1. Consider a 1D discretization with \( \Delta x = \frac{1}{3} \) and the nine grid values \( \rho_0 = 2, \rho_1 = 5, \rho_2 = 3, \rho_3 = 1, \rho_4 = -2, \rho_5 = -1, \rho_6 = 0, \rho_7 = 0, \rho_8 = 0 \). Let the locations of these grid values be \( x_0 = 0, x_1 = \Delta x, x_2 = 2\Delta x, \) etc.

(a) Construct the divided difference table for the data. Note that you will need to use \( \Delta x \) to construct this table. The first level of the table should consist of the ten values given above, and there should be three additional levels above it. Thus, your table should consist of \( 9 + 8 + 7 + 6 \) entries.

(b) Assume information is flowing to the right \( (u > 0) \). For each of the positions \( x_3, x_4, \) and \( x_5, \) use third order HJ ENO to compute a Newton polynomial at that position. Call these polynomials \( P^r_3(x), P^r_4(x), \) and \( P^r_5(x) \). You should leave your polynomials in the form of a Newton polynomial.

For \( \rho_3 \), follow the sequence LLR. This gives the polynomial

\[
P^r_3(x) = 1 - 6\left(x - \frac{3}{3}\right) - 4.5\left(x - \frac{3}{3}\right)\left(x - \frac{2}{3}\right)\left(x - \frac{1}{3}\right)
\]

For \( \rho_4 \), follow the sequence LLL. This gives the polynomial

\[
P^r_4(x) = -2 - 9\left(x - \frac{4}{3}\right) - 4.5\left(x - \frac{4}{3}\right)\left(x - \frac{3}{3}\right) - 4.5\left(x - \frac{4}{3}\right)\left(x - \frac{3}{3}\right)\left(x - \frac{2}{3}\right)
\]

For \( \rho_5 \), follow the sequence LRR. This gives the polynomial

\[
P^r_5(x) = -1 + 3\left(x - \frac{5}{3}\right) - 4.5\left(x - \frac{5}{3}\right)\left(x - \frac{4}{3}\right)\left(x - \frac{6}{3}\right)
\]
(c) These polynomials are constructed to be interpolating polynomials. Show that \( P^4(x) \) is in fact an interpolating polynomial.

\[
P^r_4(x) = -2 - 9 \left( x - \frac{4}{3} \right) - 4.5 \left( x - \frac{4}{3} \right) \left( x - \frac{3}{3} \right) - 4.5 \left( x - \frac{4}{3} \right) \left( x - \frac{3}{3} \right) \left( x - \frac{2}{3} \right)
\]

\[
P^r_4(x_1) = -2 - 9 \left( \frac{1}{3} - \frac{4}{3} \right) - 4.5 \left( \frac{1}{3} - \frac{4}{3} \right) \left( \frac{1}{3} - \frac{3}{3} \right) - 4.5 \left( \frac{1}{3} - \frac{4}{3} \right) \left( \frac{1}{3} - \frac{3}{3} \right) \left( \frac{1}{3} - \frac{2}{3} \right)
= -2 + 9 - 3 + 1 = 5 = \rho_1
\]

\[
P^r_4(x_2) = -2 - 9 \left( \frac{2}{3} - \frac{4}{3} \right) - 4.5 \left( \frac{2}{3} - \frac{4}{3} \right) \left( \frac{2}{3} - \frac{3}{3} \right) - 4.5 \left( \frac{2}{3} - \frac{4}{3} \right) \left( \frac{2}{3} - \frac{3}{3} \right) \left( \frac{2}{3} - \frac{2}{3} \right)
= -2 + 6 - 1 + 0 = 3 = \rho_2
\]

\[
P^r_4(x_3) = -2 + 3 - 0 - 0 = 1 = \rho_3
\]

\[
P^r_4(x_4) = -2 - 9 \left( \frac{4}{3} - \frac{4}{3} \right) - 4.5 \left( \frac{4}{3} - \frac{4}{3} \right) \left( \frac{4}{3} - \frac{3}{3} \right) - 4.5 \left( \frac{4}{3} - \frac{4}{3} \right) \left( \frac{4}{3} - \frac{3}{3} \right) \left( \frac{4}{3} - \frac{2}{3} \right)
= -2 - 0 - 0 - 0 = -2 = \rho_4
\]

(d) Assume instead that information is flowing to the left \((u < 0)\). Use third order HJ ENO to compute the polynomials \( P^l_3(x) \), \( P^l_4(x) \), and \( P^l_5(x) \). You should leave your polynomials in the form of a Newton polynomial.

For \( \rho_3 \), follow the sequence RLL. This gives the polynomial

\[
P^l_3(x) = 1 - 9 \left( x - \frac{3}{3} \right) - 4.5 \left( x - \frac{3}{3} \right) \left( x - \frac{4}{3} \right) - 4.5 \left( x - \frac{3}{3} \right) \left( x - \frac{4}{3} \right) \left( x - \frac{2}{3} \right)
\]

For \( \rho_4 \), follow the sequence RRR. This gives the polynomial

\[
P^l_4(x) = -2 + 3 \left( x - \frac{4}{3} \right) - 4.5 \left( x - \frac{4}{3} \right) \left( x - \frac{5}{3} \right) \left( x - \frac{6}{3} \right)
\]

For \( \rho_5 \), follow the sequence RLR. This gives the polynomial

\[
P^l_5(x) = -1 + 3 \left( x - \frac{5}{3} \right) - 4.5 \left( x - \frac{5}{3} \right) \left( x - \frac{6}{3} \right) \left( x - \frac{4}{3} \right)
\]

(e) Above you computed six Newton polynomials. They should all look distinct, but they are not all distinct polynomials. Which polynomials are actually equal and why? You should not expand out the polynomials to answer this question.

Since the polynomials interpolate the data, they will equal if they interpolate the same data. Thus, \( P^l_3(x) = P^l_4(x) = P^l_5(x) \) and \( P^l_4(x) = P^l_5(x) = P^l_4(x) \). Since \( P^l_3(0) = 8 \) and \( P^l_4(0) = 14 \), these two sets of polynomials are distinct.
2. There are multiple second order Runge Kutta schemes that one might use to evolve \( x' = f(x) \). The classical one (and the one I am referring to when I write RK2) is \( x^{n+1/2} = x^n + \frac{1}{2} \Delta t f(x^n) \), \( x^{n+1} = x^n + \Delta t f(x^{n+1/2}) \). Another second order Runge Kutta method is TVD RK2, which has the form \( \hat{x}^{n+1} = x^n + \Delta t f(x^n) \), \( x^{n+1} = \hat{x}^{n+1} + \Delta t f(\hat{x}^{n+1}) \), \( x^{n+1} = \frac{1}{2}(x^n + x^{n+2}) \).

(a) Show that these two schemes are in fact distinct schemes.

Let \( f(x) = x^2 \), \( \Delta t = 1 \), \( x^n = 1 \). Then, RK2 gives \( x^{n+1/2} = \frac{3}{2} \) and \( x^{n+1} = \frac{13}{4} \). From TVD RK2 we get \( \hat{x}^{n+1} = 2 \), \( x^{n+2} = 6 \), \( x^{n+1} = 4 \). Since the two schemes give different results, they must be different.

(b) In homework 2, question 3b, you expressed the update rule for a time integration scheme applied to \( x' = \lambda x \) (complex \( \lambda \)) in the form \( x^{n+1} = C x^n \), where \( C \) is a complex number that depends only on the value of \( \lambda \Delta t \). Compute the expression for \( C \) for both of these schemes. Let \( C_2 \) be the one you computed for RK2.

In this case, \( f(x) = \lambda x \). RK2 gives

\[
\begin{align*}
x^{n+1/2} &= x^n + \frac{1}{2} \Delta t \lambda x^n \\
x^{n+1} &= x^n + \Delta t \lambda (x^n + \frac{1}{2} \Delta t \lambda x^n)
&= x^n + \Delta t \lambda x^n + \frac{1}{2} (\Delta t \lambda)^2 x^n
\end{align*}
\]

\( C_2 = C = 1 + \Delta t \lambda + \frac{1}{2} (\Delta t \lambda)^2 \)

and TVD RK2 gives

\[
\begin{align*}
\hat{x}^{n+1} &= x^n + \Delta t \lambda x^n \\
x^{n+2} &= (x^n + \Delta t \lambda x^n) + \Delta t \lambda (x^n + \Delta t \lambda x^n)
&= x^n + 2 \Delta t \lambda x^n + (\Delta t \lambda x^n)^2 x^n \\
x^{n+1} &= \frac{1}{2}(x^n + x^n + 2 \Delta t \lambda x^n + (\Delta t \lambda x^n)^2 x^n)
&= x^n + \Delta t \lambda x^n + \frac{1}{2} (\Delta t \lambda x^n)^2 x^n
\end{align*}
\]

\( C = 1 + \Delta t \lambda + \frac{1}{2} (\Delta t \lambda)^2 \)

(c) Use this to argue that the two schemes have identical stability plots. You do not need to construct the stability plots.

The stability of any particular region was determined based on the truth of \(|C| < 1\). Since these schemes agree on \( C \) everywhere, their stability plots will agree everywhere.
(d) Let $C_1$ be the expression for $C$ that is obtained for forward Euler, $C_3$ the expression obtained for TVD RK3, and $C_4$ the value obtained for RK4. It is okay to read off $C_1$ from the answer key to the assignment where you computed this, but you will need to derive $C_3$ and $C_4$.

As before, \( C_1 = 1 + \Delta t \lambda \). For TVD RK3,

\[
\begin{align*}
\dot{x}_{n+1} &= x_n + \Delta t \lambda x^n \\
x_{n+2} &= x_{n+1} + \Delta t \lambda x_{n+1} \\
&= x^n + \Delta t \lambda x^n + \Delta t \lambda (x^n + \Delta t \lambda x^n) \\
&= x^n + 2\Delta t \lambda x^n + (\Delta t \lambda)^2 x^n \\
x_{n+1/2} &= \frac{3}{4} x^n + \frac{1}{4} x_{n+2} \\
&= \frac{3}{4} x^n + \frac{1}{4} (x^n + 2\Delta t \lambda x^n + (\Delta t \lambda)^2 x^n) \\
&= x^n + \frac{1}{2} \Delta t \lambda x^n + \frac{1}{4} (\Delta t \lambda)^2 x^n \\
x_{n+3/2} &= x_{n+1/2} + \Delta t \lambda x_{n+1/2} \\
&= (x^n + \frac{1}{2} \Delta t \lambda x^n + \frac{1}{4} (\Delta t \lambda)^2 x^n) + \Delta t \lambda (x^n + \frac{1}{2} \Delta t \lambda x^n + \frac{1}{4} (\Delta t \lambda)^2 x^n) \\
&= x^n + \frac{3}{2} \Delta t \lambda x^n + \frac{3}{4} (\Delta t \lambda)^2 x^n + \frac{1}{4} (\Delta t \lambda)^3 x^n \\
x_{n+1} &= \frac{1}{3} x^n + \frac{2}{3} x_{n+3/2} \\
x_{n+1} &= \frac{1}{3} x^n + \frac{2}{3} x^n + \frac{3}{2} \Delta t \lambda x^n + \frac{3}{4} (\Delta t \lambda)^2 x^n + \frac{1}{4} (\Delta t \lambda)^3 x^n \\
x_{n+1} &= x^n + \Delta t \lambda x^n + \frac{1}{2} (\Delta t \lambda)^2 x^n + \frac{1}{6} (\Delta t \lambda)^3 x^n \\
C_3 &= 1 + \Delta t \lambda + \frac{1}{2} (\Delta t \lambda)^2 + \frac{1}{6} (\Delta t \lambda)^3
\end{align*}
\]
Finally, for RK4 we get

\[
\begin{align*}
  k_1 &= \lambda x^n \\
  k_2 &= \lambda(x^n + \frac{1}{2}\Delta t \lambda x^n) \\
  &= \lambda x^n + \frac{1}{2}\Delta t \lambda^2 x^n \\
  k_3 &= \lambda(x^n + \frac{1}{2}\Delta t(\lambda x^n + \frac{1}{2}\Delta t \lambda^2 x^n)) \\
  &= \lambda x^n + \frac{1}{2}\Delta t \lambda^2 x^n + \frac{1}{4}\Delta t \lambda^3 x^n \\
  k_4 &= \lambda(x^n + \Delta t(\lambda x^n + \frac{1}{2}\Delta t \lambda^2 x^n + \frac{1}{4}\Delta t \lambda^3 x^n)) \\
  &= \lambda x^n + \Delta t \lambda(\lambda x^n + \frac{1}{2}\Delta t \lambda^2 x^n + \frac{1}{4}\Delta t \lambda^3 x^n) \\
  &= \lambda x^n + \Delta t \lambda^2 x^n + \frac{1}{2}\Delta t \lambda^3 x^n + \frac{1}{4}\Delta t \lambda^4 x^n \\
  x^{n+1} &= x^n + \frac{1}{6}\Delta t(k_1 + 2k_2 + 2k_3 + k_4) \\
  &= x^n + \Delta t \lambda x^n + \frac{1}{2}(\Delta t \lambda)^2 x^n + \frac{1}{6}(\Delta t \lambda)^3 x^n + \frac{1}{24}(\Delta t \lambda)^4 x^n \\
  C_4 &= 1 + \Delta t \lambda + \frac{1}{2}(\Delta t \lambda)^2 + \frac{1}{6}(\Delta t \lambda)^3 + \frac{1}{24}(\Delta t \lambda)^4
\end{align*}
\]

(e) We could continue in this way using an explicit \(n\)-order scheme to derive \(C_n\). What is \(C_\infty\) and why?

The solution the the differential equation is \(x = e^{\lambda t}\). If \(x^n = e^{\lambda t}\) then \(x^{n+1} = e^{\lambda(t+\Delta t)} = e^{\lambda \Delta t} e^{\lambda t} = e^{\lambda \Delta t} x^n\), so that

\[
C_\infty = e^{\lambda \Delta t} = 1 + \Delta t \lambda + \frac{1}{2}(\Delta t \lambda)^2 + \frac{1}{6}(\Delta t \lambda)^3 + \frac{1}{24}(\Delta t \lambda)^4 + \cdots.
\]

(f) What does the stability region for \(C_\infty\) look like? You should work out the stability region analytically. You do not need to generate a stability plot for it.

Let \(\lambda = a + bi\). \(|C_\infty| = |e^{\lambda \Delta t}| = |e^{(a+bi)\Delta t}| = |e^{a\Delta t}||e^{b\Delta t}| = e^{a\Delta t} \cdot \Delta t > 0\) implies \(a < 0\). Thus, the stability region is the region where \(\text{Re}(\lambda) < 0\), or the entire left half-plane.