Lecture 13: Fast Reinforcement Learning

Emma Brunskill

CS234 Reinforcement Learning

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- With some slides from or derived from David Silver, Examples new

Refresh Your Understanding: Deterministic Bandits

- Which of the following statements is true in a bandit problem with deterministic rewards?
  1. Upper Confidence Bound (UCB) will have sublinear regret with probability one
  2. Upper Confidence Bound (UCB) will have linear regret with some strictly positive probability
  3. a division by zero will occur in the Upper Confidence Bound (UCB) algorithm and so it cannot handle deterministic environments
  4. Not Sure

Refresh Your Understanding: Deterministic Bandits

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  2. Upper Confidence Bound (UCB) will have linear regret with some strictly positive probability
  3. a division by zero will occur in the Upper Confidence Bound (UCB) algorithm and so it cannot handle deterministic environments
  4. Not Sure

     Solutions: If the domain is deterministic, the average reward for an arm is exactly equal to its true expectation, even if we only have one sample for that arm. Therefore the confidence bounds hold with 100% probability (not just 1-δ probability) and UCB will have a sublinear regret with probability 1.

Class Survey Results

- Thank you to everyone that filled it in!
- The Good: Tutorials (+40), Check your understandings & lectures and math
- Things to Improve: Lectures and math (want more high level structure and conceptual understanding and examples)
- What I Will Change: Will try to emphasize conceptual ideas and provide concrete examples

Where We are

upper confidence bound

- Last time: Bandits and regret and UCB (fast learning)
- This time: Bandits, regret and Bayesian bandits (fast learning)

Today

- Bayesian bandits
- Thompson sampling
- Bayesian Regret

Multiarmed Bandits Notation Recap

- Multi-armed bandit is a tuple of $(\mathcal{A}, \mathcal{R})$ single state MDP
- $\mathcal{A}$: known set of m actions (arms)
- $\mathcal{R}^a(r) = \mathbb{P}[r \mid a]$ is an unknown probability distribution over rewards
- At each step t the agent selects an action $a_t \in \mathcal{A}$
- The environment generates a reward $r_t \sim \mathcal{R}^{a_t}$
- Goal: Maximize cumulative reward $\sum_{\tau=1}^{t} r_\tau$
- **Regret** is the opportunity loss for one step

$$l_t = \mathbb{E}[V^* - Q(a_t)]$$

- **Total Regret** is the total opportunity loss

$$L_t = \mathbb{E}[\sum_{\tau=1}^{t} V^* - Q(a_\tau)]$$

- Maximize cumulative reward $\Longleftrightarrow$ minimize total regret

Today

- **Bayesian bandits**
- Thompson Sampling
- Bayesian Regret

Bayesian bandits

- So far we have made no assumptions about the reward distribution $\mathcal{R}$
  - Except bounds on rewards /subG distrib
- **Bayesian bandits** exploit prior knowledge of rewards, $p[\mathcal{R}]$

Short Refresher on / Introduction to Bayesian Inference

- In Bayesian view, we start with a prior over the unknown parameters
  - Here the unknown distribution over the rewards for each arm $p(R_a)$
- Given observations / data about that parameter, update our uncertainty over the unknown parameters using Bayes Rule

Short Refresher on / Introduction to Bayesian Inference

- In Bayesian view, we start with a prior over the unknown parameters
  - Here the unknown distribution over the rewards for each arm
- Given observations / data about that parameter, update our uncertainty over the unknown parameters using Bayes Rule
- For example, let the reward of arm i be a probability distribution that depends on parameter ϕ_i ex. reward is 0 or 1 ϕ_i prob of getting 1
- Initial prior over ϕ_i is $\underline{p(\phi_i)}$
- Pull arm i and observe reward r_{i1} $\sim R(\phi_i)$
- Use Bayes rule to update estimate over ϕ_i:

$$\overline{p(\phi_i | r_{i1})} = \frac{p(\phi_i, r_{i1})}{p(r_{i1})} = \frac{p(r_{i1} | \phi_i)\, p(\phi_i)}{\int p(r_{i1} | \phi_i) p(\phi_i) d\phi_i}$$

Short Refresher on / Introduction to Bayesian Inference

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- Given observations / data about that parameter, update our uncertainty over the unknown parameters using Bayes Rule
- For example, let the reward of arm i be a probability distribution that depends on parameter ϕ_i
- Initial prior over ϕ_i is $p(\phi_i)$
- Pull arm i and observe reward r_{i1}
- Use Bayes rule to update estimate over ϕ_i:

$$p(\phi_i|r_{i1}) = \frac{p(r_{i1}|\phi_i)p(\phi_i)}{p(r_{i1})} = \frac{p(r_{i1}|\phi_i)p(\phi_i)}{\int_{\phi_i} p(r_{i1}|\phi_i)p(\phi_i)d\phi_i}$$

when is this tractable?

Short Refresher on / Introduction to Bayesian Inference

- In Bayesian view, we start with a prior over the unknown parameters
- Give observations / data about that parameter, update our uncertainty over the unknown parameters using Bayes Rule

$$p(\phi_i|r_{i1}) = \frac{p(r_{i1}|\phi_i)p(\phi_i)}{\int_{\phi_i} p(r_{i1}|\phi_i)p(\phi_i)d\phi_i}$$

- In general computing this update may be tricky to do exactly with no additional structure on the form of the prior and data likelihood

today focus on places where we can do this analytically

Short Refresher on / Introduction to Bayesian Inference: Conjugate

- In Bayesian view, we start with a prior over the unknown parameters
- Give observations / data about that parameter, update our uncertainty over the unknown parameters using Bayes Rule

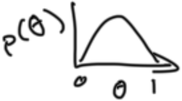
$$p(\phi_i|r_{i1}) = \frac{p(r_{i1}|\phi_i)p(\phi_i)}{\int_{\phi_i} p(r_{i1}|\phi_i)p(\phi_i)d\phi_i}$$

- In general computing this update may be tricky
- But sometimes can be done analytically
- If the parametric representation of the prior and posterior is the same, the prior and model are called **conjugate**
- For example, exponential families have conjugate priors

Short Refresher on / Introduction to Bayesian Inferencee: Bernoulli

- Consider a bandit problem where the reward of an arm is a binary outcome 0, 1, sampled from a Bernoulli with parameter θ
  - E.g. Advertisement click through rate, patient treatment success/fails, ... many binary reward applications
- The Beta distribution *Beta*(α, β) is conjugate for the Bernoulli distribution

θ is the Bernoulli parameter



$$p(\theta|\alpha,\beta) = \theta^{\alpha-1}(1-\theta)^{\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

where $\Gamma(x)$ is the Gamma family

Short Refresher on / Introduction to Bayesian Inference: Bernoulli

- Consider a bandit problem where the reward of an arm is a binary outcome 0, 1, sampled from a Bernoulli with parameter θ
  - E.g. Advertisement click through rate, patient treatment success/fails, ...
- The Beta distribution $Beta(\alpha, \beta)$ is conjugate for the Bernoulli distribution

$$p(\theta|\alpha, \beta) = \theta^{\alpha-1}(1-\theta)^{\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

  where $\Gamma(x)$ is the Gamma family
- Assume the prior over θ is $Beta(\alpha, \beta)$ as above
- Then after observed a reward $r \in \{0, 1\}$ then updated posterior over θ is $Beta(\underbrace{r+\alpha}, 1-r+\beta) \longleftarrow$ "counts" of how many times observed 0

  "counts" of how many times observed 1

Bayesian Inference for Decision Making

- Maintain distribution over reward parameters
- Use this to inform action selection

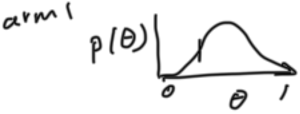
Bayesian bandits Overview

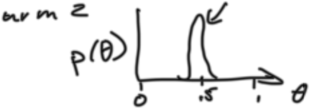
- So far we have made no assumptions about the reward distribution $\mathcal{R}$
  - Except bounds on rewards
- **Bayesian bandits** exploit prior knowledge of rewards, $p[\mathcal{R}]$
- They compute posterior distribution of rewards $p[\mathcal{R} \mid h_t]$, where $h_t = (a_1, r_1, \ldots, a_{t-1}, r_{t-1})$
- Use posterior to guide exploration
  - Upper confidence bounds (Bayesian UCB)
  - Probability matching (Thompson Sampling)
- Better performance if prior knowledge is accurate

Today

- Bandits and Probably Approximately Correct
- Bayesian bandits
- Thompson Sampling
- Bayesian Regret

Thompson Sampling





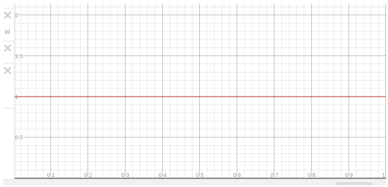
Initialize prior over each arm a, $p(\mathcal{R}_a)$
for iteration$=1, 2, \dots$ **do** for ex. $\theta_1 = .3$ $\theta_2 = .5$
  For each arm a **sample** a reward distribution $\mathcal{R}_a$ from posterior
  Compute action-value function $Q(a) = \mathbb{E}[\mathcal{R}_a]$
  $a_t = \arg \underline{\max_{a \in \mathcal{A}} Q(a)}$
  Observe reward r
  Update posterior $p(\mathcal{R}_a)$ using Bayes Rule
end for

Toy Example: Ways to Treat Broken Toes, Thompson Sampling

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose Beta(1,1) (Uniform)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,1):

Beta(1,1) distrib

$p(\theta)$



0 θ 1

Toy Example: Ways to Treat Broken Toes, Thompson Sampling[1]

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,1): 0.3 0.5 0.6
  2. Select $a = \arg\max_{a \in A} Q(a) = \arg\max_{ainA} \theta(a) =$ a_3

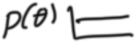
observe a 0

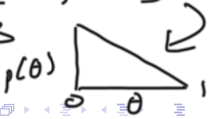
[1] Note: This is a made up example. This is not the actual expected efficacies of the various treatment options for a broken toe

Toy Example: Ways to Treat Broken Toes, Thompson Sampling

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose $\theta_i \sim$Beta(1,1)
  1. Per arm, sample a Bernoulli θ given prior: 0.3 0.5 0.6
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{ainA} \theta(a) = 3$
  3. Observe the patient outcome's outcome: 0 $r \sim \theta_3$
  4. Update the posterior over the $Q(a_t) = Q(a^3)$ value for the arm pulled

$\underbrace{\text{Beta}(1,1)}_{a1}$ $\underbrace{\text{Beta}(1,1)}_{a2}$ $\underbrace{\text{Beta}(1,2)}_{a3}$





Toy Example: Ways to Treat Broken Toes, Thompson Sampling

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  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,1): 0.3 0.5 0.6
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{ainA} \theta(a) = 3$
  3. Observe the patient outcome's outcome: 0
  4. Update the posterior over the $Q(a_t) = Q(a^1)$ value for the arm pulled
     - Beta(c_1, c_2) is the conjugate distribution for Bernoulli
     - If observe 1, $c_1 + 1$ else if observe 0 $c_2 + 1$
  5. New posterior over Q value for arm pulled is:
  6. New posterior $p(Q(a^3)) = p(\theta(a_3) = Beta(1, 2)$

Toy Example: Ways to Treat Broken Toes, Thompson Sampling

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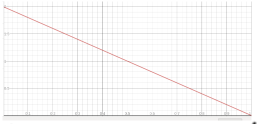
  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,1): 0.3 0.5 0.6
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{ainA} \theta(a) = 1$
  3. Observe the patient outcome's outcome: 0
  4. New posterior $p(Q(a^1)) = p(\theta(a_1) = Beta(1, 2)$

$p(\theta)$



0 θ 1

Toy Example: Ways to Treat Broken Toes, Thompson Sampling

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- Thompson sampling:
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  1. Sample a Bernoulli parameter given current prior over each arm
     Beta(1,1), Beta(1,1), Beta(1,2): 0.7, 0.5, 0.3

     uniform distrib

Toy Example: Ways to Treat Broken Toes, Thompson Sampling

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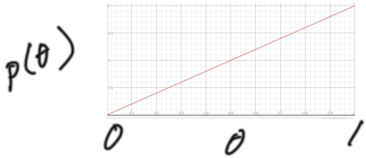
- Place a prior over each arm's parameter. Here choose $\theta_i \sim$Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,2): 0.7, 0.5, 0.3
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{ainA} \theta(a) = 1$
  3. Observe the patient outcome's outcome: 1 $\quad r \sim \theta_1$
  4. New posterior $p(Q(a^1)) = p(\theta(a_1) = Beta(2,1)$

Beta(α+r, β+1−r)



Beta(2,1) a_1 $\quad$ Beta(1,1) a_2

Beta(1,2) a_3

Toy Example: Ways to Treat Broken Toes, Thompson Sampling

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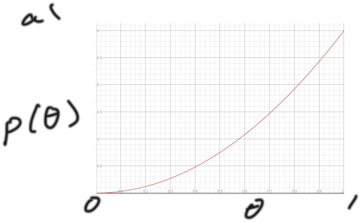
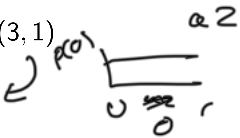
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose $\theta_i \sim$Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(2,1), Beta(1,1), Beta(1,2): 0.71, 0.65, 0.1
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{ainA} \theta(a) = 1$
  3. Observe the patient outcome's outcome: 1
  4. New posterior $p(Q(a^1)) = p(\theta(a_1) = Beta(3, 1)$





Toy Example: Ways to Treat Broken Toes, Thompson Sampling

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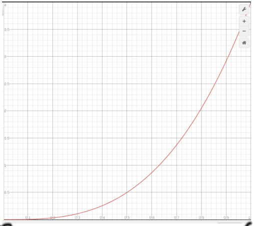
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose $\theta_i \sim$Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm
     3,1 → Beta(2,1), Beta(1,1), Beta(1,2): 0.75, 0.45, 0.4
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{ainA} \theta(a) = 1$
  3. Observe the patient outcome's outcome: 1
  4. New posterior $p(Q(a^1)) = p(\theta(a_1) = Beta(4, 1)$

a2

p(θ)



0 θ 1

Toy Example: Ways to Treat Broken Toes, Thompson Sampling vs Optimism

- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- How does the sequence of arm pulls compare in this example so far?

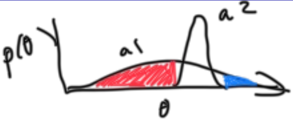
Optimism	TS
a^1	a^3
a^2	a^1
a^3	a^1
a^1	a^1
a^2	a^1

upper confid bound from prior lectures

On to General Setting for Thompson Sampling

- Now we will see how Thompson sampling works in general, and what it is doing

Probability Matching



- Assume have a parametric distribution over rewards for each arm
- **Probability matching** selects action a according to probability that a is the optimal action

history (prior arms pulled & their record outcomes)

$$\pi(a \mid h_t) = \mathbb{P}[Q(a) > Q(a'), \forall a' \neq a \mid h_t]$$

- Can be difficult to compute probability that an action is optimal analytically from posterior
- Somewhat incredibly, Thompson sampling implements probability matching

Thompson sampling implements probability matching

TS chooses arm a at time t by sampling

$\phi_a \sim \underbrace{p(\phi_a / D)}_{\text{posterior}}$ $\forall a$ D=data/history

select $a_t = \arg\max_a E[R(\phi_a)]$

prob under TS that we select arm a:

$$P(a_t = a) = \int_\phi \mathbb{1}\left[E[r_a | \phi_a] = \max_{a'} E[r_{a'} | \phi_{a'}]\right] p(\phi | D)\, d\phi$$

$$= p\left(E[r_a | \phi_a] = \max_{a'} E[r_{a'} | \phi_{a'}] \,\middle|\, D\right)$$

Thompson sampling implements probability matching

- Thompson sampling:

$$\begin{aligned}\pi(a \mid h_t) &= \mathbb{P}[Q(a) > Q(a'), \forall a' \neq a \mid h_t] \\ &= \mathbb{E}_{\mathcal{R}|h_t}\left[\mathbb{1}(a = \arg\max_{a \in \mathcal{A}} Q(a))\right]\end{aligned}$$

Probability Matching

- Assume have a parametric distribution over rewards for each arm
- **Probability matching** selects action a according to probability that a is the optimal action

$$\pi(a \mid h_t) = \mathbb{P}[Q(a) > Q(a'), \forall a' \neq a \mid h_t]$$

- Can be difficult to compute probability that an action is optimal analytically from posterior
- Somewhat incredibly, Thompson sampling implements probability matching
- Note: probability matching is often optimistic in the face of uncertainty
    - Uncertain actions have higher probability of being max

Today

- Bandits and Probably Approximately Correct
- Bayesian bandits
- Thompson Sampling
- Bayesian Regret

Framework: Regret and Bayesian Regret

- How do we evaluate performance in the Bayesian setting?
- Frequentist regret assumes a true (unknown) set of parameters

$$Regret(\mathcal{A}, T; \theta) = \mathbb{E}_{\tau}\left[\sum_{t=1}^{T} Q(a^*) - Q(a_t) | \theta\right]$$

where $\mathbb{E}_{\tau}$ denotes an expectation with respect to the history of actions taken and rewards observed given an algorithm $\mathcal{A}$.

- Bayesian regret assumes there is a prior over parameters

$$BayesRegret(\mathcal{A}, T; \theta) = \underline{\mathbb{E}_{\theta \sim p_{\theta}, \tau}}\left[\sum_{t=1}^{T} Q(a^*) - Q(a_t) | \theta\right]$$

Bounding Regret Using Optimism

- How do we evaluate performance in the Bayesian setting?
- Frequentist regret assumes a true (unknown) set of parameters

$$Regret(\mathcal{A}, T; \theta) = \mathbb{E}_\tau \left[\sum_{t=1}^{T} Q(a^*) - Q(a_t) | \theta \right] \leq \mathbb{E}_\tau \left[\sum_{t=1}^{T} U_t(a_t) - Q(a_t) | \theta \right]$$

where $\mathbb{E}_\tau$ denotes an expectation with respect to the history of actions taken and rewards observed given an algorithm $\mathcal{A}$ (under event that U_t is an upper bound).

Thompson sampling

- Frequentist bounds for standard* Thompson sampling do not* (last checked) match best bounds for frequentist algorithms
- Empirically Thompson sampling can be effective, especially in contextual multi-armed bandits

Contextual Bandit Thompson Sampling

$r(s,a)$

state

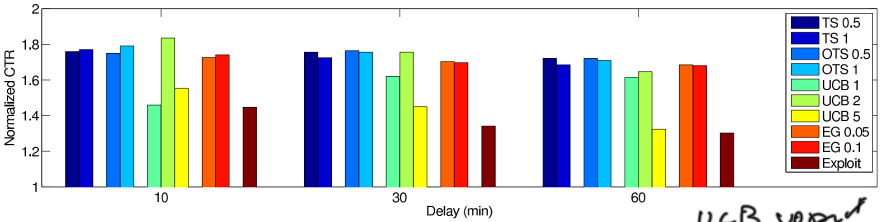
- Contextual bandit: input context which impacts reward of each arm, context sampled iid each step
- Arms = articles
- Reward = click (+1) on article ($Q(a)$=click through rate)

but actions don't impact next state

Thompson Sampling for News Article Recommendation (Chapelle and Li, 2010)

- Contextual bandit: input context which impacts reward of each arm, context sampled iid each step
- Arms = articles
- Reward = click (+1) on article ($Q(a)$=click through rate)

EG = epsilon greedy



UCB upper conf

Check Your Understanding: Thompson Sampling and Optimism

- Consider an online news website with thousands of people logging on each second. Frequently a new person will come online before we see whether the last person has clicked (or not). Select all that are true:
  1. Thompson sampling would be better than optimism here, because optimism algorithms are deterministic and would select the same action until we get feedback (click or not)
  2. Optimism algorithms would be better than TS here, because they have stronger regret bounds for this setting
  3. Thompson sampling could cause much worse performance than optimism if the initial prior is very misleading.
  4. Not sure

Check Your Understanding: Thompson Sampling and Optimism **Solutions**

- Consider an online news website with thousands of people logging on each second. Frequently a new person will come online before we see whether the last person has clicked (or not). Select all that are true:
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  3. Thompson sampling could cause much worse performance than optimism if the initial prior is very misleading.
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  Solution: (1) T (2) F (3) T. Consider prior Beta(100,1) for a Bernoulli arm with parameter 0.1. Then the prior puts large weight on high values of theta for a long time.

Optimal Policy for Bayesian bandits?

- Thompson Sampling often works well, but is it optimal?
- Given prior, and known horizon, could compute decision policy that would maximize expected rewards given the available horizon
- Computational challenge: naively this would create a decision policy that is a function of the history to the next arm to pull

Gittins Index for Bayesian bandits

- Thompson Sampling often works well, but is it optimal?
- Given prior, and known horizon, could compute decision policy that would maximize expected rewards given the available horizon
- Computational challenge: naively this would create a decision policy that is a function of the history to the next arm to pull
- **Index policy**: a decision policy that computes a "real-valued index for each arm and plays the arm with the largest index," using statistics only from that arm and the horizon (definition from Lattimore and Svespari 2019 Bandit Algorithms)
- **Gittins index**: optimal policy for maximizing expected discounted reward in a Bayesian multi-armed bandit

Today

- Bandits and Probably Approximately Correct
- Bayesian bandits
- Thompson Sampling
- Bayesian Regret

What You Should Understand

- Understand how multi-armed bandits relate to MDPs
- Be able to define regret ~~and PAC~~
- Be able to prove why UCB bandit algorithm has sublinear regret
- Understand (be able to give an example) why e-greedy and greedy and pessimism can result in linear regret
- Be able to implement UCB bandit algorithm
- Be able to implement Thompson Sampling for Bernoulli rewards

Where We are

- Last time: Bandits and regret and UCB (fast learning)
- This time: Bayesian bandits (fast learning)

Bayesian Regret Bounds for Thompson Sampling

- Regret(UCB,T)

$$BayesRegret(TS, T) = E_{\theta \sim p_\theta}\left[\sum_{t=1}^{T} f^*(a^*) - f^*(a_t)\right]$$

- Posterior sampling has the same (ignoring constants) regret bounds as UCB

Toy Example: Probably Approximately Correct and Regret

- Surgery: $\phi_1 = .95$ / Taping: $\phi_2 = .9$ / Nothing: $\phi_3 = .1$
- Let $\epsilon = 0.05$
- O = Optimism, TS = Thompson Sampling: W/in $\epsilon = \mathbb{I}(Q(a_t) \geq Q(a^*) - \epsilon)$

O	TS	Optimal	O Regret	O W/in ϵ	TS Regret	TS W/in ϵ
a^1	a^3	a^1	0	Y	0.85	N
a^2	a^1	a^1	0.05	Y	0	Y
a^3	a^1	a^1	0.85	N	0	Y
a^1	a^1	a^1	0	Y	0	Y
a^2	a^1	a^1	0.05	Y	0	Y