CS205 Homework #3

Problem 1

Consider an $n \times n$ matrix $A$.

1. Show that if $A$ has distinct eigenvalues all the corresponding eigenvectors are linearly independent.

2. Show that if $A$ has a full set of eigenvectors (i.e. any eigenvalue $\lambda$ with multiplicity $k$ has $k$ corresponding linearly independent eigenvectors), it can be written as $A = QAQ^{-1}$ where $A$ is a diagonal matrix of $A$’s eigenvalues and $Q$’s columns are $A$’s eigenvectors. Hint: show that $AQ = QA$ and that $Q$ is invertible.

3. If $A$ is symmetric show that any two eigenvectors corresponding to different eigenvalues are orthogonal.

4. If $A$ is symmetric show that it has a full set of eigenvectors. Hint: If $(\lambda, q)$ is an eigenvalue, eigenvector ($q$ normalized) pair and $\lambda$ is of multiplicity $k > 1$, show that $A - \lambda qq^T$ has an eigenvalue of $\lambda$ with multiplicity $k - 1$. To show that consider the Householder matrix $H$ such that $Hq = e_1$ and note that $HAH^{-1} = HA$ and $A$ are similar.

5. If $A$ is symmetric show that it can be written as $A = QAQ^T$ for an orthogonal matrix $Q$. (You may use the result of (4) even if you didn’t prove it)

Problem 2

[Adapted from Heath 4.23] Let $A$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Recall that these are the roots of the characteristic polynomial of $A$, defined as $f(\lambda) \equiv \det(A - \lambda I)$. Also we define the multiplicity of an eigenvalue to be the degree of it as a root of the characteristic polynomial.

1. Show that the determinant of $A$ is equal to the product of its eigenvalues, i.e. $\det(A) = \prod_{j=1}^{n} \lambda_j$.

2. The trace of a matrix is defined to be the sum of its diagonal entries, i.e., $\text{trace}(A) = \sum_{j=1}^{n} a_{jj}$. Show that the trace of $A$ is equal to the sum of its eigenvalues, i.e. $\text{trace}(A) = \sum_{j=1}^{n} \lambda_j$.

3. Recall a matrix $B$ is similar to $A$ if $B = T^{-1}AT$ for a non-singular matrix $T$. Show that two similar matrices have the same trace and determinant.
4. Consider the matrix
\[
A = \begin{pmatrix}
-\frac{a_{m-1}}{a_m} & -\frac{a_{m-2}}{a_m} & \cdots & -\frac{a_1}{a_m} & -\frac{a_0}{a_m} \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}
\]

What is \(A\)'s characteristic polynomial? Describe how you can use the power method to find the largest root (in magnitude) of an arbitrary polynomial.

**Problem 3**

Let \(A\) be a \(m \times n\) matrix and \(A = U\Sigma V^T\) its singular value decomposition.

1. Show that \(\|A\|_2 = \|\Sigma\|_2\)

2. Show that if \(m \geq n\) then for all \(x \in \mathbb{R}^n\)
\[
\sigma_{\min} \leq \frac{\|Ax\|_2}{\|x\|_2} \leq \sigma_{\max}
\]
where \(\sigma_{\min}, \sigma_{\max}\) are the smallest and largest singular values, respectively.

3. The Frobenius norm of a matrix \(A\) is defined as \(\|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2}\). Show that \(\|A\|_F = \sqrt{\sum_{i=1}^{p} \sigma_i^2}\) where \(p = \min\{m, n\}\).

4. If \(A\) is an \(m \times n\) matrix and \(b\) is an \(m\)-vector prove that the solution \(x\) of minimum Euclidean norm to the least squares problem \(Ax \cong b\) is given by
\[
x = \sum_{\sigma_i \neq 0} \frac{u_i^T b}{\sigma_i} v_i
\]
where \(u_i, v_i\) are the columns of \(U\) and \(V\), respectively.

5. Show that the columns of \(U\) corresponding to non-zero singular values form an orthogonal basis of the column space of \(A\). What space do the columns of \(U\) corresponding to zero singular values span?