CS222 / Phil358 Homework 1

Due: April 21, 2010

1. (a) Show that any relation \( R \) that is Euclidean and reflexive is also symmetric and transitive.

(b) Give a derivation in the logic \( KT \) of the \( D \) axiom, \( \neg \Box (p \land \neg p) \). In other words, show that it follows in \( K \) from the axiom \( T \) alone.

2. Show that axiom 5, \( \Diamond \phi \rightarrow \Box \Diamond \phi \) is valid on a frame \( (W, R) \), if and only if \( R \) is Euclidean.

3. Consider the following logic \( L \) in the language with two operators, \( K \) and \( B \). We take all \( S5 \) axioms for \( K \), all \( KD45 \) axioms for \( B \), and following two “bridge axioms”:

\[
K\phi \rightarrow B\phi \\
B\phi \rightarrow BK\phi
\]

We take as rules for \( L \) both modus ponens and the necessitation rule for \( K \) and \( B \) (from \( L \vdash \phi \), infer \( L \vdash K\phi \) and \( L \vdash B\phi \)). Show that \( L \vdash K\phi \leftrightarrow B\phi \).

4. Show that \( \neg K\neg K\phi \rightarrow K\neg K\neg K\phi \) is valid in all (single agent) \( KB \)-models. This can either be done directly by giving a model-theoretic argument, or by providing a derivation from the sound and complete proof system given on p.435-436 of Multiagent Systems (p.418 of the hard copy).

5. In this problem we consider a possible definition of common belief, analogous to the definition of common knowledge. Suppose we have just two agents, 1 and 2. Given a frame \( (W, R_1, R_2) \), define \( R := (R_1 \cup R_2)^* \), i.e. the transitive closure of \( R_1 \cup R_2 \). Then we define our common belief operator \( C \) as follows:

\[
M, w \vDash C\phi \iff \forall x \in W, \text{ if } w R x \text{ then } M, x \vDash \phi
\]

Provide a \( KD45 \) structure \( M \), a state \( w \), and a formula \( \phi \), such that \( M, w \vDash B_1 C\phi \), but \( M, w \nvDash C\phi \).