Lecture 5
Epipolar Geometry

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Lecture 5

Epipolar Geometry

• Why is stereo useful?
• Epipolar constraints
• Essential and fundamental matrix
• Estimating F
• Examples

Reading: [AZ] Chapter: 4 “Estimation – 2D perspective transformations
Chapter: 9 “Epipolar Geometry and the Fundamental Matrix Transformation”
Chapter: 11 “Computation of the Fundamental Matrix F”

[FP] Chapter: 7 “Stereopsis”
Chapter: 8 “Structure from Motion”
Recovering structure from a single view

From calibration rig → location/pose of the rig, K

From points and lines at infinity + orthogonal lines and planes → structure of the scene, K

Knowledge about scene (point correspondences, geometry of lines & planes, etc…)

Calibration rig

Scene

Camera K
Recovering structure from a single view

Why is it so difficult?

Intrinsic ambiguity of the mapping from 3D to image (2D)
Recovering structure from a single view

Intrinsic ambiguity of the mapping from 3D to image (2D)
Two eyes help!
Two eyes help!

\[ P = l \times l' \]

[Eq. 1]

This is called triangulation
• Find $P'$ that minimizes

$$d(p, MP^*) + d(p', M'P^*)$$  \[\text{[Eq. 2]}\]
Stereo-view geometry

- **Correspondence:** Given a point \( p \) in one image, how can I find the corresponding point \( p' \) in another one?

- **Camera geometry:** Given corresponding points in two images, find camera matrices, position and pose.

- **Scene geometry:** Find coordinates of 3D point from its projection into 2 or multiple images.
Epipolar geometry

- Epipolar Plane
- Baseline
- Epipolar Lines

- Epipoles $e$, $e'$

= intersections of baseline with image planes
= projections of the other camera center
Example of epipolar lines
Example: Parallel image planes

- Baseline intersects the image plane at infinity
- Epipoles are at infinity
- Epipolar lines are parallel to u axis
Example: Parallel Image Planes
Example: Forward translation

- The epipoles have same position in both images
- Epipole called FOE (focus of expansion)
Epipolar geometry

$p$

$p'$

Epipolar line 2

$O_1$

$O_2$
Epipolar Constraint
Epipolar Constraint

\[ M = K \begin{bmatrix} I & 0 \end{bmatrix} \]

\[ MP = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = p \quad \text{[Eq. 3]} \]

\[ M' = K' \begin{bmatrix} R^T & -R^T T \end{bmatrix} \]

\[ M' P = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = p' \quad \text{[Eq. 4]} \]
Epipolar Constraint

\[ K_{\text{canonical}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ M = K \begin{bmatrix} I & 0 \end{bmatrix} \]

\[ M' = K' \begin{bmatrix} R^T & -R^T T \end{bmatrix} \]

\[ M = \begin{bmatrix} I & 0 \end{bmatrix} \quad \text{[Eq. 5]} \]

\[ M' = \begin{bmatrix} R^T & -R^T T \end{bmatrix} \quad \text{[Eq. 6]} \]
The cameras are related by $\mathbf{R}, \mathbf{T}$

$\mathbf{T} = \mathbf{O}_2$ in the camera 1 reference system

$\mathbf{R}$ is the rotation matrix such that a vector $\mathbf{p}'$ in the camera 2 is equal to $\mathbf{R} \mathbf{p}'$ in camera 1.
Epipolar Constraint

\[ T \times ((R \ p') + T) = T \times (R \ p') \] is perpendicular to epipolar plane

\[ \rightarrow p^T \cdot [T \times (R \ p')] = 0 \] [Eq. 7]
Cross product as matrix multiplication

\[ \mathbf{a} \times \mathbf{b} = \begin{bmatrix}
0 & -a_z & a_y \\
-az & 0 & -a_x \\
-ay & ax & 0 \\
\end{bmatrix} \begin{bmatrix}
b_x \\
b_y \\
b_z \\
\end{bmatrix} = [\mathbf{a}_x] \mathbf{b} \]
Epipolar Constraint

\[ p^T \cdot [T \times (R \ p')] = 0 \rightarrow p^T \cdot [T_x] \cdot R \ p' = 0 \]

[Eq. 8] [Eq. 9]

\[ E = \text{Essential matrix} \]

(Longuet-Higgins, 1981)
\( p^T \cdot E \cdot p' = 0 \)

- \( l = Ep' \) is the epipolar line associated with \( p' \)
- \( l' = E^T p \) is the epipolar line associated with \( p \)
- \( E \cdot e' = 0 \) and \( E^T e = 0 \)
- \( E \) is a 3x3 matrix; 5 DOF
- \( E \) is singular (rank two)
Epipolar Constraint

\[ M = K[I \ 0] \]

\[ M' = K'[R^T \ -R^T T] \]

\[ p_c = K^{-1} p \quad [\text{Eq. 11}] \]

\[ p'_c = K'^{-1} p' \quad [\text{Eq. 12}] \]
$$p_c = K^{-1} p$$  \[\text{[Eq. 11]}\]

$$p'_c = K'^{-1} p'$$  \[\text{[Eq. 12]}\]

$$p_c^T \cdot [T_x] \cdot R \ p'_c = 0 \rightarrow (K^{-1} p)^T \cdot [T_x] \cdot R \ K'^{-1} \ p' = 0$$  \[\text{[Eq. 9]}\]

$$p^T K^{-T} \cdot [T_x] \cdot R \ K'^{-1} \ p' = 0 \rightarrow p^T \boxed{F} p' = 0$$  \[\text{[Eq. 13]}\]
Epipolar Constraint

\[ \mathbf{p}^T \mathbf{F} \mathbf{p}' = 0 \]  

\[ \mathbf{F} = \mathbf{K}^{-T} \cdot [\mathbf{T}_x] \cdot \mathbf{R} \mathbf{K}'^{-1} \]

\( \mathbf{F} = \text{Fundamental Matrix} \)  

(Faugeras and Luong, 1992)
Epipolar Constraint

\[ p^T \cdot F \cdot p' = 0 \]

- \( l = F \cdot p' \) is the epipolar line associated with \( p' \)
- \( l' = F^T \cdot p \) is the epipolar line associated with \( p \)
- \( F \cdot e' = 0 \) and \( F^T \cdot e = 0 \)
- \( F \) is a 3x3 matrix; 7 DOF
- \( F \) is singular (rank two)
Why F is useful?

- Suppose F is known
- No additional information about the scene and camera is given
- Given a point on left image, we can compute the corresponding epipolar line in the second image

\[ l' = F^T p \]
Why F is useful?

- F captures information about the epipolar geometry of 2 views + camera parameters

- **MORE IMPORTANTLY:** F gives constraints on how the scene changes under view point transformation (without reconstructing the scene!)

- Powerful tool in:
  - 3D reconstruction
  - Multi-view object/scene matching
Estimating $F$

The Eight-Point Algorithm

(Longuet-Higgins, 1981)

(Hartley, 1995)

\[ p^T F p' = 0 \]
Estimating $F$

[Eq. 13] \[ p^T \mathbf{F} p' = 0 \]

\[
\begin{pmatrix}
F_{11} & F_{12} & F_{13} \\
F_{21} & F_{22} & F_{23} \\
F_{31} & F_{32} & F_{33}
\end{pmatrix}
\begin{pmatrix}
u'\\v'\\1\end{pmatrix} = 0
\]

\[ p = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \quad p' = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} \]

\[
\begin{pmatrix}
u\\u'\\v'\\1
\end{pmatrix} = 0
\]

Let's take 8 corresponding points
Estimating F
Estimating $F$

\[
\begin{pmatrix}
F_{11} \\
F_{12} \\
F_{13} \\
F_{21} \\
F_{22} \\
F_{23} \\
F_{31} \\
F_{32} \\
F_{33}
\end{pmatrix} = 0
\]

[Eq. 14]
Estimating $F$

- **Homogeneous system** $W f = 0$

- **Rank 8** $\rightarrow$ A non-zero solution exists (unique)

- **If $N>8$** $\rightarrow$ Lsq. solution by SVD! $\rightarrow \hat{F}$

$$\|f\| = 1$$

$$W = \begin{pmatrix}
    u_1u'_1 & u_1v'_1 & u_1 & v_1u'_1 & v_1v'_1 & v_1 & u'_1 & v'_1 & 1 \\
    u_2u'_2 & u_2v'_2 & u_2 & v_2u'_2 & v_2v'_2 & v_2 & u'_2 & v'_2 & 1 \\
    u_3u'_3 & u_3v'_3 & u_3 & v_3u'_3 & v_3v'_3 & v_3 & u'_3 & v'_3 & 1 \\
    u_4u'_4 & u_4v'_4 & u_4 & v_4u'_4 & v_4v'_4 & v_4 & u'_4 & v'_4 & 1 \\
    u_5u'_5 & u_5v'_5 & u_5 & v_5u'_5 & v_5v'_5 & v_5 & u'_5 & v'_5 & 1 \\
    u_6u'_6 & u_6v'_6 & u_6 & v_6u'_6 & v_6v'_6 & v_6 & u'_6 & v'_6 & 1 \\
    u_7u'_7 & u_7v'_7 & u_7 & v_7u'_7 & v_7v'_7 & v_7 & u'_7 & v'_7 & 1 \\
    u_8u'_8 & u_8v'_8 & u_8 & v_8u'_8 & v_8v'_8 & v_8 & u'_8 & v'_8 & 1 \\
\end{pmatrix}$$

$$\begin{bmatrix}
    F_{11} \\
    F_{12} \\
    F_{13} \\
    F_{21} \\
    F_{22} \\
    F_{23} \\
    F_{31} \\
    F_{32} \\
    F_{33}
\end{bmatrix} = 0 \quad [\text{Eqs. 15}]$$
\( \hat{F} \) satisfies: \( p^T \hat{F} p' = 0 \)

and estimated \( \hat{F} \) may have full rank (\( \det(\hat{F}) \neq 0 \))

**But remember:** fundamental matrix is Rank2

Find \( F \) that minimizes \( \| F - \hat{F} \| = 0 \)

Subject to \( \det(F)=0 \)

**Frobenius norm \((*)\)**

SVD (again!) can be used to solve this problem

\((*)\) Sq. root of the sum of squares of all entries
Find $F$ that minimizes $\|F - \hat{F}\| = 0$

Subject to $\det(F) = 0$

$$
F = U \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T
$$

Where:

$$
U \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} V^T = \text{SVD}(\hat{F})
$$

[HZ] pag 281, chapter 11, “Computation of $F$”
Mean errors:
10.0 pixel
9.1 pixel
Problems with the 8-Point Algorithm

- Recall the structure of $W$:
  - do we see any potential (numerical) issue?

\[ \mathbf{W} \mathbf{f} = 0, \quad \| \mathbf{f} \| = 1 \]

Lsq solution by SVD

\[ \mathbf{F} \]
Problems with the 8-Point Algorithm

\[ \mathbf{W}_f = 0 \]

\[
\begin{pmatrix}
  u_1u'_{1} & u_1v'_{1} & 1 & v_1u'_{1} & v_1v'_{1} & 1 & u'_{1} & v'_{1} & 1 \\
  u_2u'_{2} & u_2v'_{2} & 1 & v_2u'_{2} & v_2v'_{2} & 1 & u'_{2} & v'_{2} & 1 \\
  u_3u'_{3} & u_3v'_{3} & 1 & v_3u'_{3} & v_3v'_{3} & 1 & u'_{3} & v'_{3} & 1 \\
  u_4u'_{4} & u_4v'_{4} & 1 & v_4u'_{4} & v_4v'_{4} & 1 & u'_{4} & v'_{4} & 1 \\
  u_5u'_{5} & u_5v'_{5} & 1 & v_5u'_{5} & v_5v'_{5} & 1 & u'_{5} & v'_{5} & 1 \\
  u_6u'_{6} & u_6v'_{6} & 1 & v_6u'_{6} & v_6v'_{6} & 1 & u'_{6} & v'_{6} & 1 \\
  u_7u'_{7} & u_7v'_{7} & 1 & v_7u'_{7} & v_7v'_{7} & 1 & u'_{7} & v'_{7} & 1 \\
  u_8u'_{8} & u_8v'_{8} & 1 & v_8u'_{8} & v_8v'_{8} & 1 & u'_{8} & v'_{8} & 1 \\
\end{pmatrix}
\begin{pmatrix}
  F_{11} \\
  F_{12} \\
  F_{13} \\
  F_{21} \\
  F_{22} \\
  F_{23} \\
  F_{31} \\
  F_{32} \\
  F_{33} \\
\end{pmatrix}
= 0
\]

- Highly un-balanced (not well conditioned)
- Values of \( W \) must have similar magnitude
- This creates problems during the SVD decomposition
Normalization

IDEA: Transform image coordinates such that the matrix $W$ becomes better conditioned (pre-conditioning)

For each image, apply a following transformation $T$ (translation and scaling) acting on image coordinates such that:

- Origin = centroid of image points
- Mean square distance of the image points from origin is $\sim 2$ pixels
Example of normalization

- Origin = centroid of image points
- Mean square distance of the image points from origin is ~2 pixels
Normalization

\[ q_i = T p_i \]

\[ q'_i = T' p'_i \]
The Normalized Eight-Point Algorithm

0. Compute $T$ and $T'$ for image 1 and 2, respectively

1. Normalize coordinates in images 1 and 2:
   \[ q_i = T \, p_i \quad q'_i = T' \, p'_i \]

2. Use the eight-point algorithm to compute $\hat{F}_q$ from the corresponding points $q$ and $q'$.

1. Enforce the rank-2 constraint.
   \[ \quad \rightarrow \quad F_q \quad \text{such that:} \]
   \[ \begin{align*}
   q^T \, F_q \, q' &= 0 \\
   \det( F_q ) &= 0
   \end{align*} \]

2. De-normalize $F_q$:
   \[ F = T^T \, F_q \, T' \]
With normalization

Without normalization

Mean errors:
10.0 pixel
9.1 pixel

Mean errors:
1.0 pixel
0.9 pixel
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Example: Parallel image planes

\[ K_1 = K_2 = \text{known} \]

\[ x \text{ parallel to } O_1O_2 \]

\[ E = ? \]

Hint:
\[ R = I \quad T = (T, 0, 0) \]
Essential matrix for parallel images

\[
E = \begin{bmatrix}
0 & -T_z & T_y \\
T_z & 0 & -T_x \\
-T_y & T_x & 0 \\
\end{bmatrix}
\]

\[
R = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -T \\
0 & T & 0 \\
\end{bmatrix}
\]

\[
T = \begin{bmatrix}
T & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
R = I
\]

[Eq. 20]
What are the directions of epipolar lines?

$$l = E \cdot p' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -T & 0 \\ 0 & T & 0 & 0 \end{bmatrix} \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -T T v' \end{bmatrix}$$

horizontal!
Example: Parallel image planes

How are \( p \) and \( p' \) related?

\[
p^T \cdot E \ p' = 0
\]
Example: Parallel image planes

How are $p$ and $p'$ related?

$$\begin{bmatrix} u & v & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix} \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} u & v & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -T \\ T v' \end{bmatrix} = 0 \Rightarrow T v = T v' \Rightarrow v = v'$$
Example: Parallel image planes

Rectification: making two images “parallel”

Why it is useful?
• Epipolar constraint $\rightarrow v = v'$
• New views can be synthesized by linear interpolation
Application: view morphing

Rectification
From its reflection!
The Fundamental Matrix Song