# Lecture 2: Making Sequences of Good Decisions Given a Model of the World

Emma Brunskill

CS234 Reinforcement Learning

Winter 2026

# L2N1 Quick Check Your Understanding 1. Participation Poll

In a Markov decision process, a large discount factor $\gamma$ means that short term rewards are much more influential than long term rewards. [Enter your answer in participation poll ]

- True
- False
- Don't know

Question for today's lecture (not for poll): Can we construct algorithms for computing decision policies so that we can guarantee with additional computation / iterations, we monotonically improve the decision policy? Do all algorithms satisfy this property?

# L2N1 Quick Check Your Understanding 1. Participation Poll

In a Markov decision process, a large discount factor $\gamma$ means that short term rewards are much more influential than long term rewards. [Enter your answer in the poll]

- True
- False
- Don't know

Question for today's lecture (not for poll): Can we construct algorithms for computing decision policies so that we can guarantee with additional computation / iterations, we monotonically improve the decision policy? Do all algorithms satisfy this property?

# Class Tasks and Updates

- **Friendly reminder: tutorial sign up! These start next week.**
- Homework 1 out by Friday. Due next Friday at 6pm.
- Office hours will start next week. See Ed for days, times of group and 1:1 office hours and we will also share information about location and/or zoom links.

# Today's Plan

- Last Time:
    - Introduction
    - Components of an agent: model, value, policy
- This Time:
    - Making good decisions given a Markov decision process
- Next Time:
    - Policy evaluation when don't have a model of how the world works

# Today: Given a model of the world

- Markov Processes (last time)
- Markov Reward Processes (MRPs) (continue from last time)
- Markov Decision Processes (MDPs)
- Evaluation and Control in MDPs

# Iterative Algorithm for Computing Value of a MRP

- Dynamic programming
- Initialize $V_0(s) = 0$ for all $s$
- For $k = 1$ until convergence
  - For all $s$ in $S$
$$V_k(s) = R(s) + \gamma \sum_{s' \in S} P(s'|s) V_{k-1}(s')$$
- Computational complexity: $O(|S|^2)$ for each iteration ($|S| = N$)

# Markov Decision Process (MDP)

- Markov Decision Process is Markov Reward Process + actions
- Definition of MDP
  - $S$ is a (finite) set of Markov states $s \in S$
  - $A$ is a (finite) set of actions $a \in A$
  - $P$ is dynamics/transition model for **each action**, that specifies $P(s_{t+1} = s' | s_t = s, a_t = a)$
  - $R$ is a reward function[1]

$$R(s_t = s, a_t = a) = \mathbb{E}[r_t | s_t = s, a_t = a]$$

  - Discount factor $\gamma \in [0, 1]$
- MDP is a tuple: $(S, A, P, R, \gamma)$

[1] Reward is sometimes defined as a function of the current state, or as a function of the (state, action, next state) tuple. Most frequently in this class, we will assume reward is a function of state and action

# Example: Mars Rover MDP

| $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ | $s_7$ |
|---|---|---|---|---|---|---|
| | | |  | | | |

$$P(s'|s,a_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad P(s'|s,a_2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- 2 deterministic actions

# MDP Policies

- Policy specifies what action to take in each state
  - Can be deterministic or stochastic
- For generality, consider as a conditional distribution
  - Given a state, specifies a distribution over actions
- Policy: $\pi(a|s) = P(a_t = a|s_t = s)$

# MDP + Policy

- MDP + $\pi(a|s)$ = Markov Reward Process
- Precisely, it is the MRP $(S, R^\pi, P^\pi, \gamma)$, where

$$R^\pi(s) = \sum_{a \in A} \pi(a|s) R(s,a)$$
$$P^\pi(s'|s) = \sum_{a \in A} \pi(a|s) P(s'|s,a)$$

- Implies we can use same techniques to evaluate the value of a policy for a MDP as we could to compute the value of a MRP, by defining a MRP with $R^\pi$ and $P^\pi$

# MDP Policy Evaluation, Iterative Algorithm

- Initialize $V_0(s) = 0$ for all $s$
- For $k = 1$ until convergence
  - For all $s$ in $S$

$$V_k^\pi(s) = \sum_a \pi(a|s) \left[ R(s,a) + \gamma \sum_{s' \in S} p(s'|s,a) V_{k-1}^\pi(s') \right]$$

- This is a **Bellman backup** for a particular policy
- Note that if the policy is deterministic then the above update simplifies to

$$V_k^\pi(s) = R(s,\pi(s)) + \gamma \sum_{s' \in S} p(s'|s,\pi(s)) V_{k-1}^\pi(s')$$

# Exercise L2E1: MDP 1 Iteration of Policy Evaluation, Mars Rover Example

- Dynamics: $p(s_6|s_6, a_1) = 0.5$, $p(s_7|s_6, a_1) = 0.5$, $\ldots$
- Reward: for all actions, $+1$ in state $s_1$, $+10$ in state $s_7$, 0 otherwise
- Let $\pi(s) = a_1 \; \forall s$, assume $V_k =$[1 0 0 0 0 0 10] and $k = 1$, $\gamma = 0.5$
- Compute $V_{k+1}(s_6)$

See answer at the end of the slide deck. If you'd like practice, work this out and then check your answers.

# Check Your Understanding Poll L2N2

| $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ | $s_7$ |
|---|---|---|---|---|---|---|
| | | |  | | | |

- We will shortly be interested in not just evaluating the value of a single policy, but finding an optimal policy. Given this it is informative to think about properties of the potential policy space.
- First for the Mars rover example [ 7 discrete states (location of rover); 2 actions: Left or Right]
- How many deterministic policies are there?
- Select answer on the participation poll: 2 / 14 / $7^2$ / $2^7$ / Not sure
- Is the optimal policy (one with highest value) for a MDP unique?
- Select answer on the participation poll: Yes / No / Not sure

# Check Your Understanding L2N2

| $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ | $s_7$ |
|---|---|---|---|---|---|---|
| | | |  | | | |

- 7 discrete states (location of rover)
- 2 actions: Left or Right
- How many deterministic policies are there?

- Is the highest reward policy for a MDP always unique?

# MDP Control

- Compute the optimal policy

$$\pi^*(s) = \arg\max_{\pi} V^{\pi}(s)$$

- There **exists a unique optimal value function**
- Optimal policy for a MDP in an infinite horizon problem is deterministic

# MDP Control

- Compute the optimal policy

$$\pi^*(s) = \arg\max_{\pi} V^{\pi}(s)$$

- There exists a unique optimal value function
- Optimal policy for a MDP in an infinite horizon problem (agent acts forever is
  - Deterministic
  - Stationary (does not depend on time step)
  - Unique? Not necessarily, may have two policies with identical (optimal) values

# Policy Search

- One option is searching to compute best policy
- Number of deterministic policies is $|A|^{|S|}$
- Policy iteration is generally more efficient than enumeration

# MDP Policy Iteration (PI)

- Set $i = 0$
- Initialize $\pi_0(s)$ randomly for all states $s$
- While $i == 0$ or $\|\pi_i - \pi_{i-1}\|_1 > 0$ (L1-norm, measures if the policy changed for any state):
  - $V^{\pi_i} \leftarrow$ MDP V function policy **evaluation** of $\pi_i$
  - $\pi_{i+1} \leftarrow$ Policy **improvement**
  - $i = i + 1$

# New Definition: State-Action Value Q

- State-action value of a policy

$$Q^{\pi}(s, a) = R(s, a) + \gamma \sum_{s' \in S} P(s'|s, a) V^{\pi}(s')$$

- Take action $a$, then follow the policy $\pi$

# Policy Improvement

- Compute state-action value of a policy $\pi_i$
  - For $s$ in $S$ and $a$ in $A$:

$$Q^{\pi_i}(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_i}(s')$$

- Compute new policy $\pi_{i+1}$, for all $s \in S$

$$\pi_{i+1}(s) = \arg\max_a Q^{\pi_i}(s,a) \; \forall s \in S$$

# MDP Policy Iteration (PI)

- Set $i = 0$
- Initialize $\pi_0(s)$ randomly for all states $s$
- While $i == 0$ or $\|\pi_i - \pi_{i-1}\|_1 > 0$ (L1-norm, measures if the policy changed for any state):
  - $V^{\pi_i} \leftarrow$ MDP V function policy **evaluation** of $\pi_i$
  - $\pi_{i+1} \leftarrow$ Policy **improvement**
  - $i = i + 1$

# Delving Deeper Into Policy Improvement Step

$$Q^{\pi_i}(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_i}(s')$$

# Delving Deeper Into Policy Improvement Step

$$Q^{\pi_i}(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_i}(s')$$

$$\max_a Q^{\pi_i}(s,a) \geq R(s,\pi_i(s)) + \gamma \sum_{s' \in S} P(s'|s,\pi_i(s)) V^{\pi_i}(s') = V^{\pi_i}(s)$$

$$\pi_{i+1}(s) = \arg\max_a Q^{\pi_i}(s,a)$$

- Suppose we take $\pi_{i+1}(s)$ for one action, then follow $\pi_i$ forever
  - Our expected sum of rewards is at least as good as if we had always followed $\pi_i$
- But new proposed policy is to always follow $\pi_{i+1}$ ...

# Monotonic Improvement in Policy

- Definition

$$V^{\pi_1} \geq V^{\pi_2} : V^{\pi_1}(s) \geq V^{\pi_2}(s), \forall s \in S$$

- Proposition: $V^{\pi_{i+1}} \geq V^{\pi_i}$ with strict inequality if $\pi_i$ is suboptimal, where $\pi_{i+1}$ is the new policy we get from policy improvement on $\pi_i$

# Proof: Monotonic Improvement in Policy

$$\begin{aligned} V^{\pi_i}(s) &\leq \max_a Q^{\pi_i}(s,a) \\ &= \max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_i}(s') \end{aligned}$$

# Proof: Monotonic Improvement in Policy

# Check Your Understanding L2N3: Policy Iteration (PI)

- Note: all the below is for finite state-action spaces
- Set $i = 0$
- Initialize $\pi_0(s)$ randomly for all states $s$
- While $i == 0$ or $\|\pi_i - \pi_{i-1}\|_1 > 0$ (L1-norm, measures if the policy changed for any state):
  - $V^{\pi_i} \leftarrow$ MDP V function policy **evaluation** of $\pi_i$
  - $\pi_{i+1} \leftarrow$ Policy **improvement**
  - $i = i + 1$
- **If policy doesn't change, can it ever change again?**
- Select on participation poll: Yes / No / Not sure
- **Is there a maximum number of iterations of policy iteration?**
- Select on participation poll: Yes / No / Not sure

# Lecture Break after Policy Iteration

# Results for Check Your Understanding L2N3 Policy Iteration

- Note: all the below is for finite state-action spaces
- Set $i = 0$
- Initialize $\pi_0(s)$ randomly for all states $s$
- While $i == 0$ or $\|\pi_i - \pi_{i-1}\|_1 > 0$ (L1-norm, measures if the policy changed for any state):
  - $V^{\pi_i} \leftarrow$ MDP V function policy **evaluation** of $\pi_i$
  - $\pi_{i+1} \leftarrow$ Policy **improvement**
  - $i = i + 1$
- **If policy doesn't change, can it ever change again?**

- **Is there a maximum number of iterations of policy iteration?**

# Check Your Understanding Explanation of Policy Not Changing

- Suppose for all $s \in S$, $\pi_{i+1}(s) = \pi_i(s)$
- Then for all $s \in S$, $Q^{\pi_{i+1}}(s, a) = Q^{\pi_i}(s, a)$
- Recall policy improvement step

$$Q^{\pi_i}(s, a) = R(s, a) + \gamma \sum_{s' \in S} P(s'|s, a) V^{\pi_i}(s')$$

$$\pi_{i+1}(s) = \arg\max_a Q^{\pi_i}(s, a)$$

$$\pi_{i+2}(s) = \arg\max_a Q^{\pi_{i+1}}(s, a) = \arg\max_a Q^{\pi_i}(s, a)$$

# MDP: Computing Optimal Policy and Optimal Value

- Policy iteration computes infinite horizon value of a policy and then improves that policy
- Value iteration is another technique
  - Idea: Maintain optimal value of starting in a state $s$ if have a finite number of steps $k$ left in the episode
  - Iterate to consider longer and longer episodes

# Bellman Equation and Bellman Backup Operators

- Value function of a policy must satisfy the Bellman equation

$$V^\pi(s) = R^\pi(s) + \gamma \sum_{s' \in S} P^\pi(s'|s) V^\pi(s')$$

- Bellman backup operator
  - Applied to a value function
  - Returns a new value function
  - Improves the value if possible

$$BV(s) = \max_a \left[ R(s,a) + \gamma \sum_{s' \in S} p(s'|s,a) V(s') \right]$$

  - $BV$ yields a value function over all states $s$

# Value Iteration (VI)

- Set $k = 1$
- Initialize $V_0(s) = 0$ for all states $s$
- Loop until convergence: (for ex. $||V_{k+1} - V_k||_\infty \leq \epsilon$)
  - For each state $s$

$$V_{k+1}(s) = \max_a \left[ R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') \right]$$

  - View as Bellman backup on value function

$$V_{k+1} = BV_k$$

$$\pi_{k+1}(s) = \arg\max_a \left[ R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') \right]$$

# Policy Iteration as Bellman Operations

- Bellman backup operator $B^\pi$ for a particular policy is defined as

$$B^\pi V(s) = R^\pi(s) + \gamma \sum_{s' \in S} P^\pi(s'|s)V(s')$$

- Policy evaluation amounts to computing the fixed point of $B^\pi$
- To do policy evaluation, repeatedly apply operator until $V$ stops changing

$$V^\pi = B^\pi B^\pi \cdots B^\pi V$$

# Policy Iteration as Bellman Operations

- Bellman backup operator $B^\pi$ for a particular policy is defined as

$$B^\pi V(s) = R^\pi(s) + \gamma \sum_{s' \in S} P^\pi(s'|s) V(s')$$

- To do policy improvement

$$\pi_{k+1}(s) = \arg\max_a \left[ R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_k}(s') \right]$$

# Going Back to Value Iteration (VI)

- Set $k = 1$
- Initialize $V_0(s) = 0$ for all states $s$
- Loop until convergence: (for ex. $||V_{k+1} - V_k||_\infty \leq \epsilon$)
  - For each state $s$

$$V_{k+1}(s) = \max_a \left[ R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') \right]$$

  - Equivalently, in Bellman backup notation

$$V_{k+1} = BV_k$$

- To extract optimal policy if can act for $k+1$ more steps,

$$\pi(s) = \arg\max_a \left[ R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_{k+1}(s') \right]$$

# Contraction Operator

- Let $O$ be an operator,and $|x|$ denote (any) norm of $x$
- If $|OV - OV'| \leq |V - V'|$, then $O$ is a contraction operator

# Will Value Iteration Converge?

- Yes, if discount factor $\gamma < 1$, or end up in a terminal state with probability 1
- Bellman backup is a contraction if discount factor, $\gamma < 1$
- If apply it to two different value functions, distance between value functions shrinks after applying Bellman equation to each

# Proof: Bellman Backup is a Contraction on $V$ for $\gamma < 1$

- Let $\|V - V'\| = \max_s |V(s) - V'(s)|$ be the infinity norm

$$\|BV_k - BV_j\| = \left\| \max_a \left( R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') \right) - \max_{a'} \left( R(s,a') + \gamma \sum_{s' \in S} P(s'|s,a') V_j(s') \right) \right\|$$

# Proof: Bellman Backup is a Contraction on $V$ for $\gamma < 1$

# Opportunities for Out-of-Class Practice

- Prove value iteration converges to a unique solution for discrete state and action spaces with $\gamma < 1$
- Does the initialization of values in value iteration impact anything?
- Is the value of the policy extracted from value iteration at each round guaranteed to monotically improve (if executed in the real infinite horizon problem), like policy iteartion?

# Value Iteration for Finite Horizon $H$

$V_k$ = optimal value if making $k$ more decisions
$\pi_k$ = optimal policy if making $k$ more decisions

- Initialize $V_0(s) = 0$ for all states $s$
- For $k = 1 : H$
  - For each state $s$

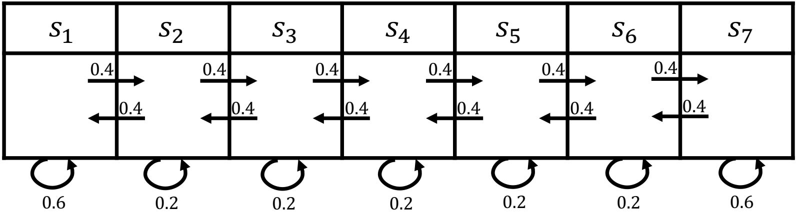
$$V_{k+1}(s) = \max_a \left[ R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') \right]$$

$$\pi_{k+1}(s) = \arg\max_a \left[ R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') \right]$$

# Computing the Value of a Policy in a Finite Horizon

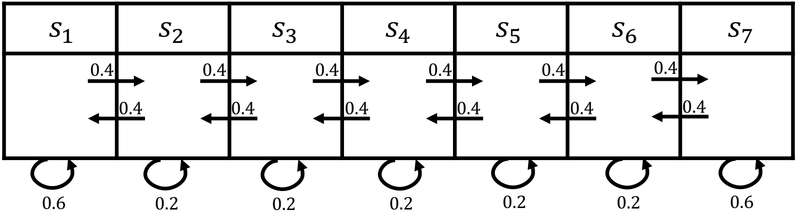
- Alternatively can estimate by simulation
  - Generate a large number of episodes
  - Average returns
  - Concentration inequalities bound how quickly average concentrates to expected value
  - Requires **no assumption** of Markov structure

# Example: Mars Rover




- Reward: $+1$ in $s_1$, $+10$ in $s_7$, 0 in all other states
- Sample returns for sample 4-step ($H$=4) episodes, $\gamma = 1/2$
  - $s_4, s_5, s_6, s_7$: $0 + \frac{1}{2} \times 0 + \frac{1}{4} \times 0 + \frac{1}{8} \times 10 = 1.25$

# Example: Mars Rover




- Reward: $+1$ in $s_1$, $+10$ in $s_7$, 0 in all other states
- Sample returns for sample 4-step ($H$=4) episodes, start state $s_4$ , $\gamma = 1/2$
  - $s_4, s_5, s_6, s_7$: $0 + \frac{1}{2} \times 0 + \frac{1}{4} \times 0 + \frac{1}{8} \times 10 = 1.25$
  - $s_4, s_4, s_5, s_4$: $0 + \frac{1}{2} \times 0 + \frac{1}{4} \times 0 + \frac{1}{8} \times 0 = 0$
  - $s_4, s_3, s_2, s_1$: $0 + \frac{1}{2} \times 0 + \frac{1}{4} \times 0 + \frac{1}{8} \times 1 = 0.125$

# Question: Finite Horizon Policies

- Set $k = 1$
- Initialize $V_0(s) = 0$ for all states $s$
- Loop until $k == H$:
  - For each state $s$

$$V_{k+1}(s) = \max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

$$\pi_{k+1}(s) = \arg\max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

Is optimal policy stationary (independent of time step) in finite horizon tasks?

# Question: Finite Horizon Policies

- Set $k = 1$
- Initialize $V_0(s) = 0$ for all states $s$
- Loop until $k == H$:
  - For each state $s$

$$V_{k+1}(s) = \max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

$$\pi_{k+1}(s) = \arg\max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

Is optimal policy stationary (independent of time step) in finite horizon tasks?

# Value vs Policy Iteration

- Value iteration:
  - Compute optimal value for horizon $= k$
    - Note this can be used to compute optimal policy if horizon $= k$
  - Increment $k$
- Policy iteration
  - Compute infinite horizon value of a policy
  - Use to select another (better) policy
  - Closely related to a very popular method in RL: policy gradient

# RL Terminology: Models, Policies, Values

- **Model**: Mathematical models of dynamics and reward
- **Policy**: Function mapping states to actions
- **Value function**: future rewards from being in a state and/or action when following a particular policy

# What You Should Know

- Define MP, MRP, MDP, Bellman operator, contraction, model, Q-value, policy
- Be able to implement
  - Value Iteration
  - Policy Iteration
- Give pros and cons of different policy evaluation approaches
- Be able to prove contraction properties
- Limitations of presented approaches and Markov assumptions
  - Which policy evaluation methods require the Markov assumption?

# Where We Are

- Last Time:
  - Introduction
  - Components of an agent: model, value, policy
- This Time:
  - Making good decisions given a Markov decision process
- Next Time:
  - Policy evaluation when don't have a model of how the world works

# Exercise L2E1: MDP 1 Iteration of Policy Evaluation, Mars Rover Example, Answer

- Dynamics: $p(s_6|s_6, a_1) = 0.5$, $p(s_7|s_6, a_1) = 0.5$, ...
- Reward: for all actions, $+1$ in state $s_1$, $+10$ in state $s_7$, 0 otherwise
- Let $\pi(s) = a_1 \ \forall s$, assume $V_k$ =[1 0 0 0 0 0 10] and $k = 1$, $\gamma = 0.5$
- Compute $V_{k+1}(s_6)$

$$\begin{aligned} V_{k+1}(s_6) &= r(s_6) + \gamma \sum_{s'} p(s'|s_6, a_1) V_k(s') & (1) \\ &= 0 + 0.5 * (0.5 * 10 + 0.5 * 0) & (2) \\ &= 2.5 & (3) \end{aligned}$$

# Check Your Understanding L2N1: MDP 1 Iteration of Policy Evaluation, Mars Rover Example

- Dynamics: $p(s_6|s_6, a_1) = 0.5$, $p(s_7|s_6, a_1) = 0.5$, ...
- Reward: for all actions, $+1$ in state $s_1$, $+10$ in state $s_7$, 0 otherwise
- Let $\pi(s) = a_1\ \forall s$, assume $V_k =$[1 0 0 0 0 0 10] and $k = 1$, $\gamma = 0.5$
- $$V_k^{\pi}(s) = r(s, \pi(s)) + \gamma \sum_{s' \in S} p(s'|s, \pi(s)) V_{k-1}^{\pi}(s')$$

# Return & Value Function

- Definition of Horizon ($H$)
  - Number of time steps in each episode
  - Can be infinite
  - Otherwise called **finite** Markov reward process
- Definition of Return, $G_t$ (for a MRP)
  - Discounted sum of rewards from time step $t$ to horizon $H$

$$G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \cdots + \gamma^{H-1} r_{t+H-1}$$

- Definition of State Value Function, $V(s)$ (for a MRP)
  - Expected return from starting in state $s$

$$V(s) = \mathbb{E}[G_t | s_t = s] = \mathbb{E}[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \cdots + \gamma^{H-1} r_{t+H-1} | s_t = s]$$

# Computing the Value of an Infinite Horizon Markov Reward Process

- Markov property provides structure
- MRP value function satisfies

$$V(s) = \underbrace{R(s)}_{\text{Immediate reward}} + \underbrace{\gamma \sum_{s' \in S} P(s'|s) V(s')}_{\text{Discounted sum of future rewards}}$$

# Matrix Form of Bellman Equation for MRP

- For finite state MRP, we can express $V(s)$ using a matrix equation

$$\begin{pmatrix} V(s_1) \\ \vdots \\ V(s_N) \end{pmatrix} = \begin{pmatrix} R(s_1) \\ \vdots \\ R(s_N) \end{pmatrix} + \gamma \begin{pmatrix} P(s_1|s_1) & \cdots & P(s_N|s_1) \\ P(s_1|s_2) & \cdots & P(s_N|s_2) \\ \vdots & \ddots & \vdots \\ P(s_1|s_N) & \cdots & P(s_N|s_N) \end{pmatrix} \begin{pmatrix} V(s_1) \\ \vdots \\ V(s_N) \end{pmatrix}$$

$$V = R + \gamma PV$$

# Analytic Solution for Value of MRP

- For finite state MRP, we can express $V(s)$ using a matrix equation

s

$$\begin{pmatrix} V(s_1) \\ \vdots \\ V(s_N) \end{pmatrix} = \begin{pmatrix} R(s_1) \\ \vdots \\ R(s_N) \end{pmatrix} + \gamma \begin{pmatrix} P(s_1|s_1) & \cdots & P(s_N|s_1) \\ P(s_1|s_2) & \cdots & P(s_N|s_2) \\ \vdots & \ddots & \vdots \\ P(s_1|s_N) & \cdots & P(s_N|s_N) \end{pmatrix} \begin{pmatrix} V(s_1) \\ \vdots \\ V(s_N) \end{pmatrix}$$

$$\begin{aligned} V &= R + \gamma P V \\ V - \gamma P V &= R \\ (I - \gamma P) V &= R \\ V &= (I - \gamma P)^{-1} R \end{aligned}$$

- Solving directly requires taking a matrix inverse $\sim O(N^3)$
- Requires that $(I - \gamma P)$ is invertible