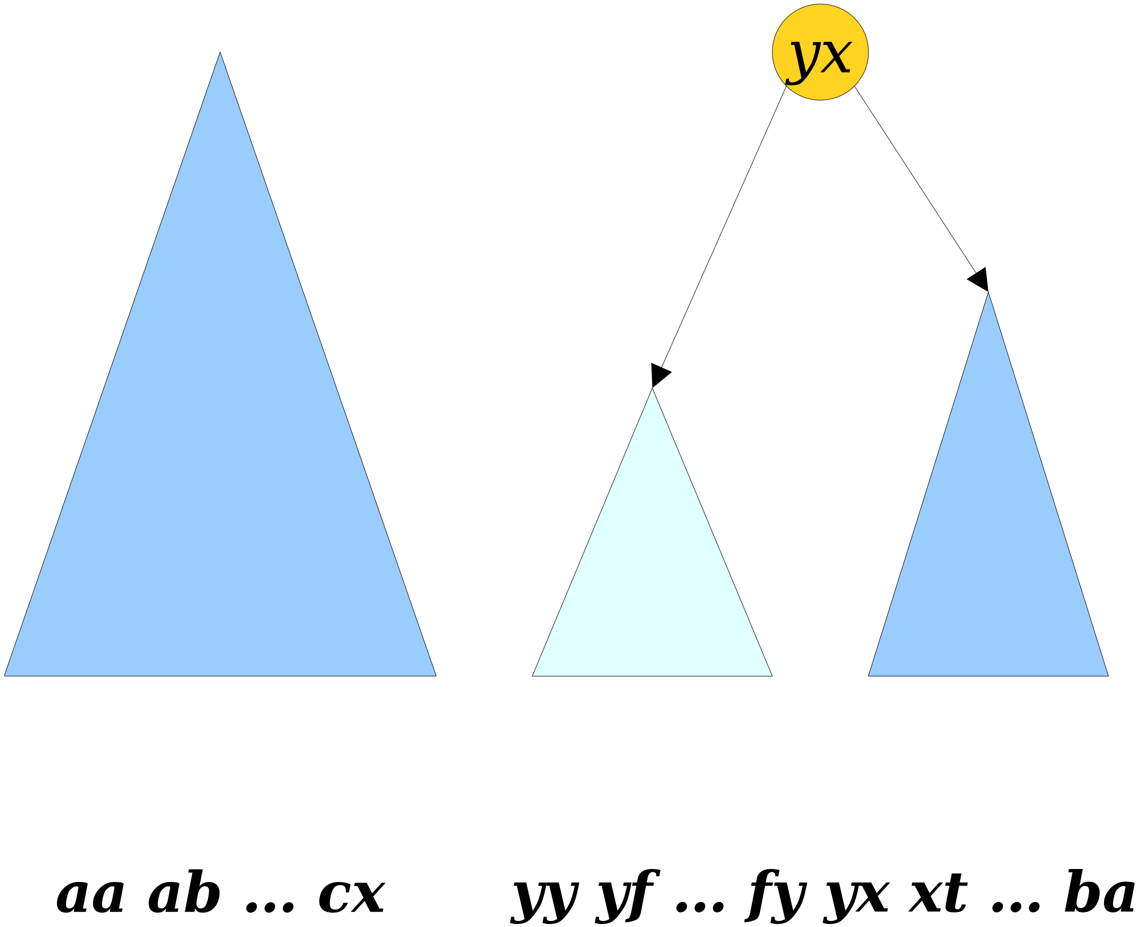

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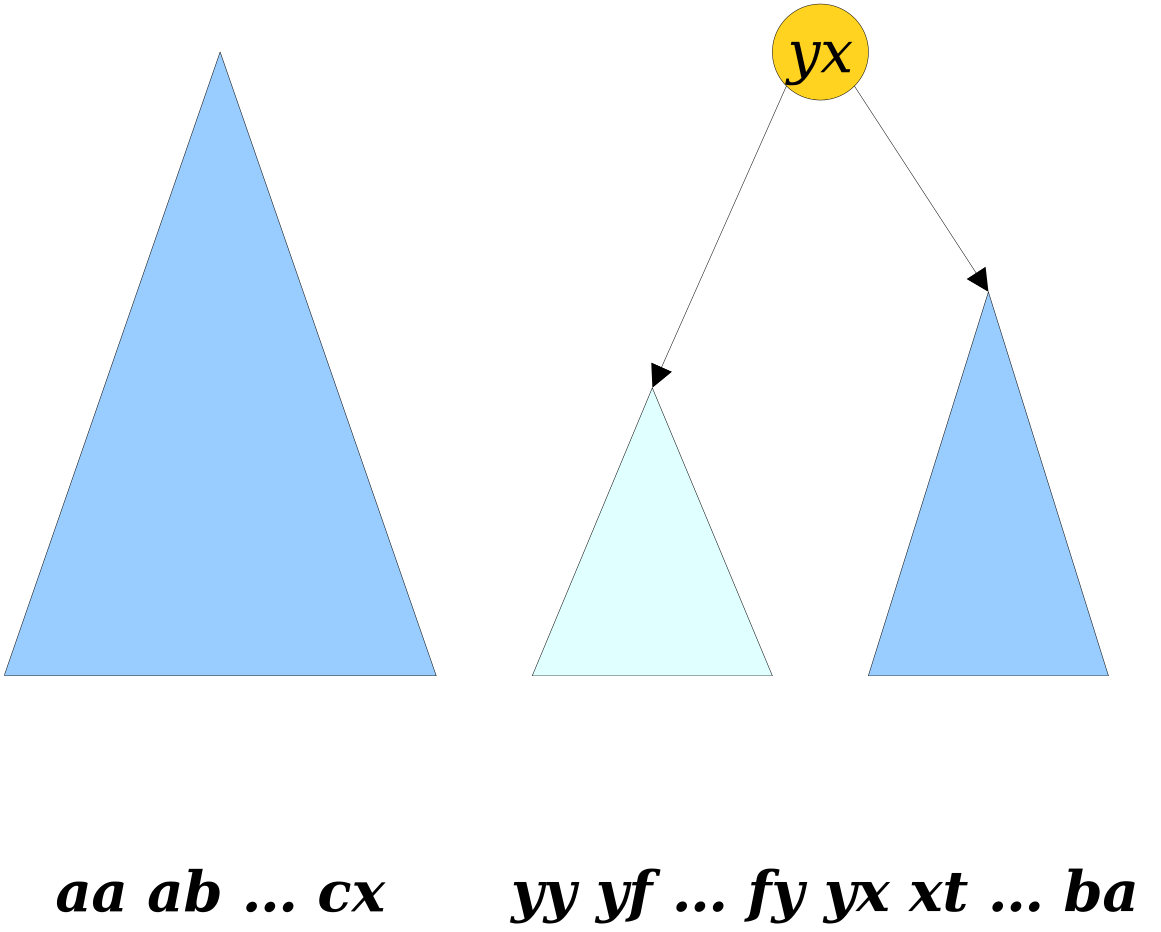

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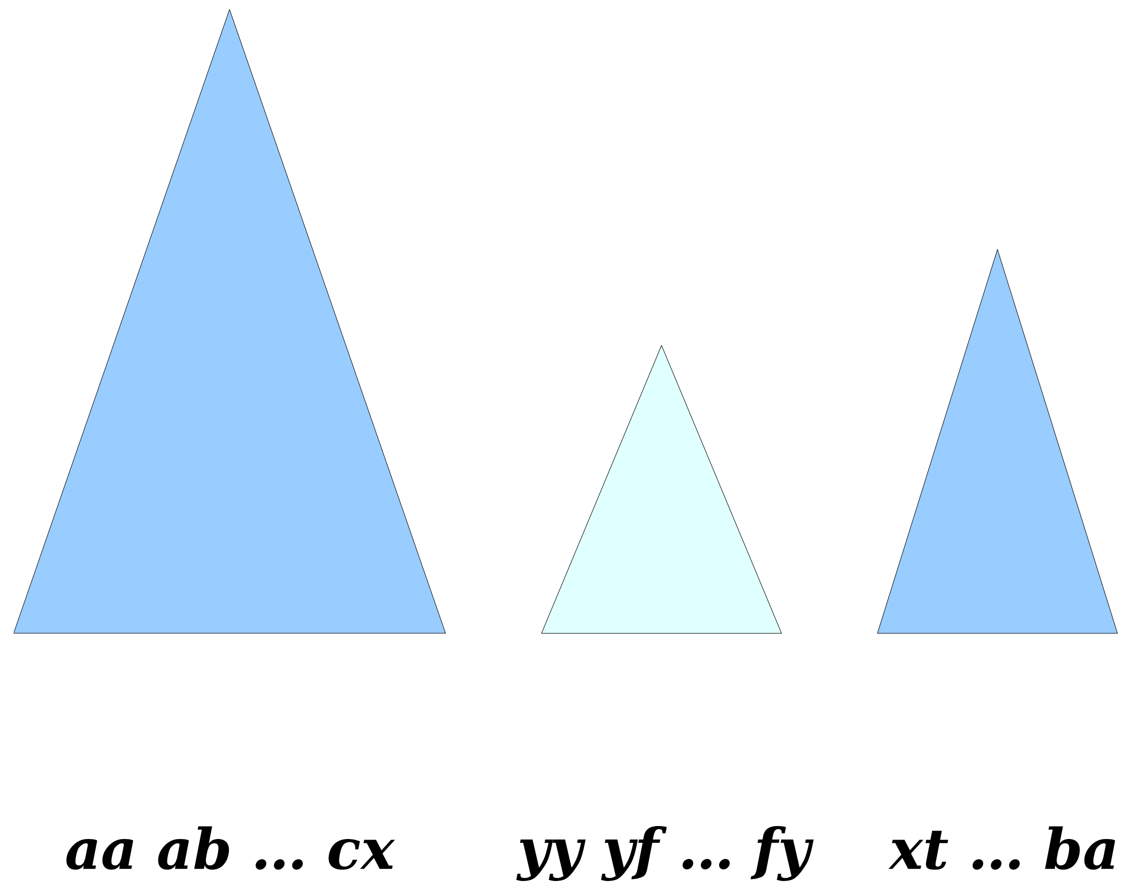

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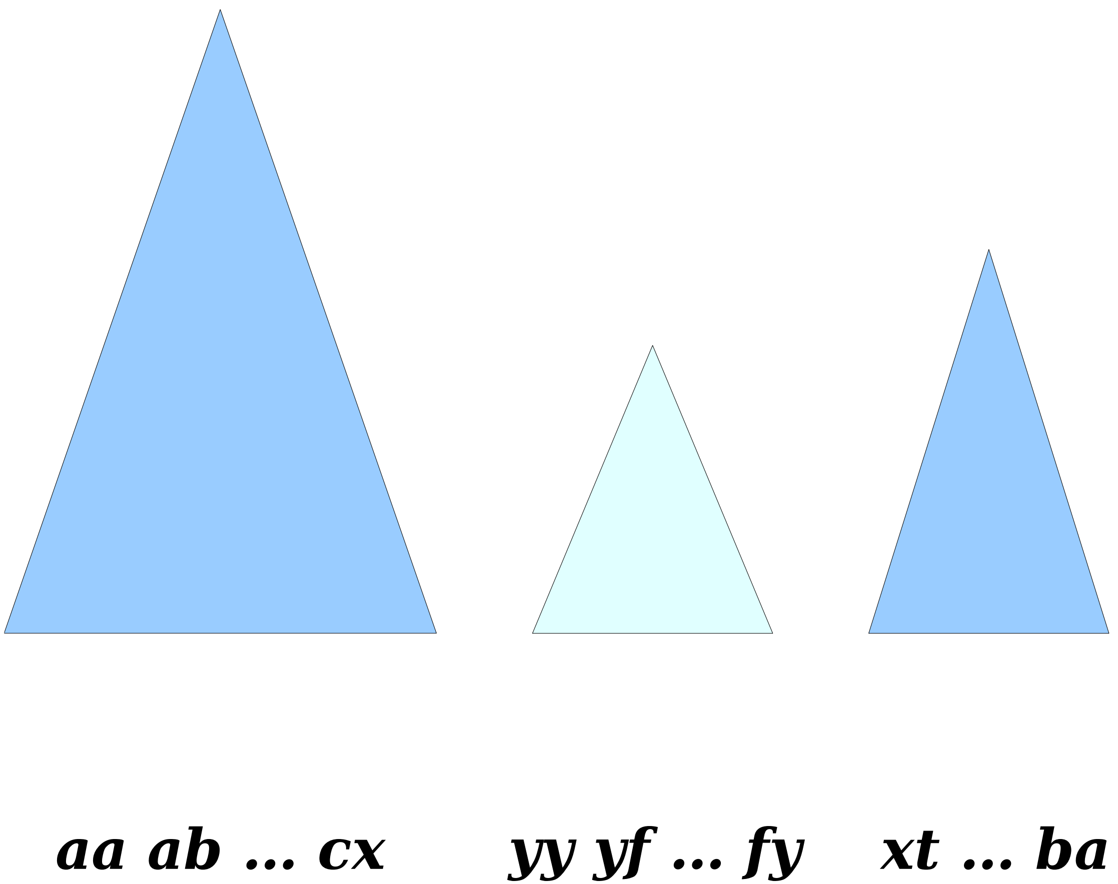

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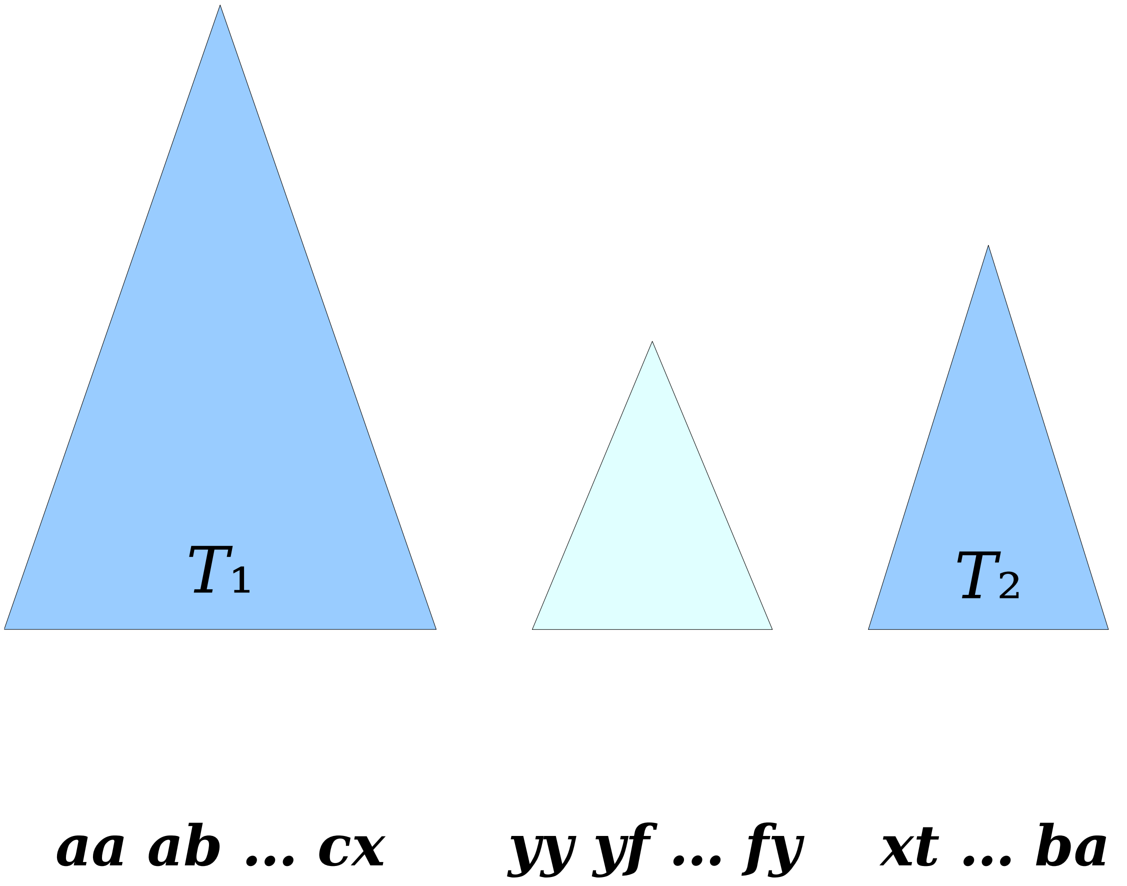

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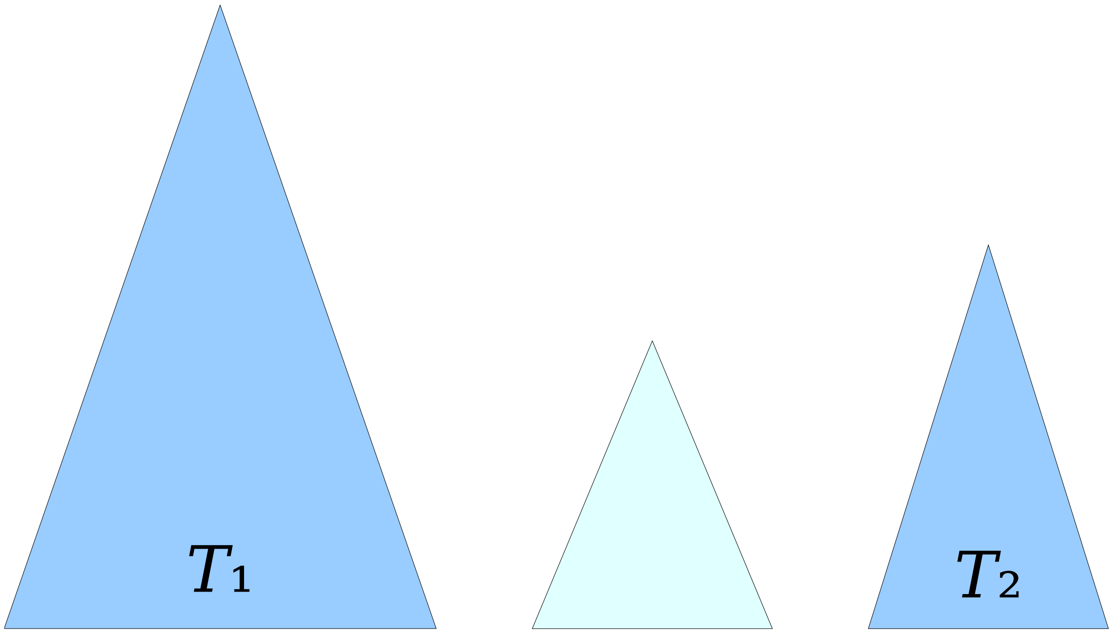

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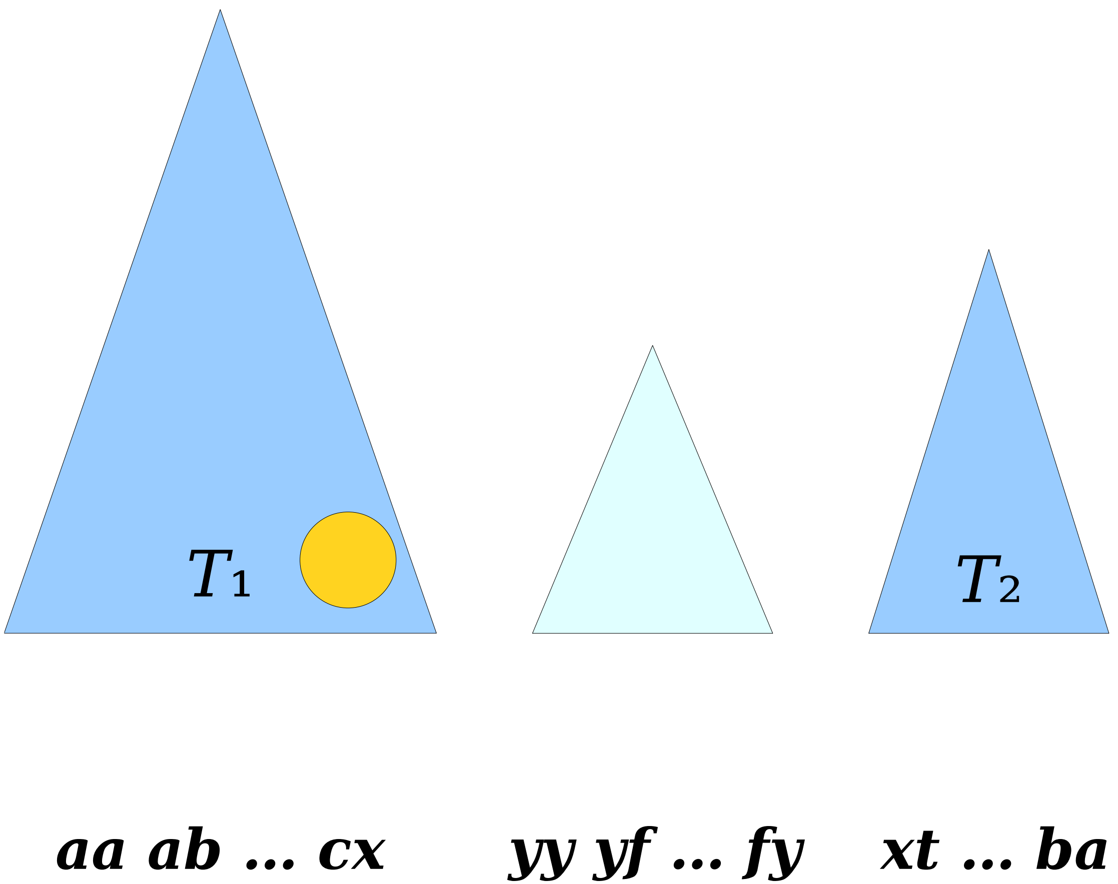

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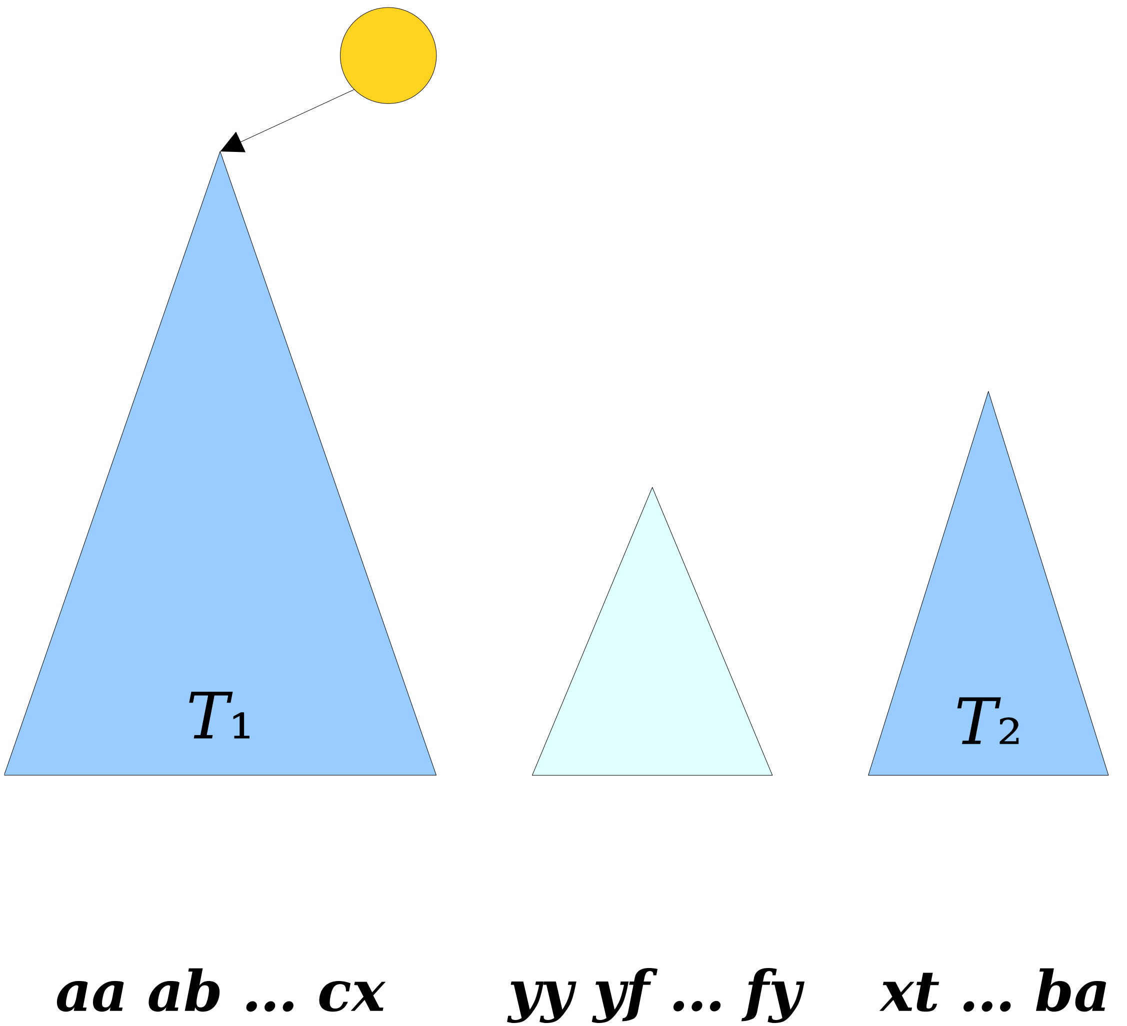

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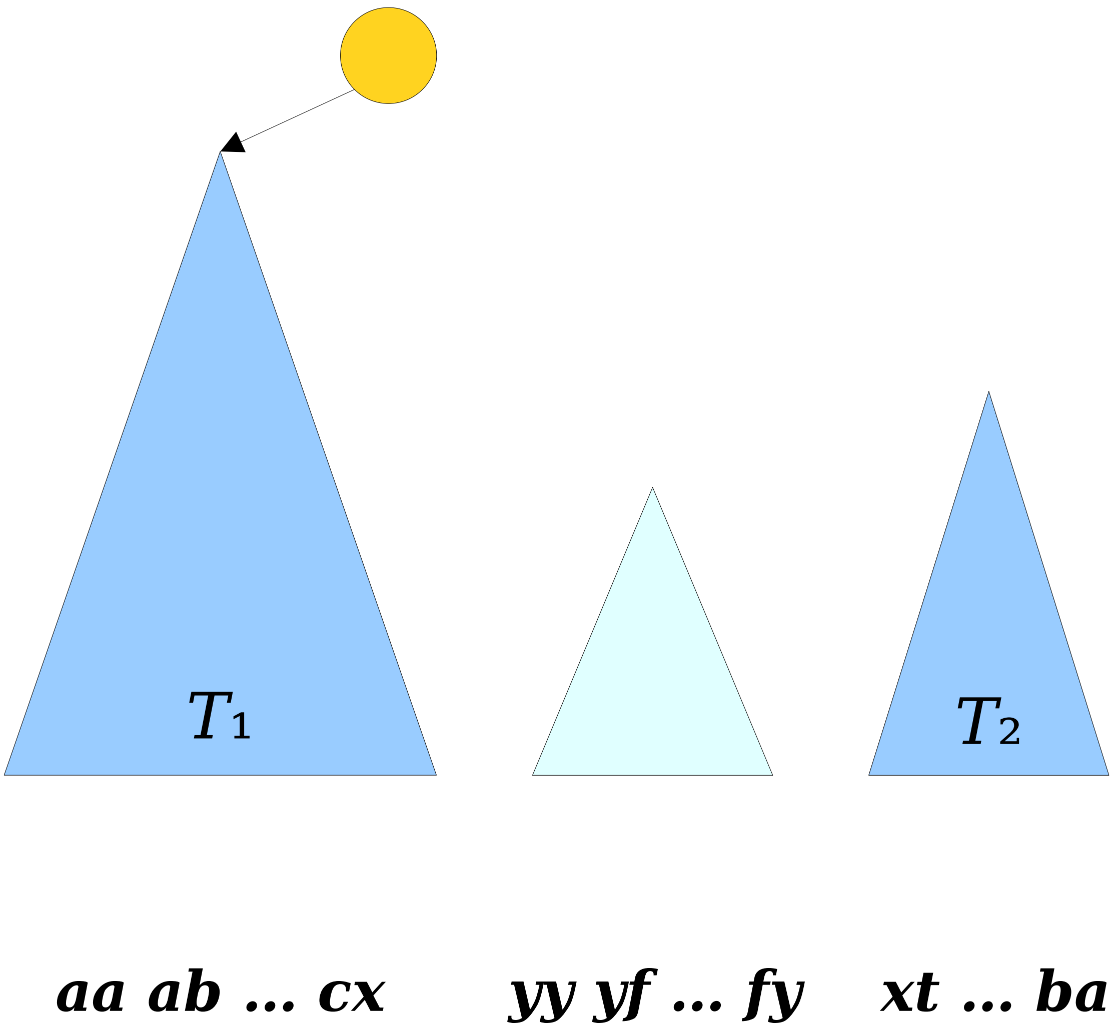

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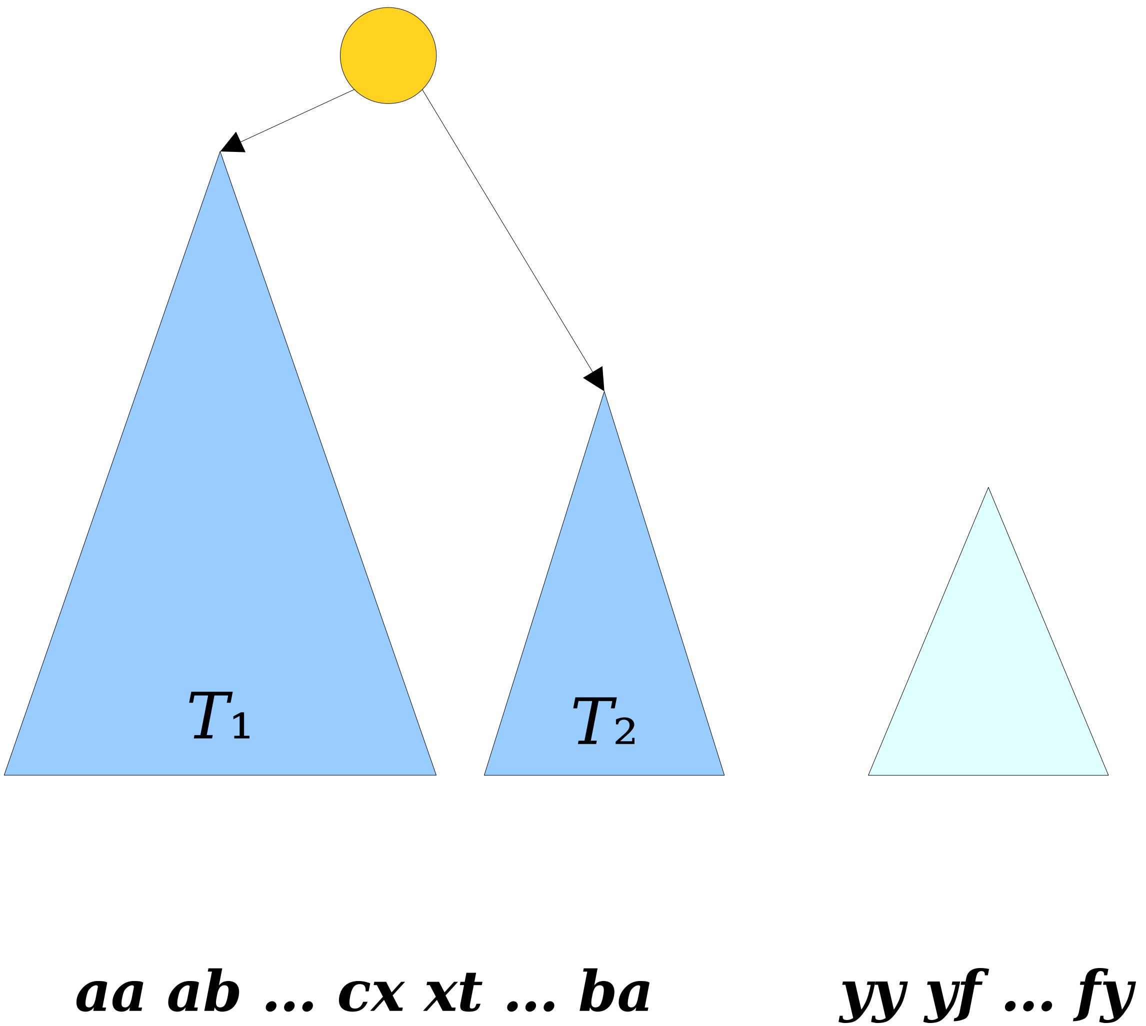

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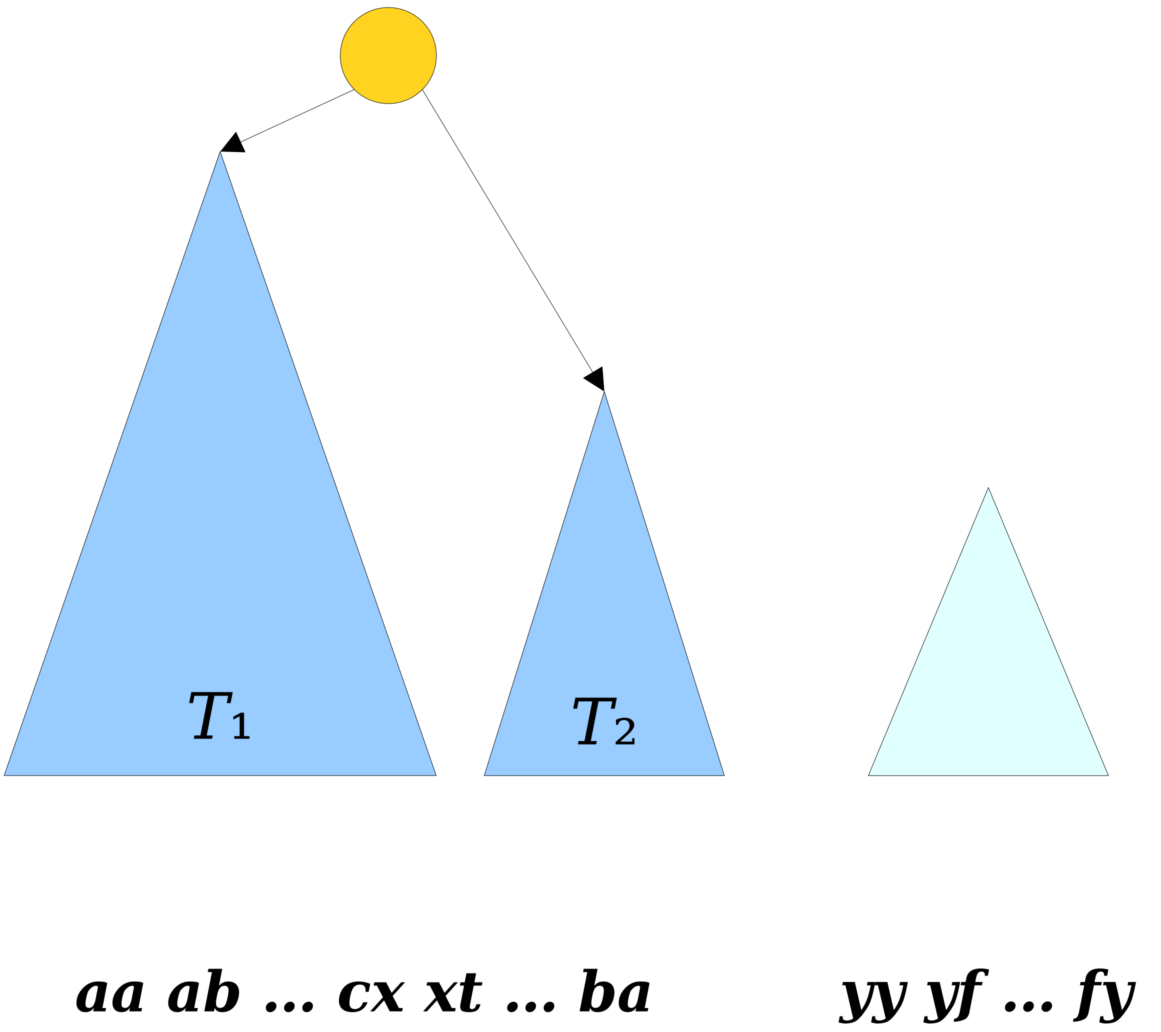
- Amortized cost: **O(log *n*)**.



Euler Tour Trees

- With all things said and done, we get the following amortized runtimes for each operation:
  - ***are-connected***: O(log *n*)
  - ***link***: O(log *n*)
  - ***cut***: O(log *n*)
- These bounds can be made worst-case efficient using different types of balanced BSTs instead of splay trees, but splaying is probably the fastest way to do this.

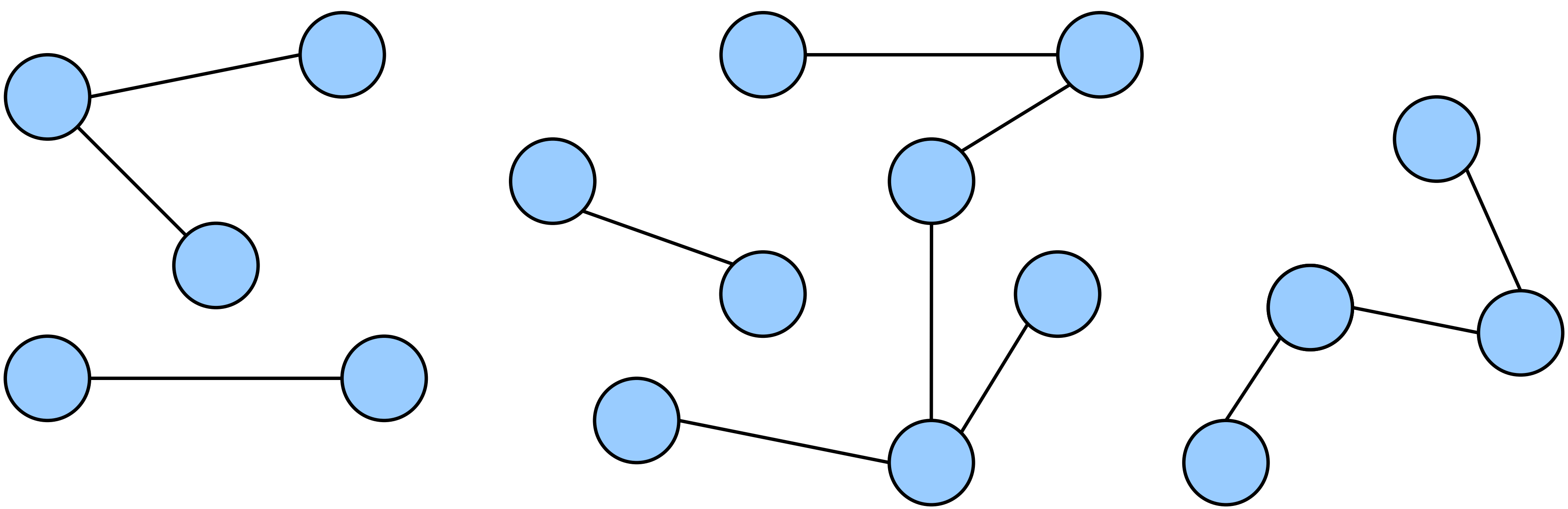
Extending Euler Tour Trees

Extending Euler Tour Trees

- We now have a (relatively) simple and fast data structure for solving dynamic connectivity in forests.
- What else can we do with them?

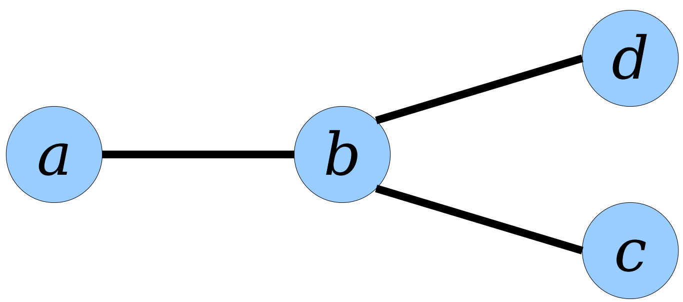
Extending Euler Tour Trees

- Suppose we want to add an operation ***size***(x) that returns the number of nodes in the tree containing x.
- How might we accomplish this?



Tree Sizes

- We can determine ***size***(x) as follows:
  - Figure out which Euler tour xx is in.
  - Count how many nodes of the form zz it contains.
- A naive implementation of this algorithm might take time $\Theta(n)$ if all nodes are in the same tree. Can we do better?



aa ab bb bc cc cb bd dd db ba

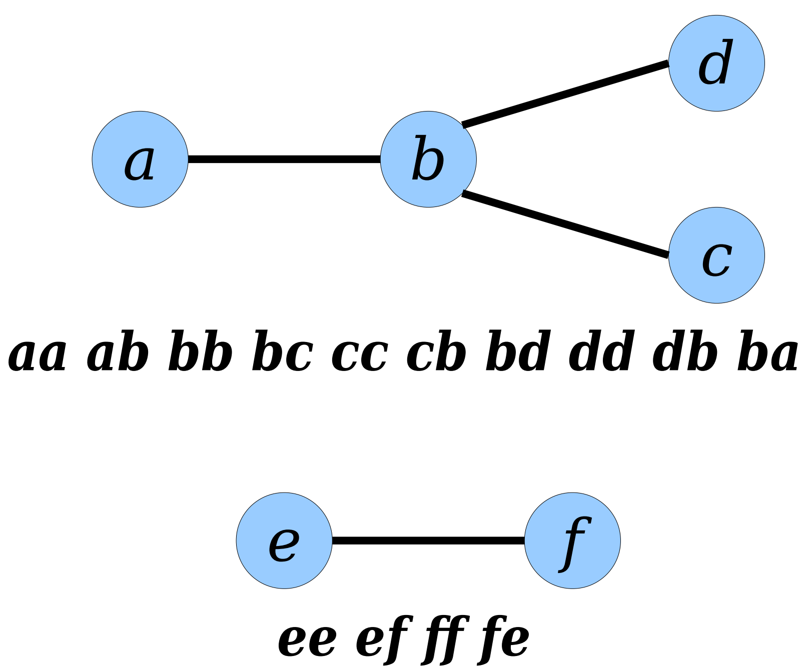
ee ef ff fe

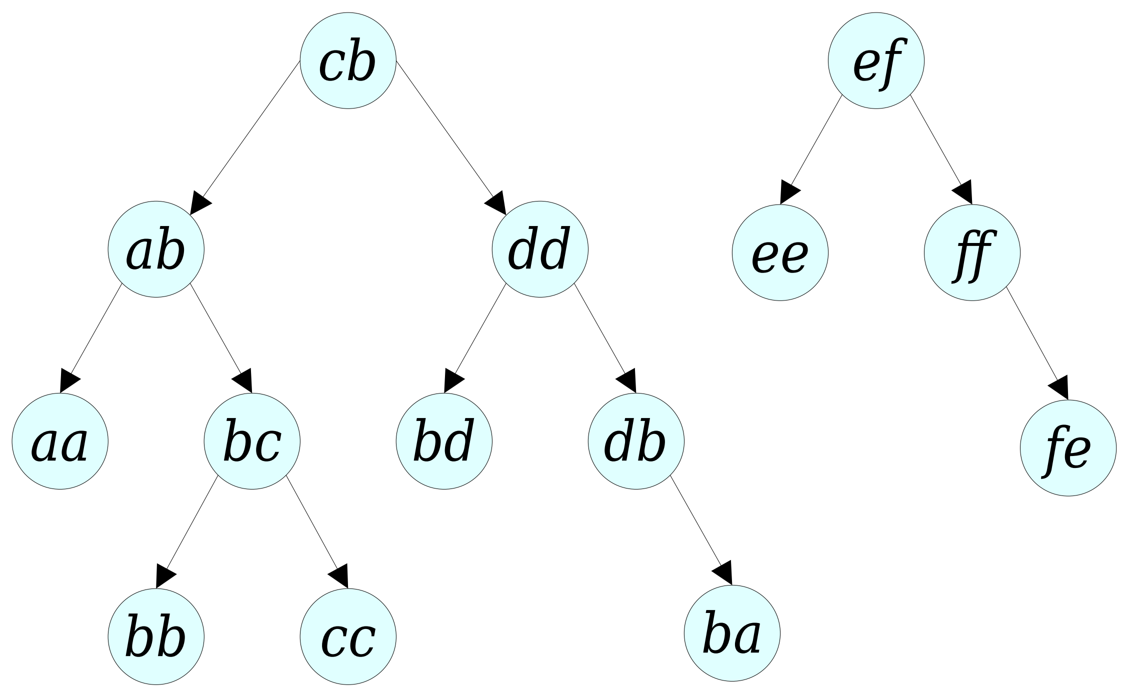
Tree Sizes

- We’re storing our Euler tours in balanced BSTs.
- We want to be able to answer the following question about a given BST:

How many nodes of the form xx are in this BST?

- This can be done in time O(log n). How?





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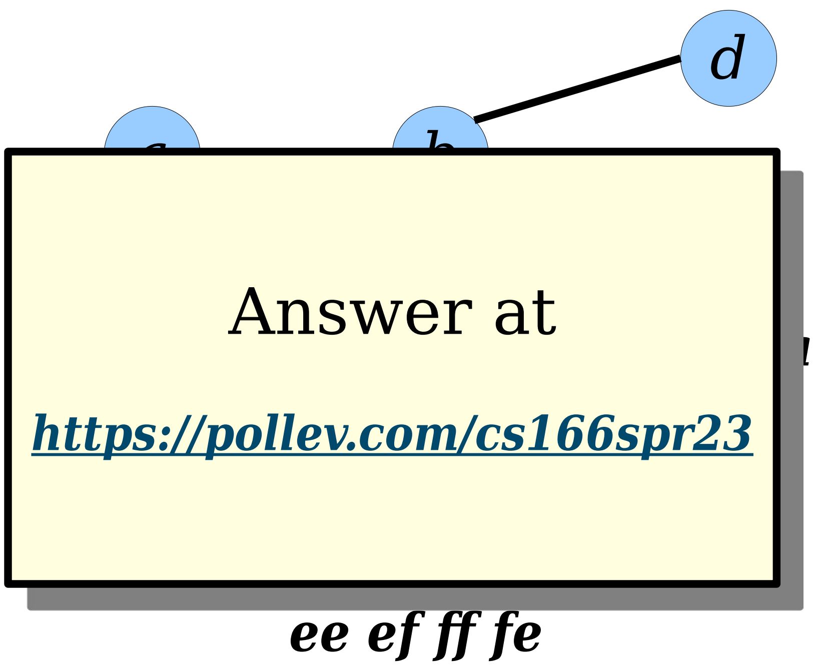
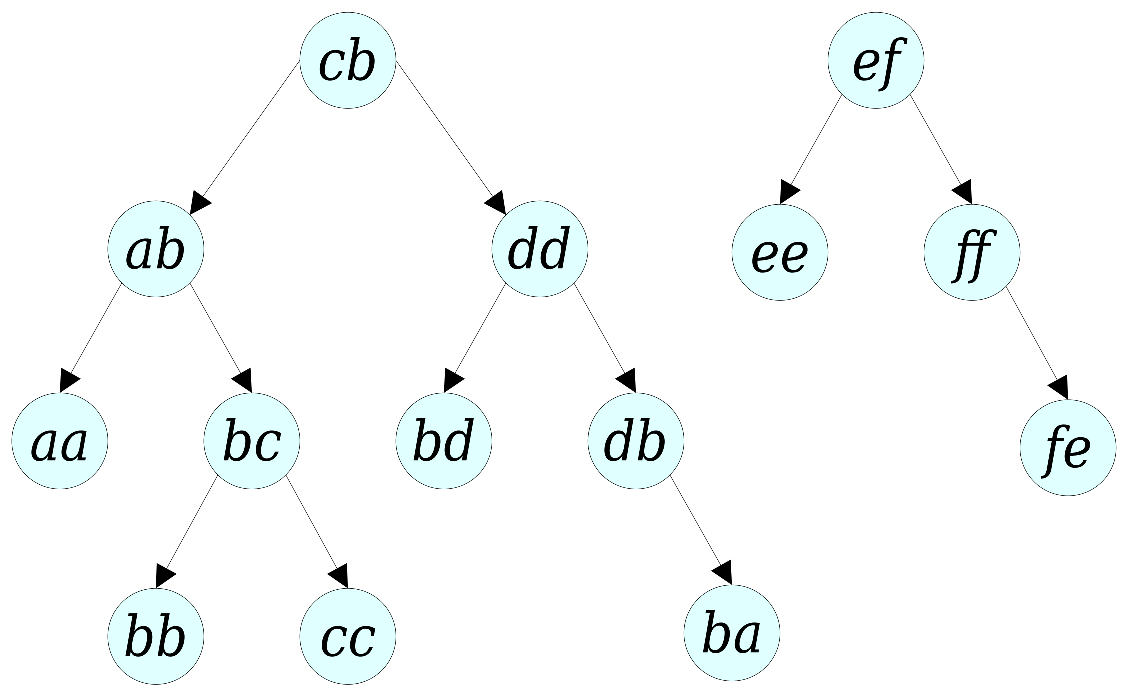
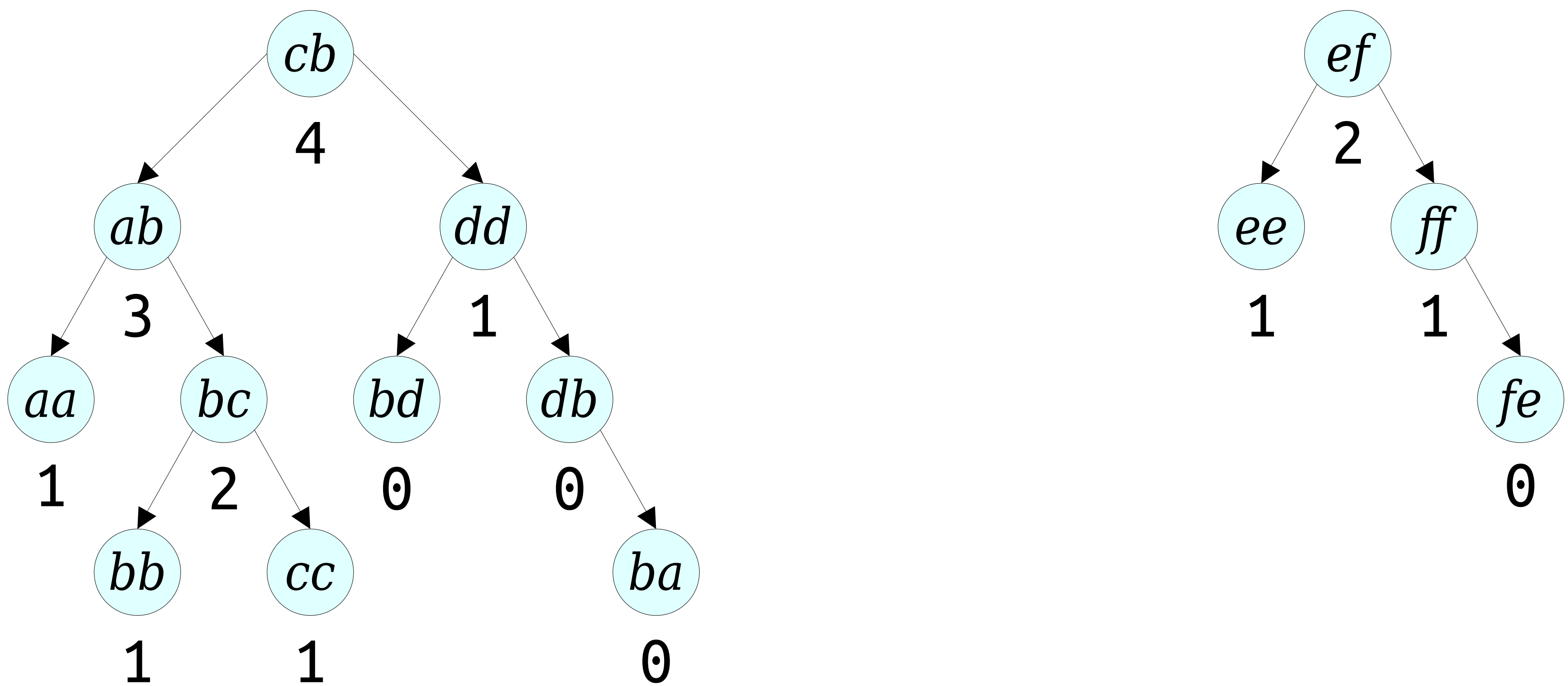




Tree Sizes

- ***Idea:*** Augment the BSTs holding our Euler tours.
- Specifically, each node stores the number of self-loops at or below it in the tree.
- This information can be maintained through rotations and after each splay tree operation.



Tree Sizes

- To determine ***size***(x):

Tree Sizes

- To determine ***size***(x):
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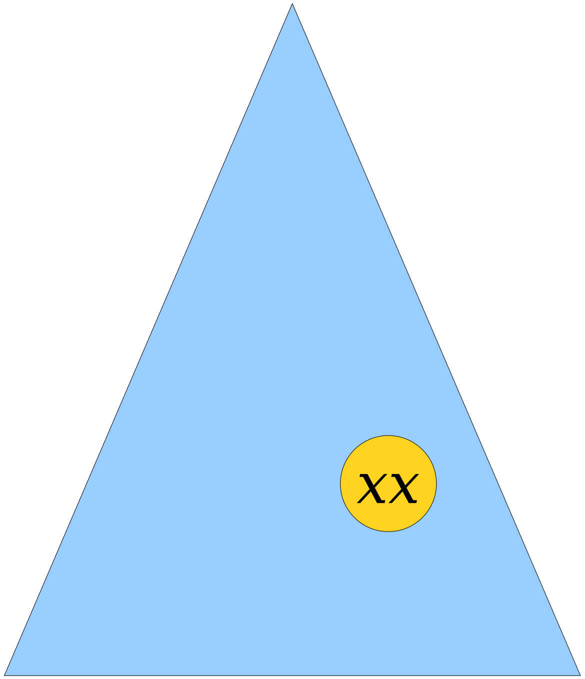

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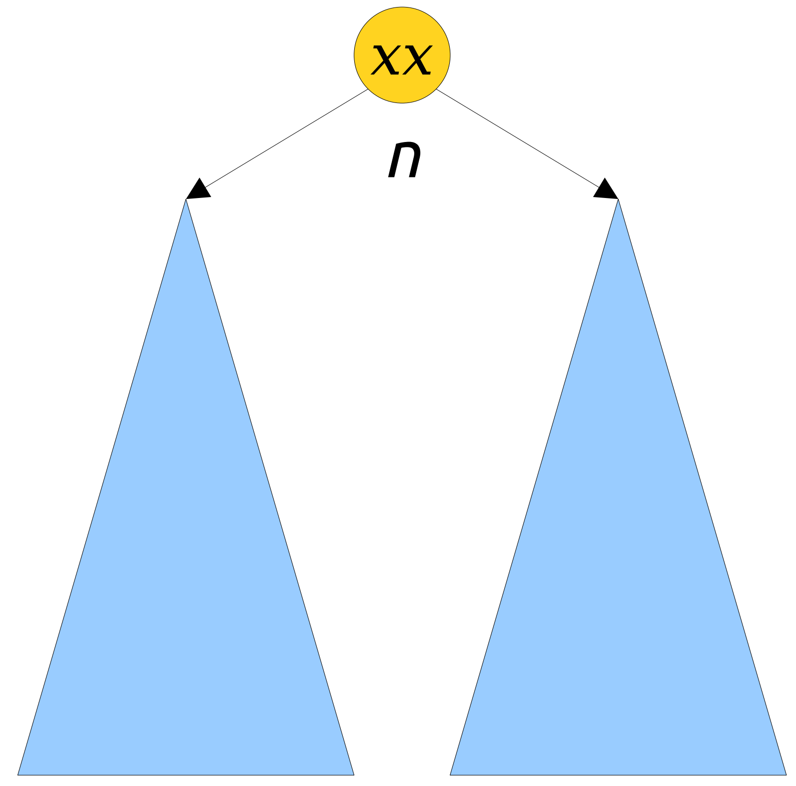

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- To determine ***size***(x):
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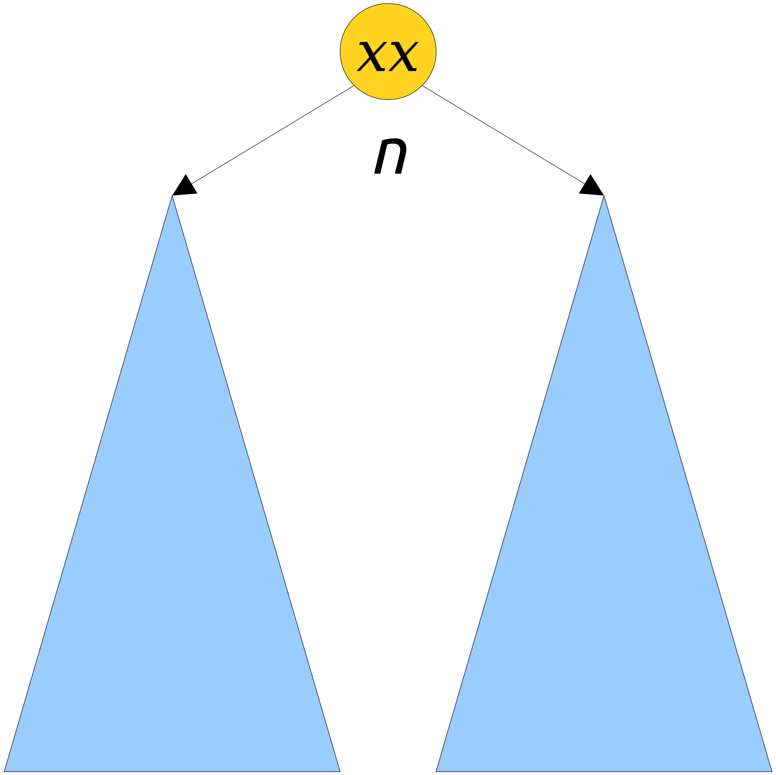

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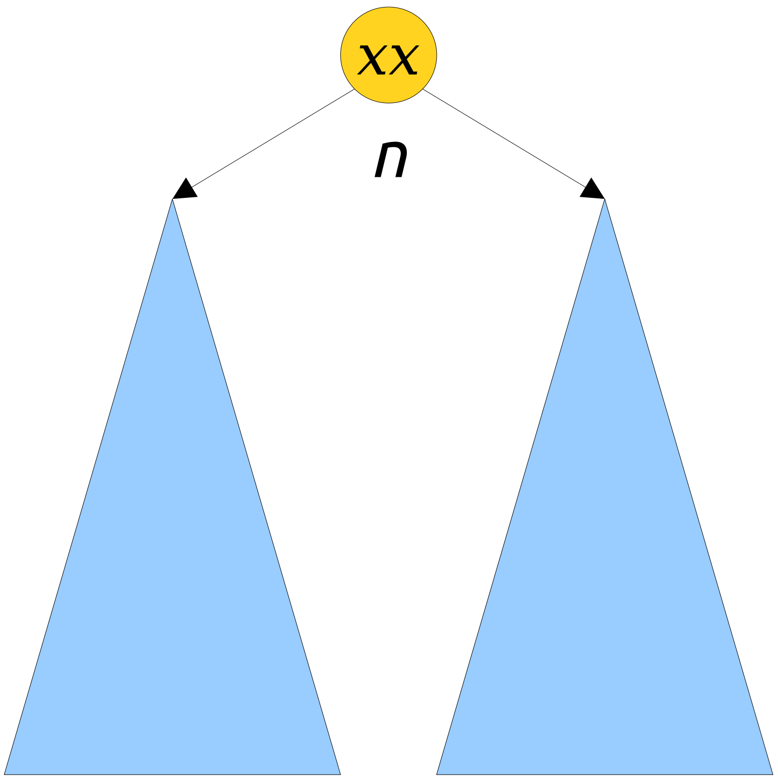

Extending Euler Tour Trees

- Suppose that each node represents a network router.
- We want to add these two operations:
  - ***add-packet***(x, p), which attaches packet p to node x; and
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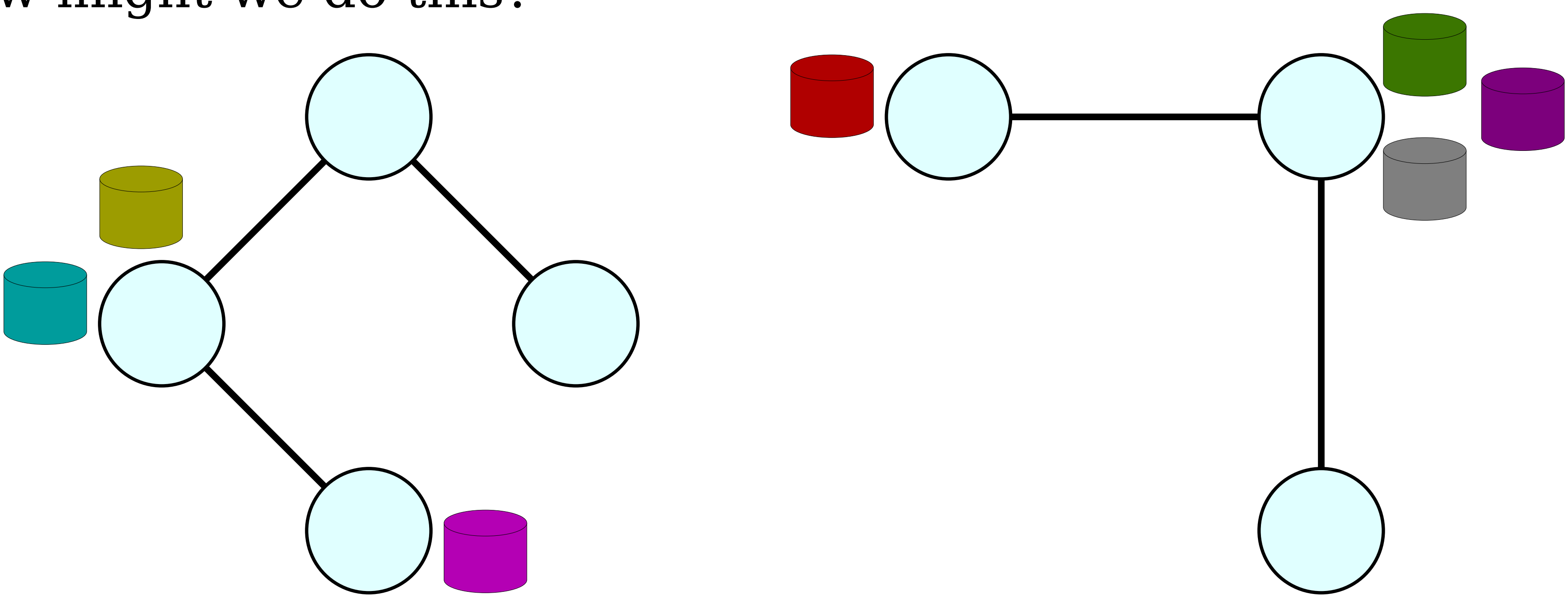
- How might we do this?



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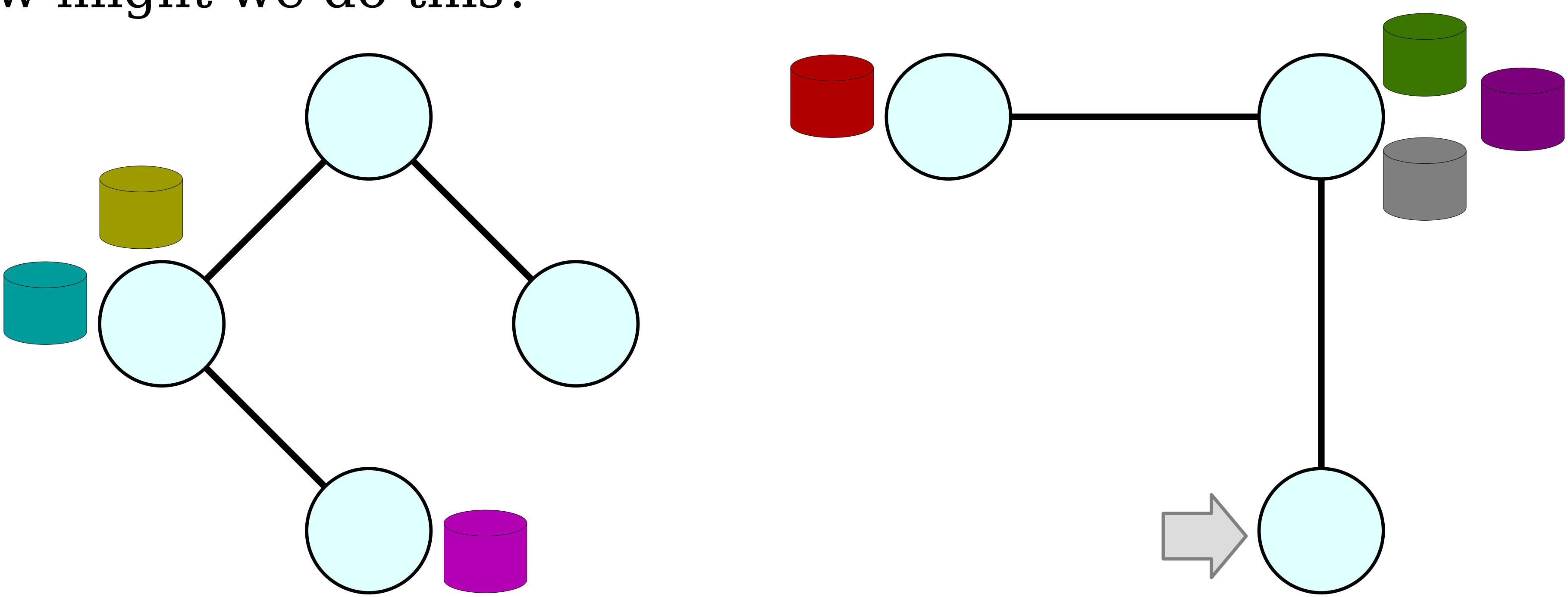

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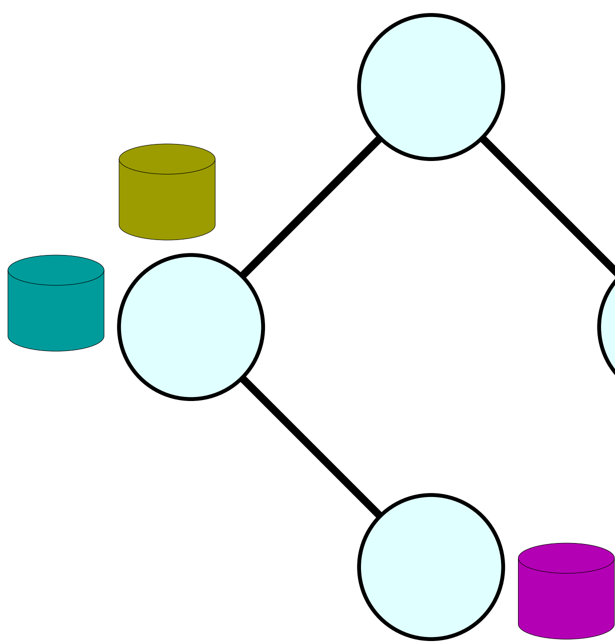

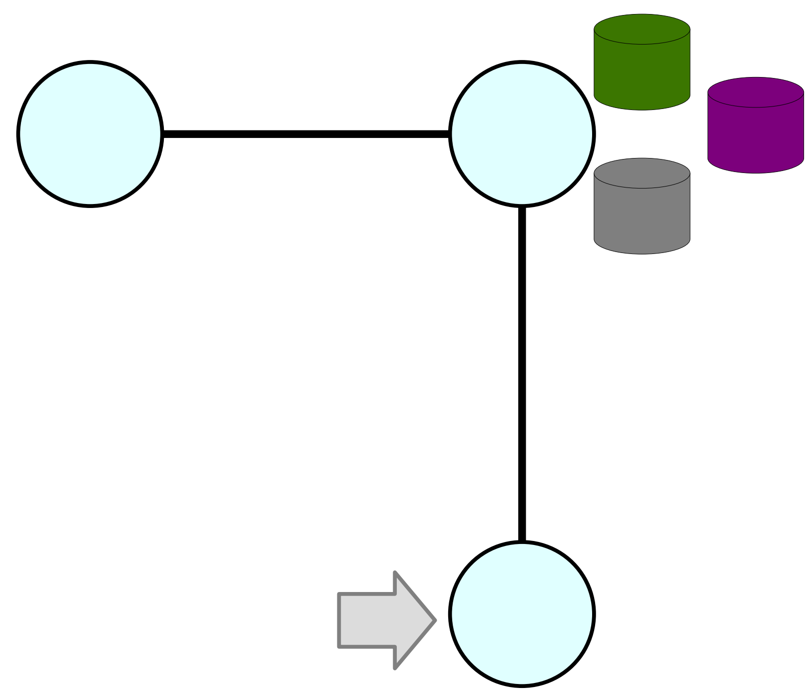

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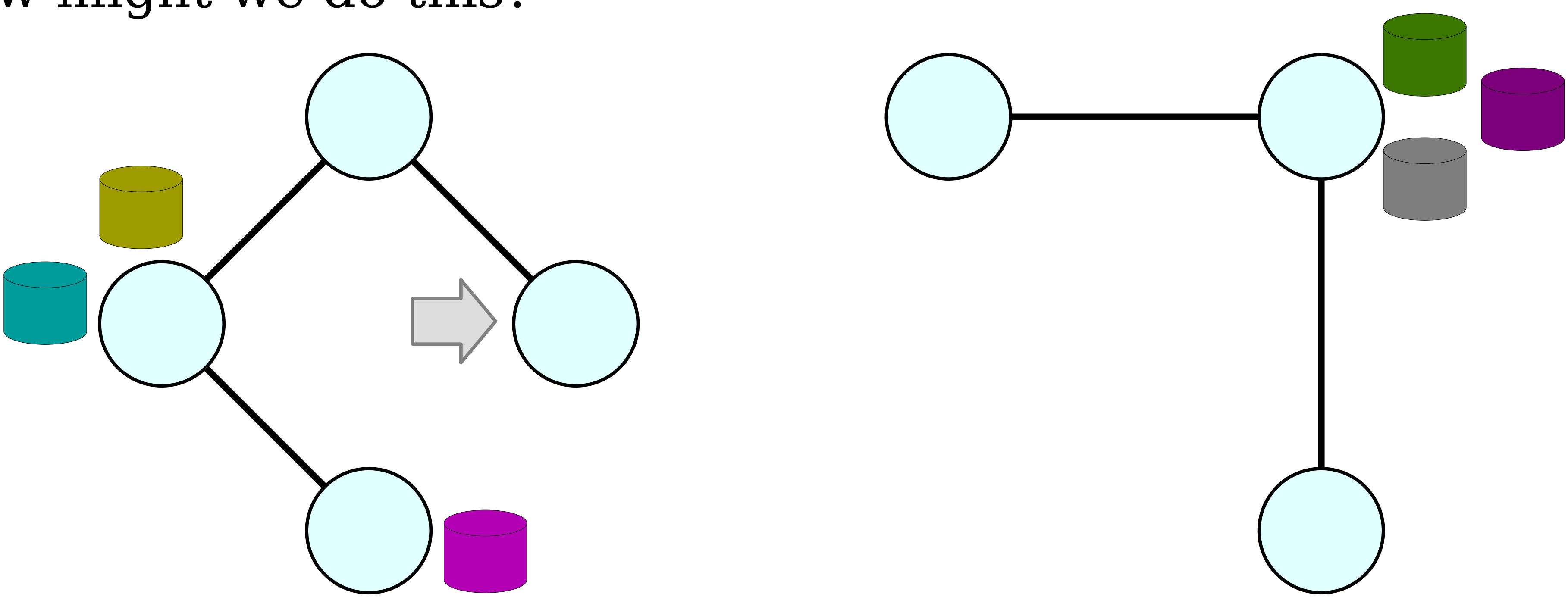

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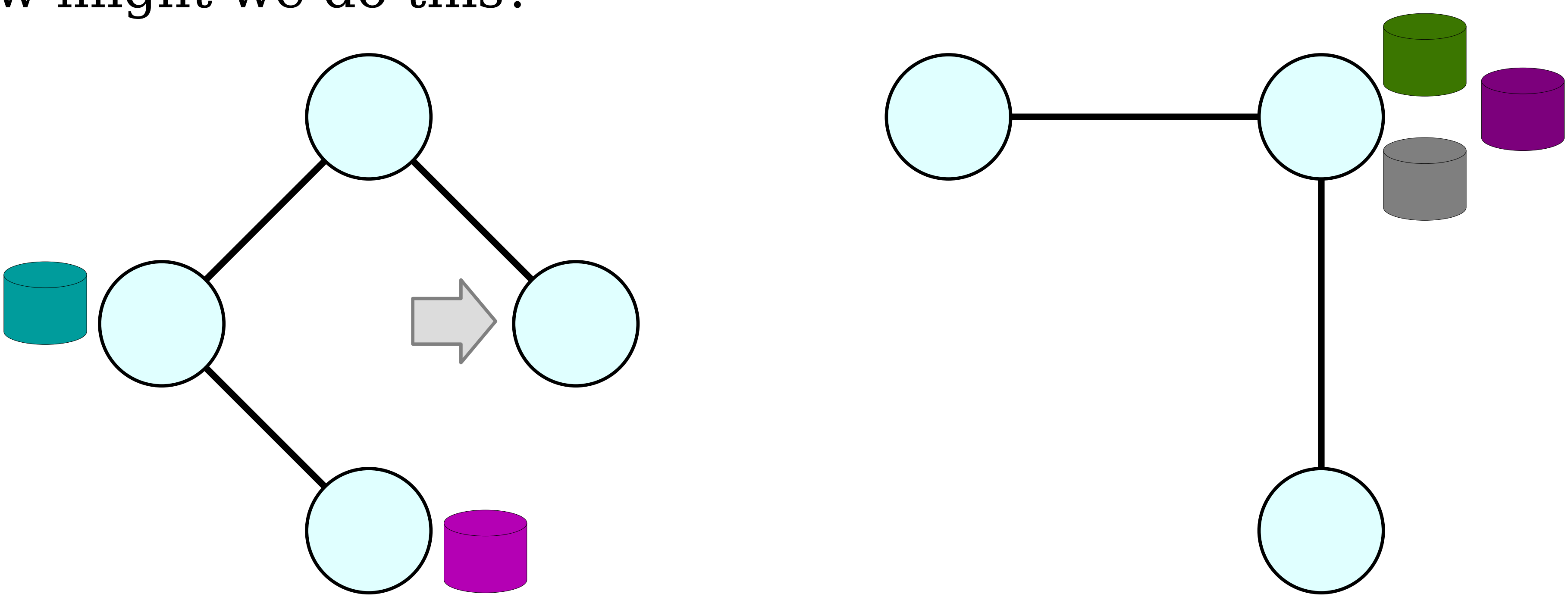

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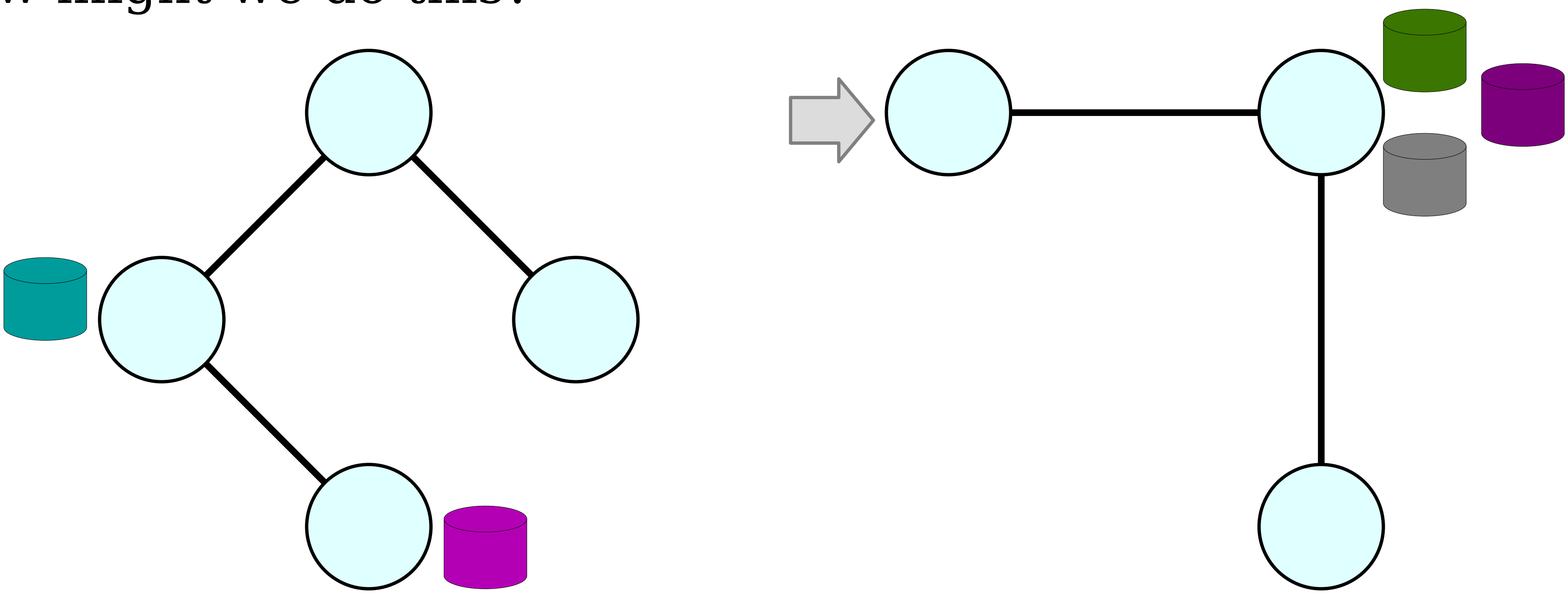

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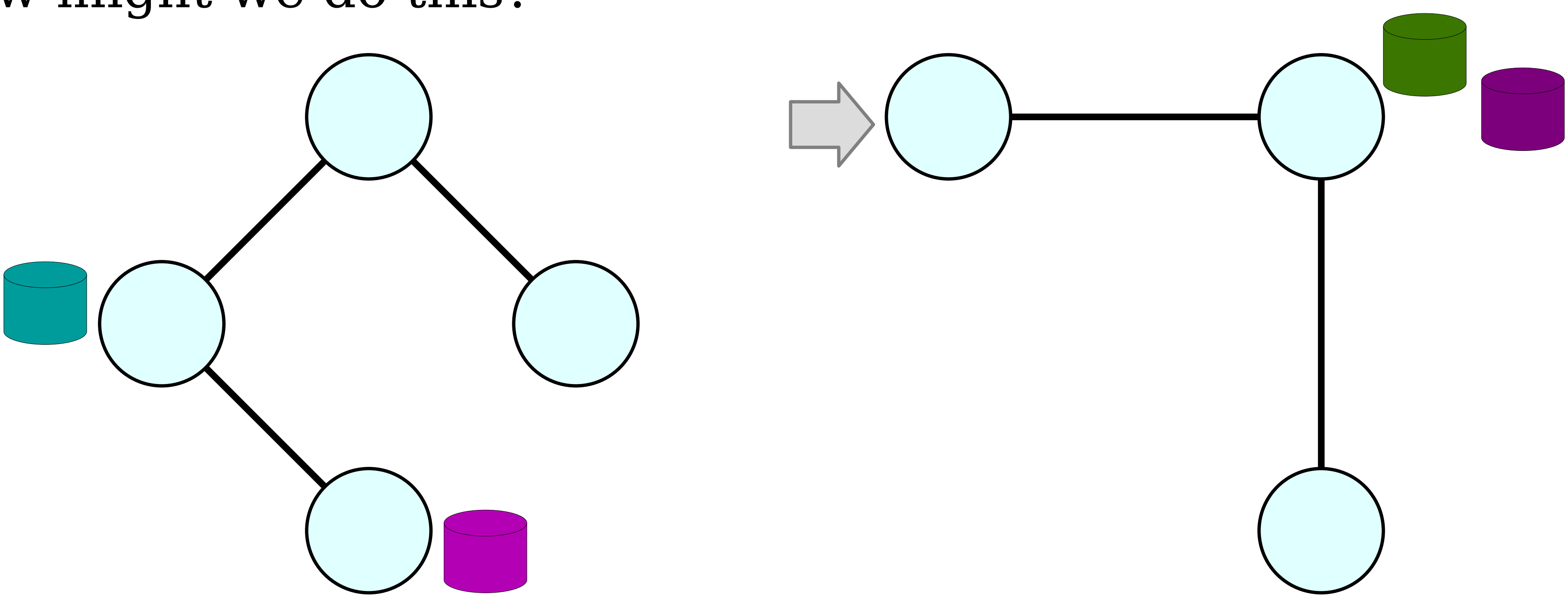

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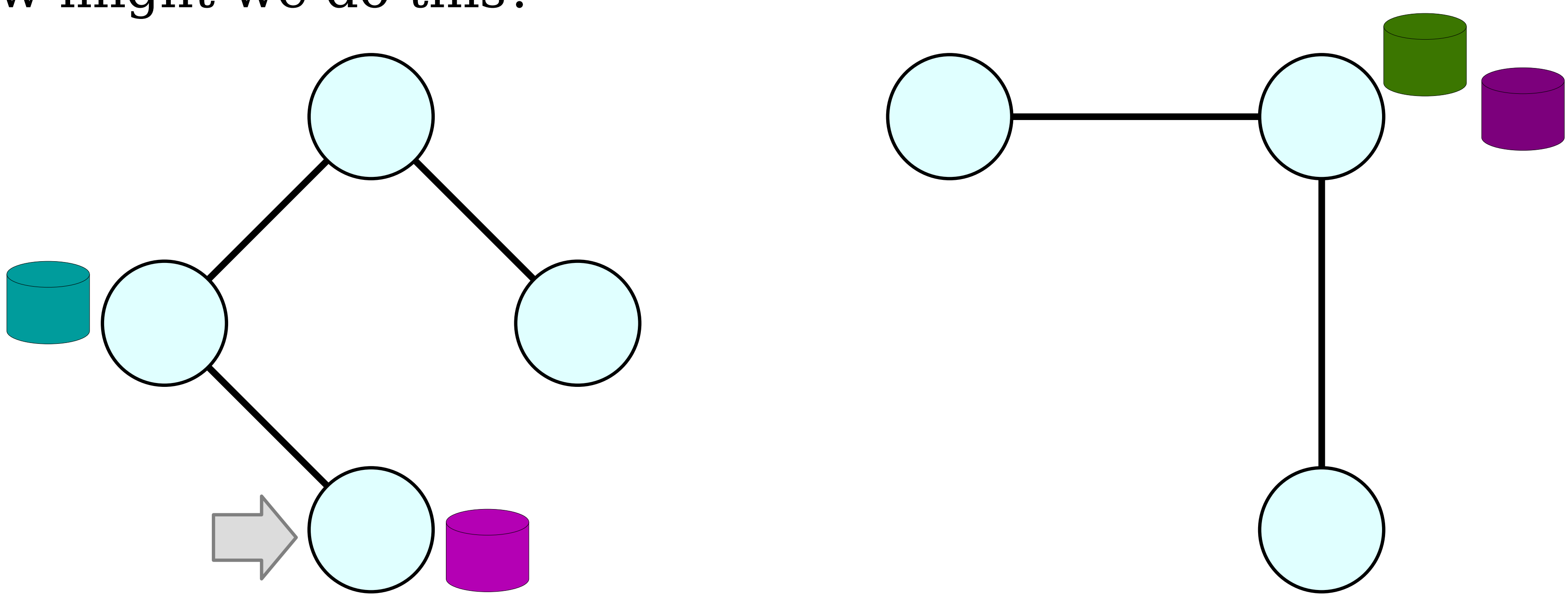

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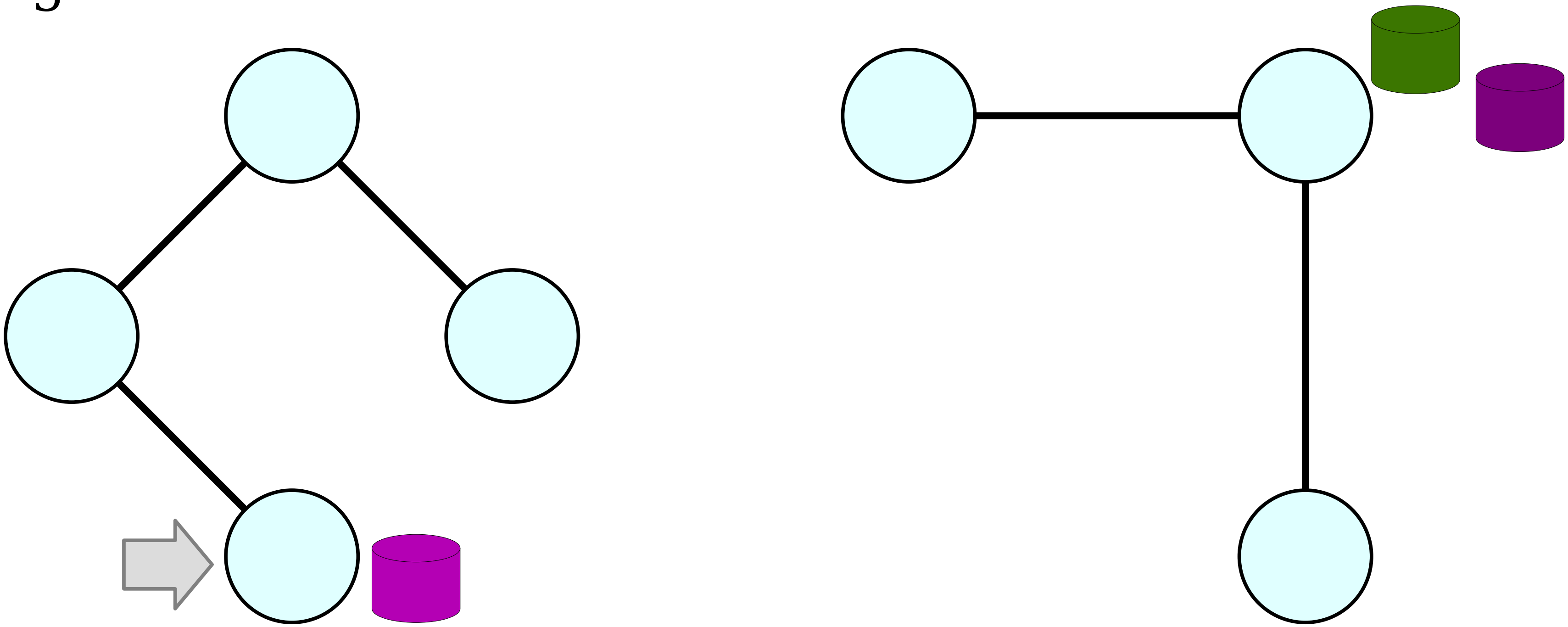


Packet Finding

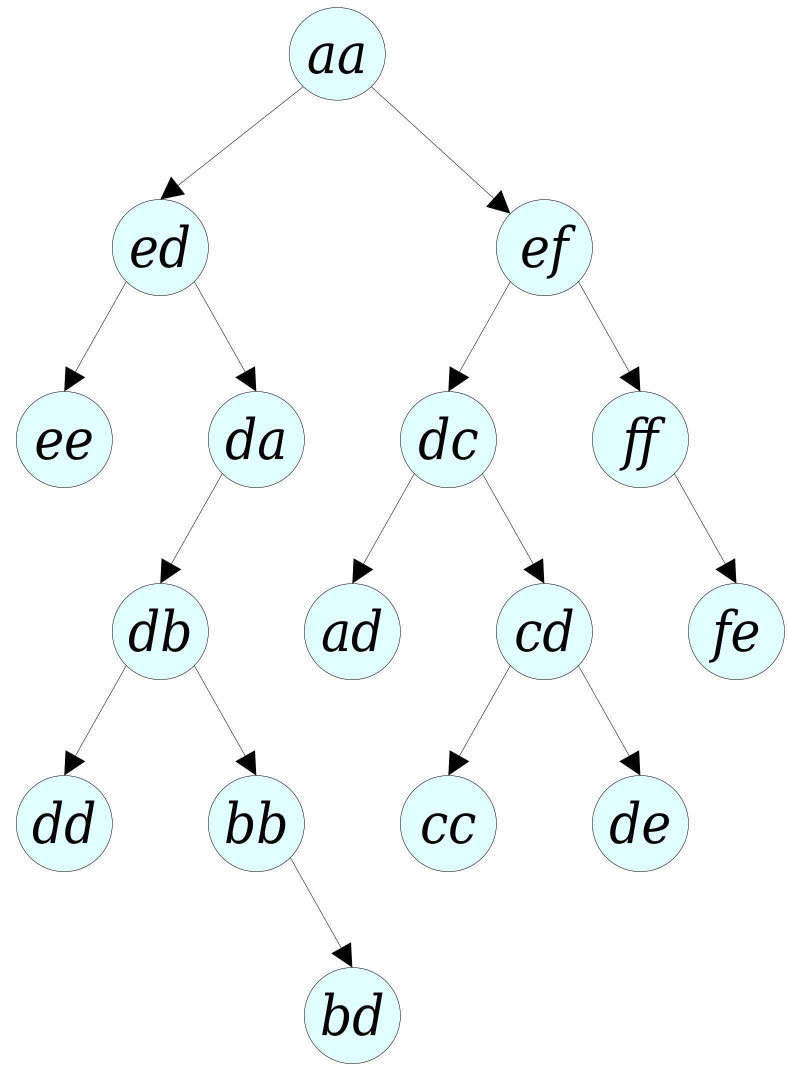
- Given the Euler tour representation of our trees, this essentially boils down to the following:

Augment a BST containing nodes and edges so that we can quickly identify a node with a packet.

- How might we do this?

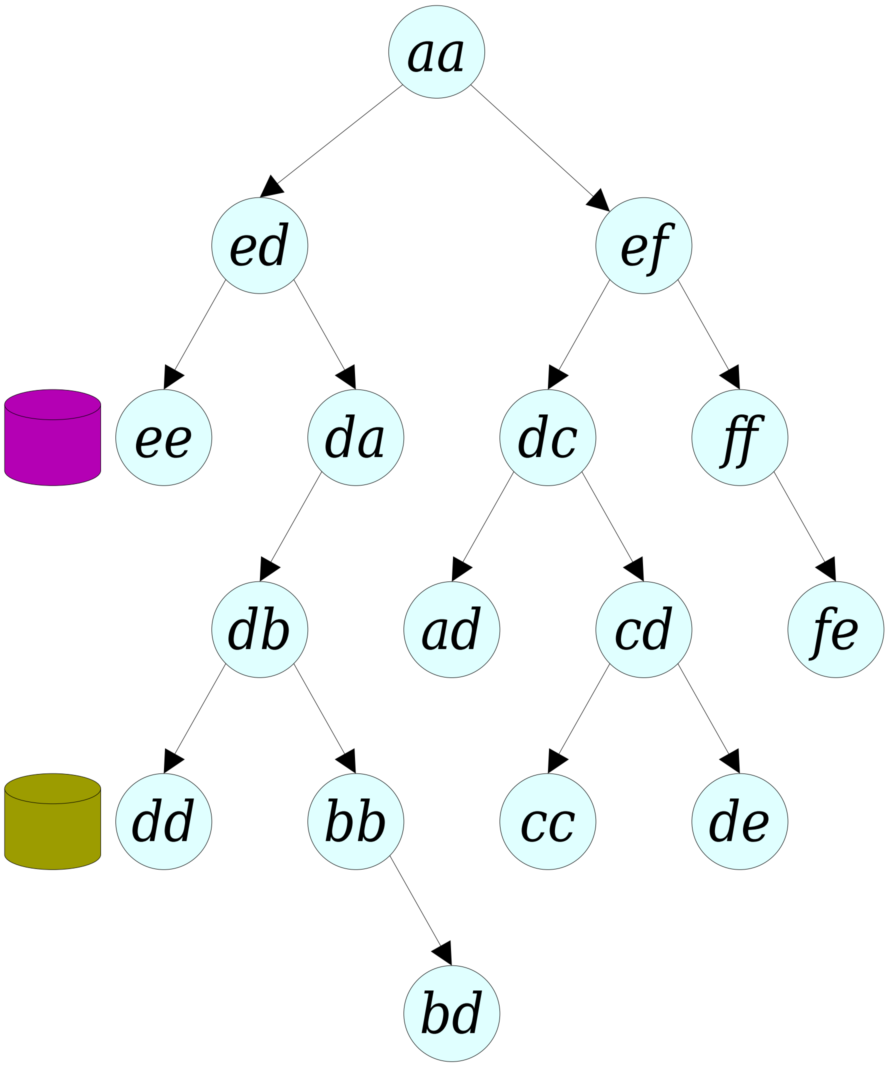
Packet Finding

- Augment each node *xx* with a list of the packets it stores.



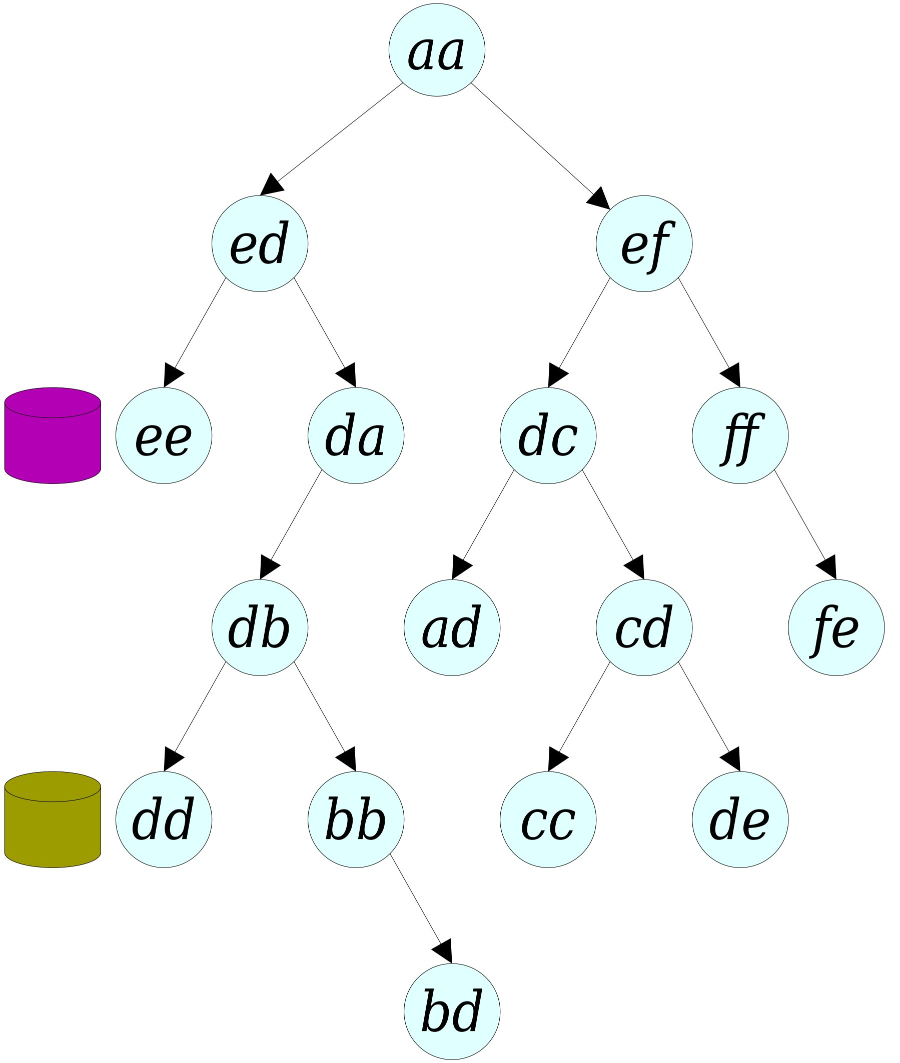
Packet Finding

- Augment each node *xx* with a list of the packets it stores.



Packet Finding

- Augment each node *xx* with a list of the packets it stores.
- Augment each tree node with a bit indicating whether there's a packet in its subtree.



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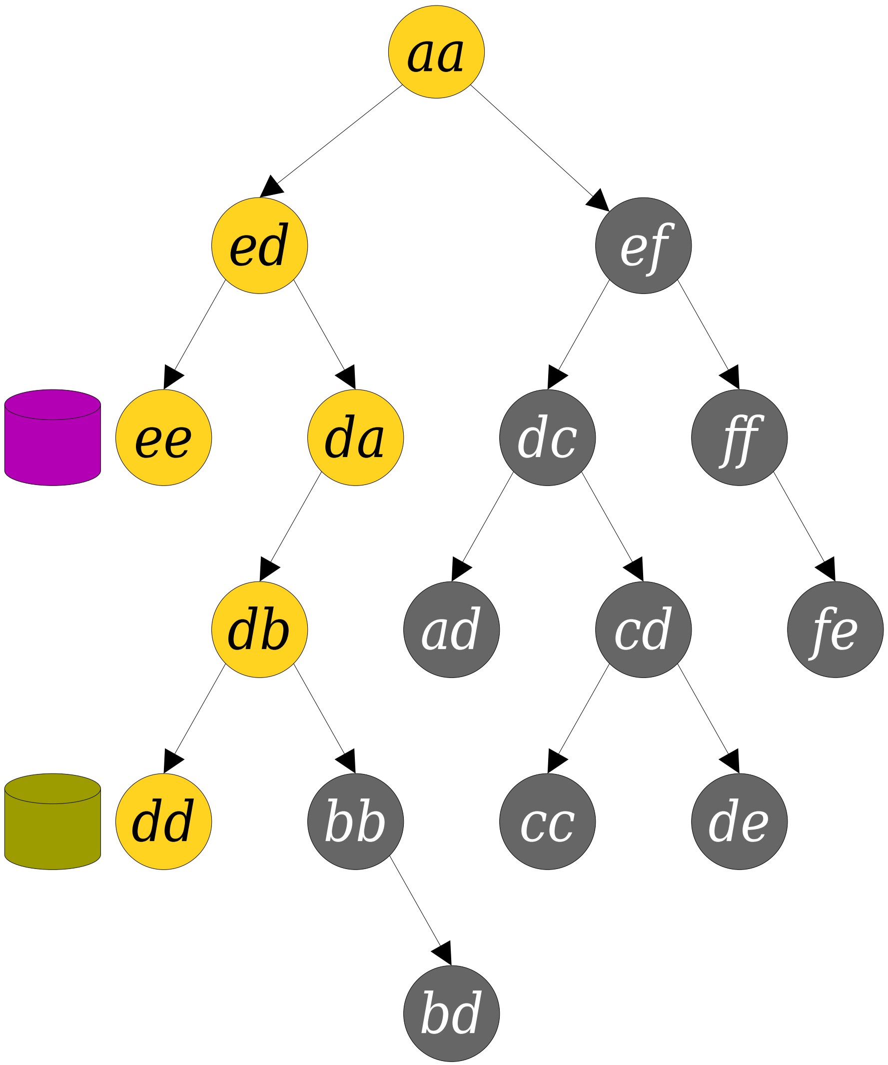
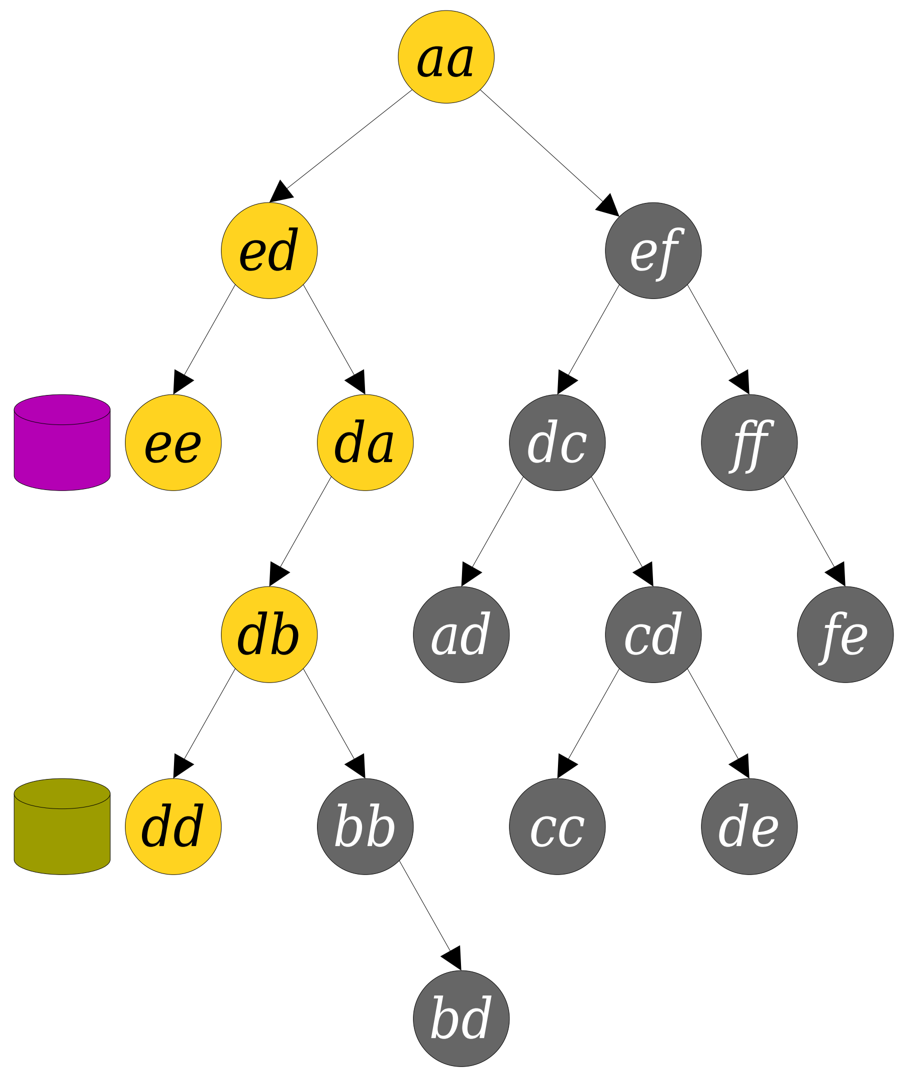


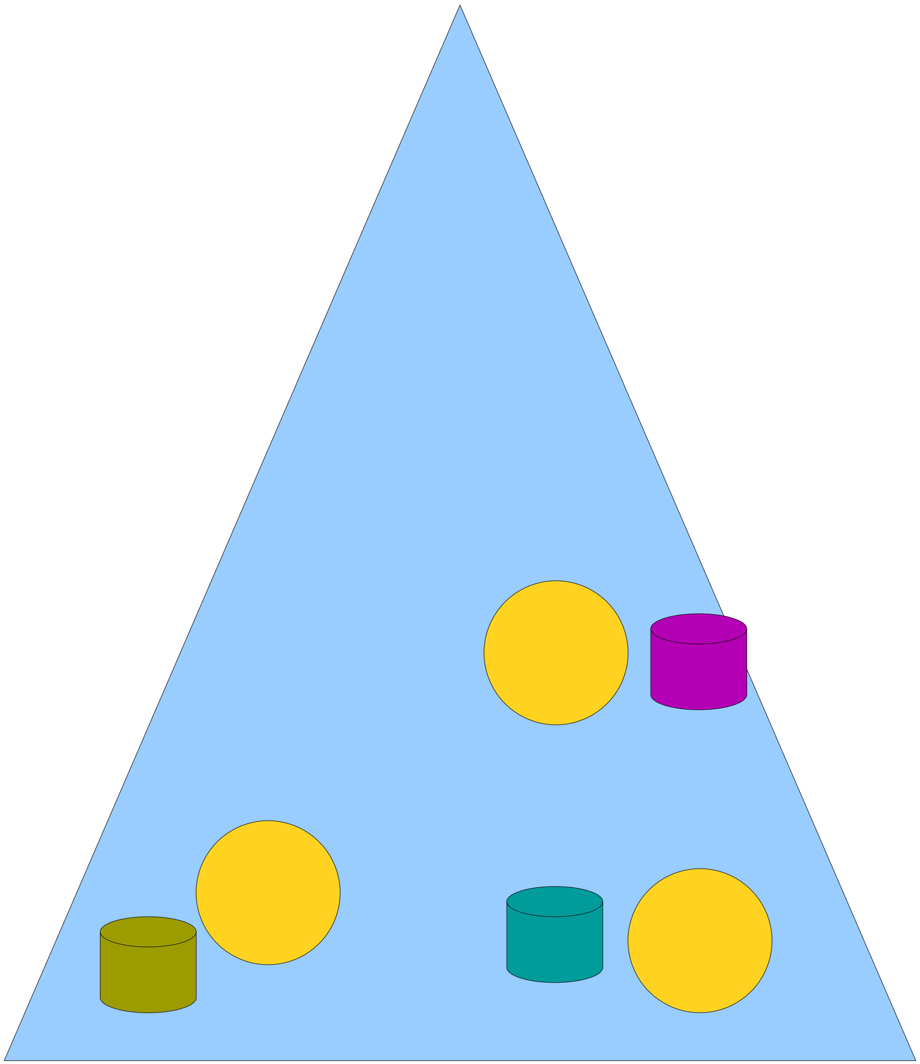
Packet Finding

- Augment each node *xx* with a list of the packets it stores.
- Augment each tree node with a bit indicating whether there's a packet in its subtree.
- We can use this latter information to quickly find nodes holding packets.



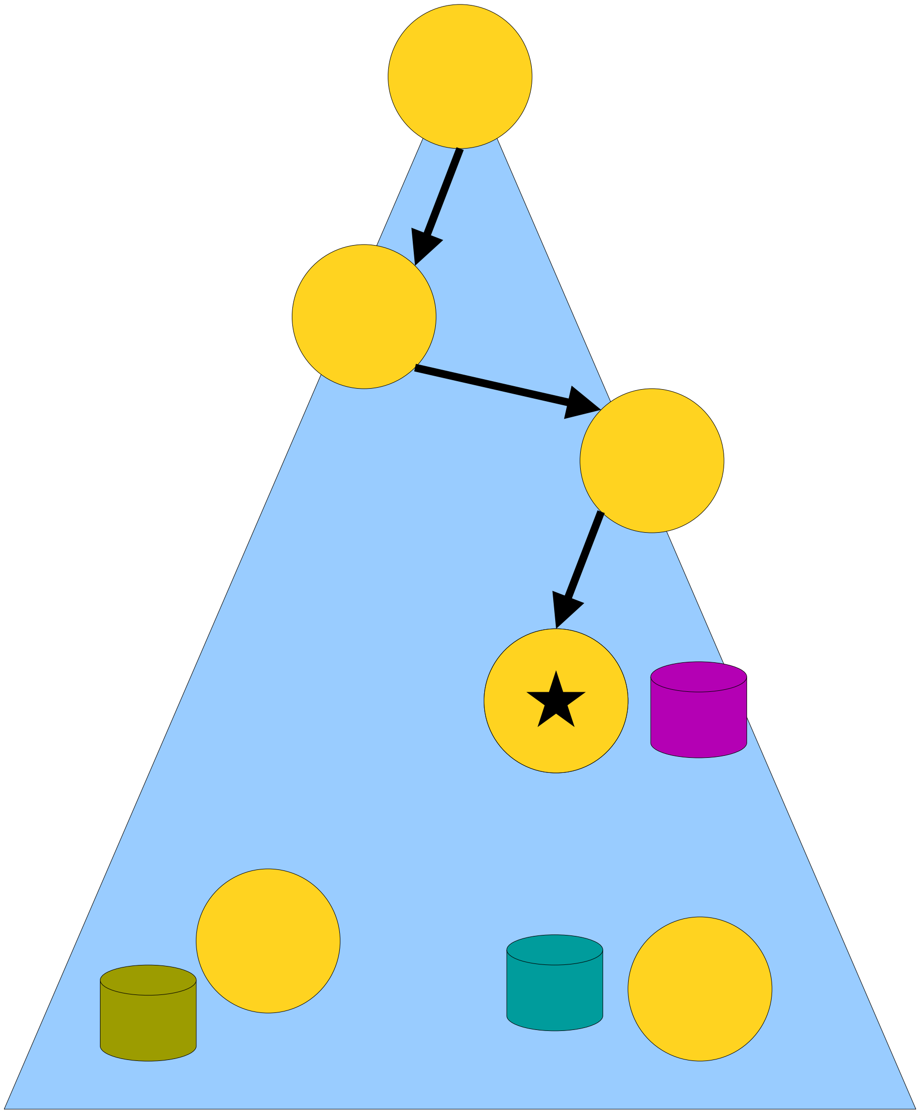
Packet Finding

- To find and remove a packet:



Packet Finding

- To find and remove a packet:
  - Walk from the root to any node containing a packet, using the augmentation to guide the search.



Packet Finding

- To find and remove a packet:
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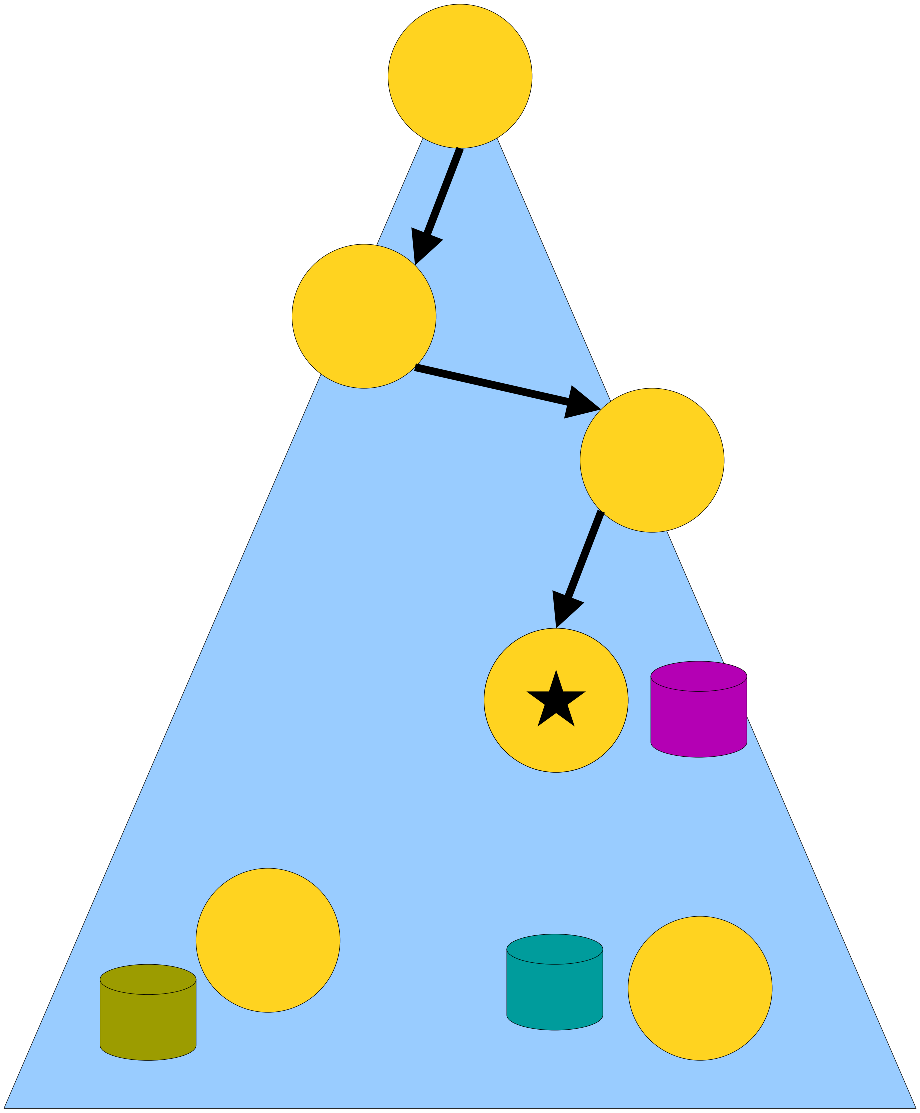

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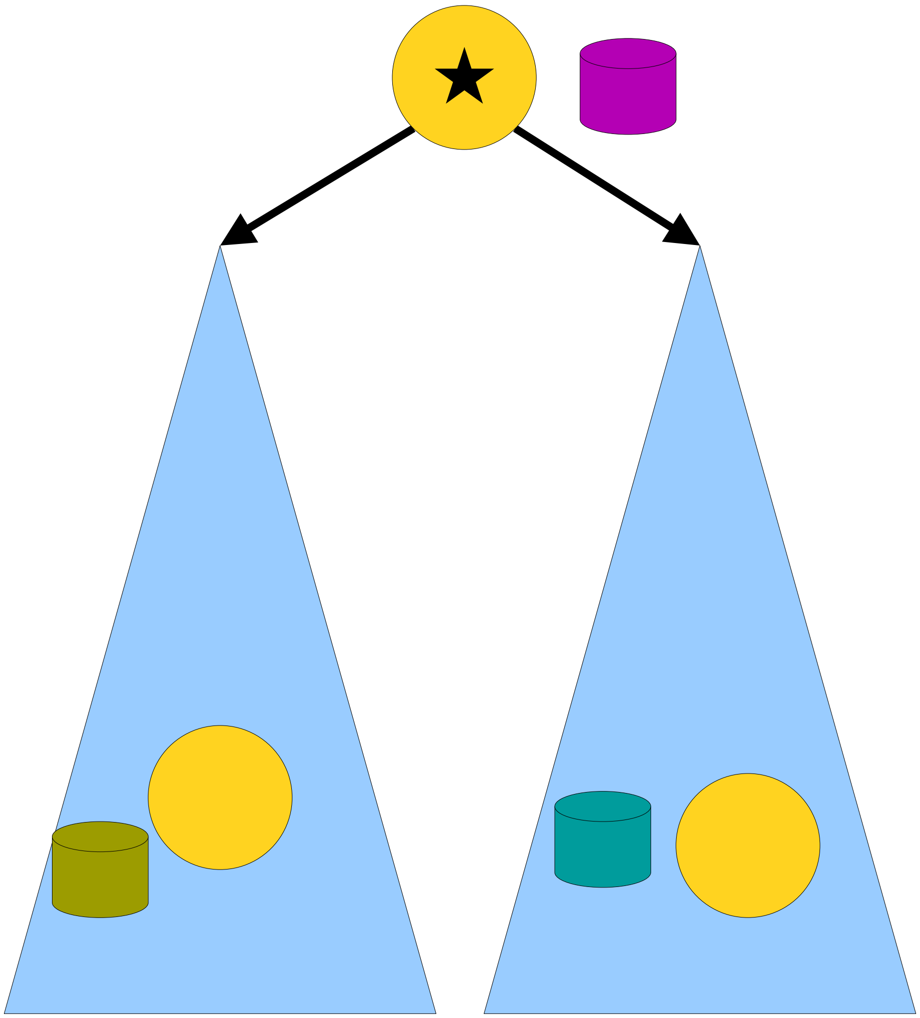

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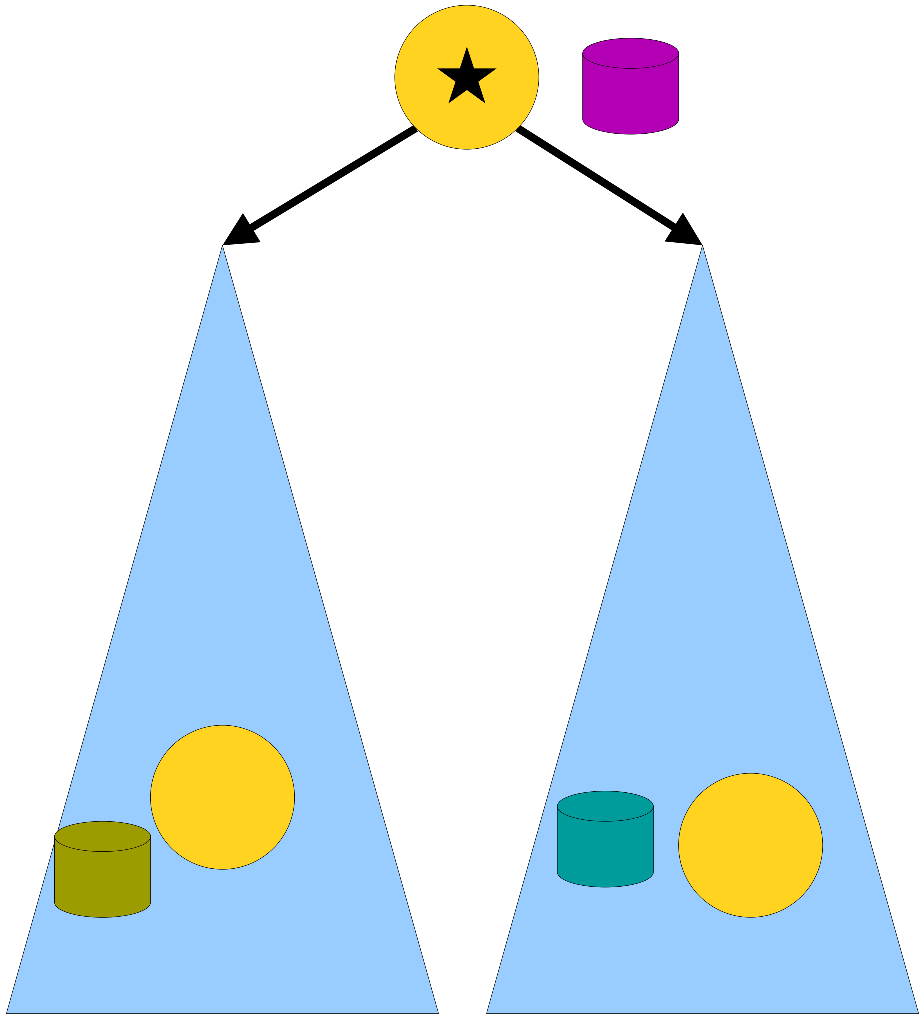
  - Remove a packet from it, updating the root's augmentation.



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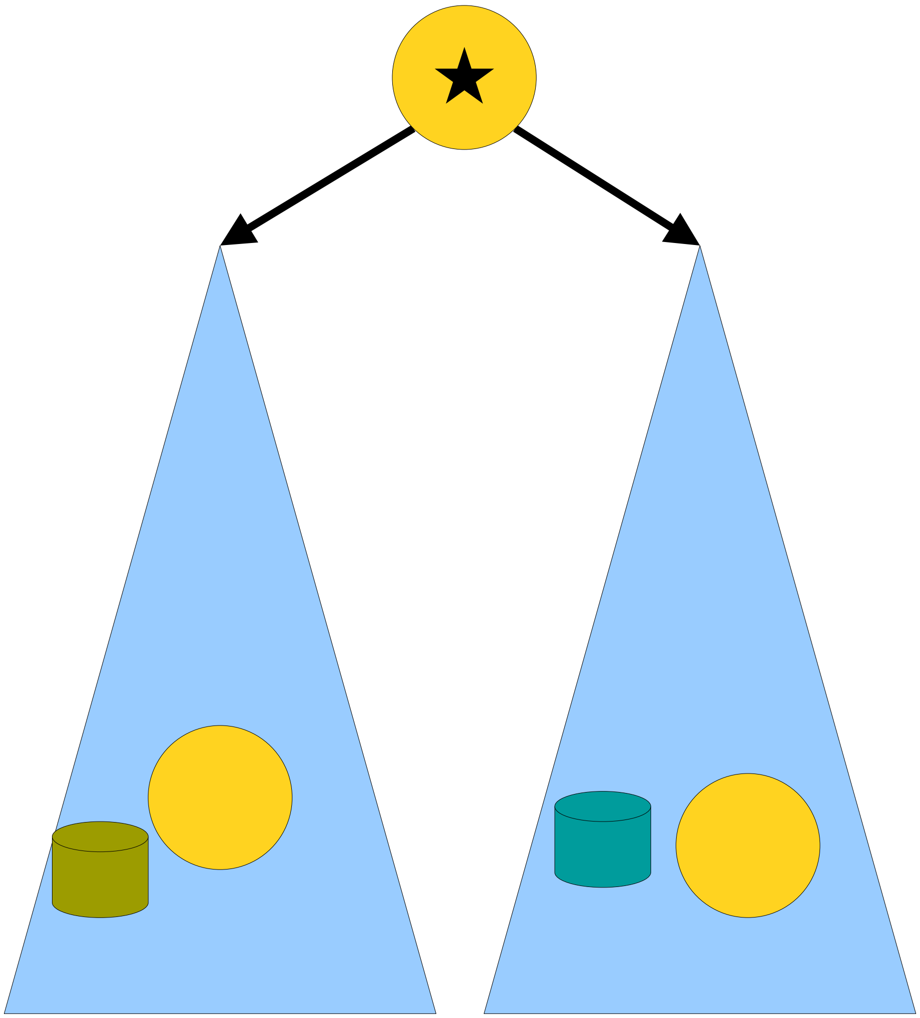

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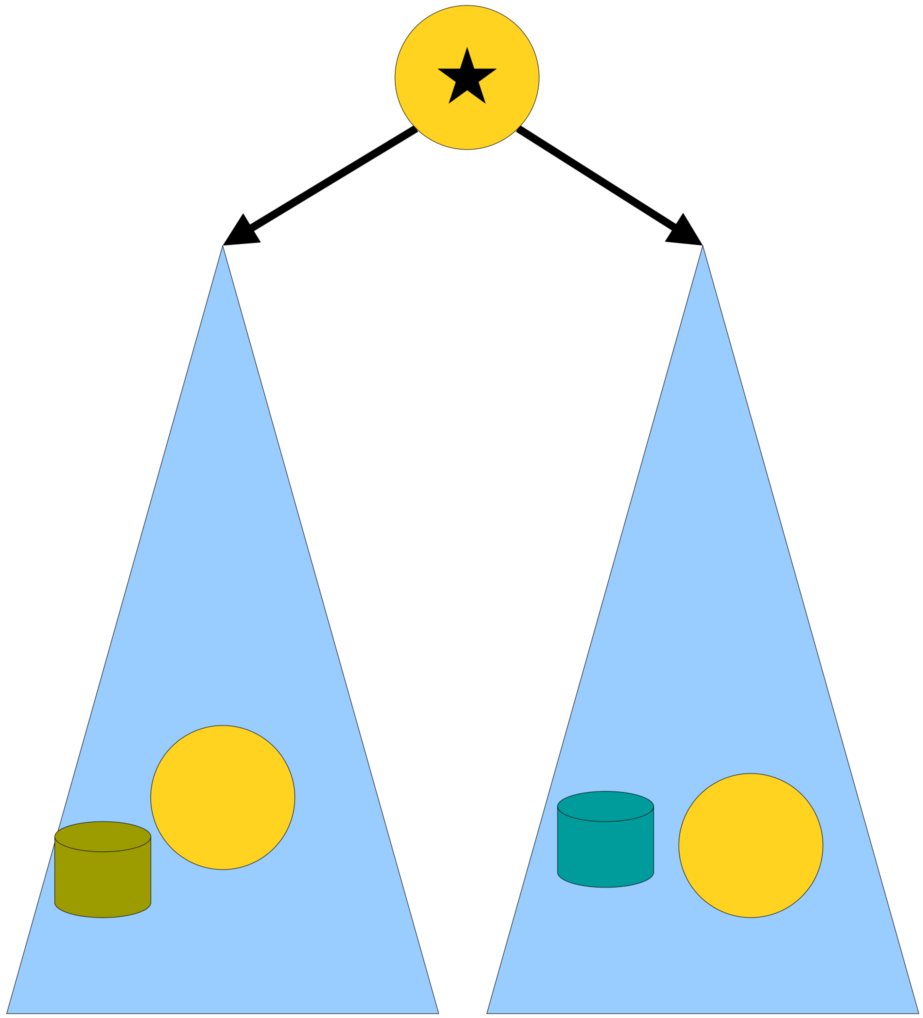

Generalizing This Idea

- More generally, Euler tour trees play well with augmentations that care about global properties of individual trees.
- There's another way to use splay trees to encode dynamic trees (***st-trees***, also called ***link/cut trees***, though the later name is ambiguous) that works well for augmenting over ***paths*** in trees rather than trees as a whole.
- (Check out the Sleator/Tarjan paper for more details.)

Next Time

- ***Fully-Dynamic Connectivity***
  - Solving connectivity in general graphs, not just forests.
- ***“Blame It On The Little Guy”***
  - A surprisingly versatile algorithmic strategy.
- ***Holm’s Structure***
  - An elegant way to solve dynamic connectivity by harnessing augmented ETTs.