Lecture 3: Model-Free Policy Evaluation: Policy Evaluation Without Knowing How the World Works

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CS234 Reinforcement Learning

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- Material builds on structure from David Silver's Lecture 4: Model-Free Prediction. Other resources: Sutton and Barto Jan 1 2018 draft Chapter/Sections: 5.1; 5.5; 6.1-6.3

L3N1 Refresh Your Knowledge [Polleverywhere Poll]

- In a tabular MDP asymptotically value iteration will always yield a policy with the same value as the policy returned by policy iteration
  1. True.
  2. False
  3. Not sure

  T

- Can value iteration require more iterations than $|A|^{|S|}$ to compute the optimal value function? (Assume $|A|$ and $|S|$ are small enough that each round of value iteration can be done exactly).
  1. True.
  2. False
  3. Not sure

L3N1 Refresh Your Knowledge

- In a tabular MDP asymptotically value iteration will always yield a policy with the same value as the policy returned by policy iteration

  Answer. True. Both are guaranteed to converge to the optimal value function and a policy with an optimal value

- Can value iteration require more iterations than $|A|^{|S|}$ to compute the optimal value function? (Assume $|A|$ and $|S|$ are small enough that each round of value iteration can be done exactly).

  Answer: True. As an example, consider a single state, single action MDP where $r(s,a) = 1$, $\gamma = .9$ and initialize $V_0(s) = 0$. $V^*(s) = \frac{1}{1-\gamma}$ but after the first iteration of value iteration, $V_1(s) = 1$.

Today's Plan

- Last Time:
  - Markov reward / decision processes
  - Policy evaluation & control when have true model (of how the world works)
- **Today**
  - **Policy evaluation without known dynamics & reward models**
- Next Time:
  - Control when don't have a model of how the world works

Evaluation through Direct Experience

- Estimate expected return of policy π
- Only using data from environment[1] (direct experience)
- Why is this important?
- What properties do we want from policy evaluation algorithms?

[1] Assume today this experience comes from executing the policy π. Later will consider how to do policy evaluation using data gathered from other policies.

This Lecture: Policy Evaluation

- Estimating the expected return of a particular policy if don't have access to true MDP models
- Monte Carlo policy evaluation
  - Policy evaluation when don't have a model of how the world works
    - Given on-policy samples
- Temporal Difference (TD)
- Certainty Equivalence with dynamic programming
- Batch policy evaluation

Recall

- Definition of Return, G_t (for a MRP)
  - Discounted sum of rewards from time step t to horizon

$$G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$$

- Definition of State Value Function, $V^\pi(s)$
  - Expected return starting in state s under policy π

$$V^\pi(s) = \mathbb{E}_\pi[G_t | s_t = s] = \mathbb{E}_\pi[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots | s_t = s]$$

- Definition of State-Action Value Function, $Q^\pi(s, a)$
  - Expected return starting in state s, taking action a and following policy π

$$\begin{aligned} Q^\pi(s,a) &= \mathbb{E}_\pi[G_t | s_t = s, a_t = a] \\ &= \mathbb{E}_\pi[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots | s_t = s, a_t = a] \end{aligned}$$

Recall: Dynamic Programming for Policy Evaluation

- In a Markov decision process

$$\begin{aligned} V^\pi(s) &= \mathbb{E}_\pi[G_t|s_t = s] \\ &= \mathbb{E}_\pi[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots | s_t = s] \\ &= R(s, \pi(s)) + \gamma \sum_{s' \in S} P(s'|s, \pi(s)) V^\pi(s') \end{aligned}$$

- If given dynamics and reward models, can do policy evaluation through dynamic programming

$$V_k^\pi(s) = r(s, \pi(s)) + \gamma \sum_{s' \in S} p(s'|s, \pi(s)) V_{k-1}^\pi(s') \tag{1}$$

- **Note**: before convergence, V_k^π is an estimate of V^π
- In Equation 1 we are substituting $\sum_{s' \in S} p(s'|s, \pi(s)) V_{k-1}^\pi(s')$ for $\mathbb{E}_\pi[r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \cdots | s_t = s]$.
- This substitution is an instance of **bootstrapping**

This Lecture: Policy Evaluation

- Estimating the expected return of a particular policy if don't have access to true MDP models
- **Monte Carlo policy evaluation**
  - Policy evaluation when don't have a model of how the world work
    - Given on-policy samples
- Temporal Difference (TD)
- Certainty Equivalence with dynamic programming
- Batch policy evaluation

Monte Carlo (MC) Policy Evaluation

- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots + \gamma^{T_i - t} r_{T_i}$ in MDP M under policy π
- $V^\pi(s) = \mathbb{E}_{\tau \sim \pi}[G_t | s_t = s]$
  - Expectation over trajectories τ generated by following π
- Simple idea: Value = mean return
- If trajectories are all finite, sample set of trajectories & average returns
- Note: all trajectories may not be same length (e.g. consider MDP with terminal states)

Monte Carlo (MC) Policy Evaluation

- If trajectories are all finite, sample set of trajectories & average returns
- Does not require MDP dynamics/rewards
- Does not assume state is Markov
- Can be applied to episodic MDPs
  - Averaging over returns from a complete episode
  - Requires each episode to terminate

First-Visit Monte Carlo (MC) On Policy Evaluation

Initialize $N(s) = 0,\ G(s) = 0\ \forall s \in S$
Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}, , a_{i,T_i}, r_{i,T_i}$
- Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$ as return from time step t onwards in ith episode
- For each time step t until T_i (the end of the episode i)
  - If this is the **first** time t that state s is visited in episode i
    - Increment counter of total first visits: $N(s) = N(s) + 1$
    - Increment total return $G(s) = G(s) + G_{i,t}$
    - Update estimate $V^\pi(s) = G(s)/N(s)$

Every-Visit Monte Carlo (MC) On Policy Evaluation

Initialize $N(s) = 0$, $G(s) = 0$ $\forall s \in S$
Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}, , a_{i,T_i}, r_{i,T_i}$
- Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$ as return from time step t onwards in ith episode
- For each time step t until T_i (the end of the episode i)
  - state s is the state visited at time step t in episodes i
  - Increment counter of total visits: $N(s) = N(s) + 1$
  - Increment total return $G(s) = G(s) + G_{i,t}$
  - Update estimate $V^\pi(s) = G(s)/N(s)$

Optional Worked Example: MC On Policy Evaluation

Initialize $N(s) = 0$, $G(s) = 0 \ \forall s \in S$
Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}, a_{i,T_i}, r_{i,T_i}$
- $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$
- For each time step t until T_i (the end of the episode i)
  - If this is the **first** time t that state s is visited in episode i (for first visit MC)
    - Increment counter of total first visits: $N(s) = N(s) + 1$
    - Increment total return $G(s) = G(s) + G_{i,t}$
    - Update estimate $V^\pi(s) = G(s)/N(s)$

- Mars rover: R(s) = [1 0 0 0 0 0 +10]
- Trajectory = (s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1, terminal)
- Let $\gamma < 1$. Compute the first visit & every visit MC estimates of s_2.
- See solutions at the end of the slides

Incremental Monte Carlo (MC) On Policy Evaluation

After each episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots$

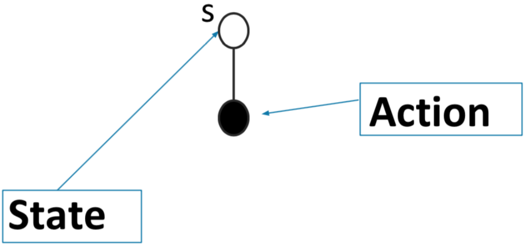
- Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots$ as return from time step t onwards in ith episode
- For state s visited at time step t in episode i
  - Increment counter of total visits: $N(s) = N(s) + 1$
  - Update estimate

$$V^\pi(s) = V^\pi(s)\frac{N(s)-1}{N(s)} + \frac{G_{i,t}}{N(s)} = V^\pi(s) + \frac{1}{N(s)}(G_{i,t} - V^\pi(s))$$

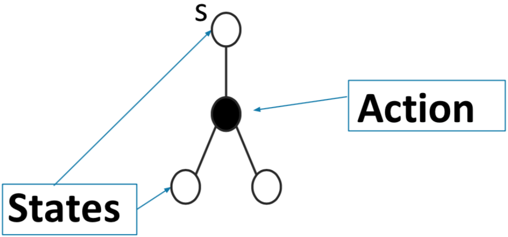
Incremental Monte Carlo (MC) On Policy Evaluation

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}, , a_{i,T_i}, r_{i,T_i}$
- $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$
- for $t = 1 : T_i$ where T_i is the length of the i-th episode
  - $V^\pi(s_{it}) = V^\pi(s_{it}) + \alpha(G_{i,t} - V^\pi(s_{it}))$
- We will see many algorithms of this form with a learning rate, target, and incremental update

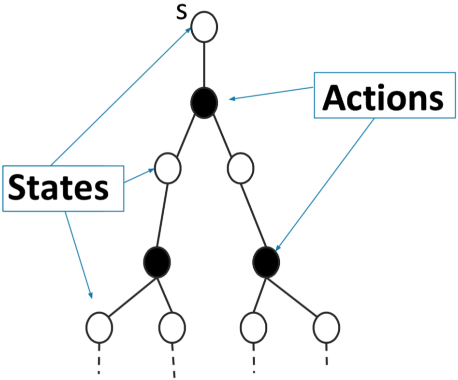
Policy Evaluation Diagram



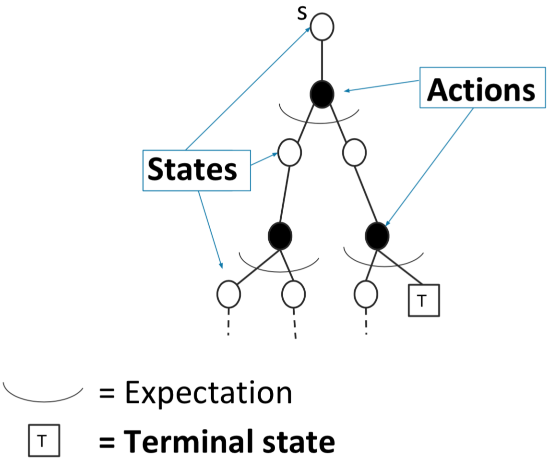
Policy Evaluation Diagram



Policy Evaluation Diagram

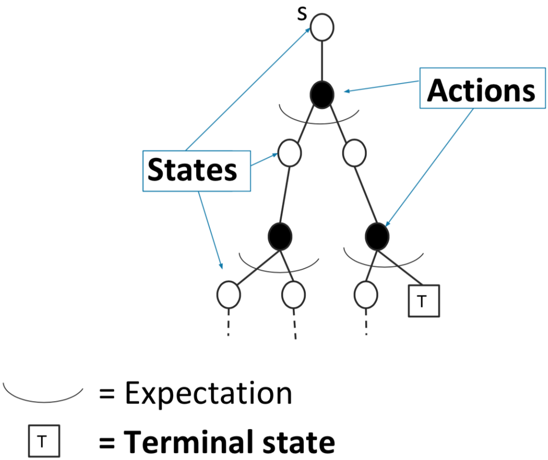


Policy Evaluation Diagram



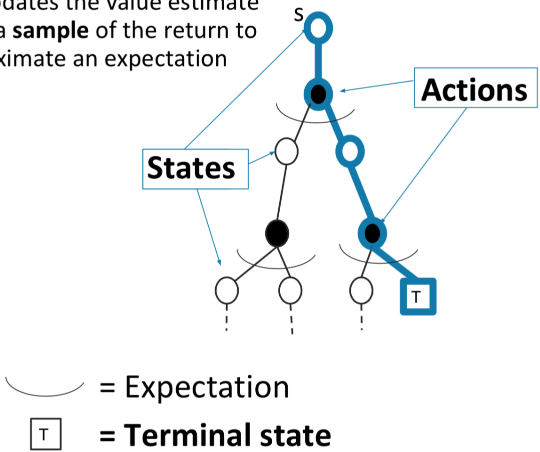
MC Policy Evaluation

$$V^\pi(s) = V^\pi(s) + \alpha(G_{i,t} - V^\pi(s))$$



$$V^\pi(s) = V^\pi(s) + \alpha(G_{i,t} - V^\pi(s))$$

MC updates the value estimate using a **sample** of the return to approximate an expectation



Evaluation of the Quality of a Policy Estimation Approach

- Consistency: with enough data, does the estimate converge to the true value of the policy?
- Computational complexity: as get more data, computational cost of updating estimate
- Memory requirements
- Statistical efficiency (intuitively, how does the accuracy of the estimate change with the amount of data)
- Empirical accuracy, often evaluated by mean squared error

Evaluation of the Quality of a Policy Estimation Approach: Bias, Variance and MSE

- Consider a statistical model that is parameterized by θ and that determines a probability distribution over observed data $P(x|\theta)$
- Consider a statistic $\hat{\theta}$ that provides an estimate of θ and is a function of observed data x
  - E.g. for a Gaussian distribution with known variance, the average of a set of i.i.d data points is an estimate of the mean of the Gaussian
- Definition: the bias of an estimator $\hat{\theta}$ is:

$$Bias_\theta(\hat{\theta}) = \mathbb{E}_{x|\theta}[\hat{\theta}] - \theta$$

- Definition: the variance of an estimator $\hat{\theta}$ is:

$$Var(\hat{\theta}) = \mathbb{E}_{x|\theta}[(\hat{\theta} - \mathbb{E}[\hat{\theta}])^2]$$

- Definition: mean squared error (MSE) of an estimator $\hat{\theta}$ is:

$$MSE(\hat{\theta}) = Var(\hat{\theta}) + Bias_\theta(\hat{\theta})^2$$

Evaluation of the Quality of a Policy Estimation Approach: Consistent Estimator

- Consider a statistical model that is parameterized by θ and that determines a probability distribution over observed data $P(x|\theta)$
- Consider a statistic $\hat{\theta}$ that provides an estimate of θ and is a function of observed data x
- Definition: the bias of an estimator $\hat{\theta}$ is:

$$Bias_{\theta}(\hat{\theta}) = \mathbb{E}_{x|\theta}[\hat{\theta}] - \theta$$

- Let n be the number of data points x used to estimate the parameter θ and call the resulting estimate of θ using that data $\hat{\theta}_n$
- Then the estimator $\hat{\theta}_n$ is consistent if, for all $\epsilon > 0$

$$\lim_{n \to \infty} Pr(|\hat{\theta}_n - \theta| > \epsilon) = 0$$

- If an estimator is unbiased (bias $= 0$) is it consistent?

Properties of Monte Carlo On Policy Evaluators

Properties:

- First-visit Monte Carlo
  - V^π estimator is an unbiased estimator of true $\mathbb{E}_\pi[G_t|s_t = s]$
  - By law of large numbers, as $N(s) \to \infty$, $V^\pi(s) \to \mathbb{E}_\pi[G_t|s_t = s]$
- Every-visit Monte Carlo
  - V^π every-visit MC estimator is a **biased** estimator of V^π
  - But consistent estimator and often has better MSE
- Incremental Monte Carlo
  - Properties depends on the learning rate α

Properties of Monte Carlo On Policy Evaluators

- Update is: $V^\pi(s_{it}) = V^\pi(s_{it}) + \alpha_k(s_j)(G_{i,t} - V^\pi(s_{it}))$
- where we have allowed α to vary (let k be the total number of updates done so far, for state $s_{it} = s_j$)
- If

$$\begin{aligned}\sum_{n=1}^{\infty} \alpha_n(s_j) &= \infty, \\ \sum_{n=1}^{\infty} \alpha_n^2(s_j) &< \infty\end{aligned}$$

- then incremental MC estimate will converge to true policy value $V^\pi(s_j)$

Monte Carlo (MC) Policy Evaluation Key Limitations

- Generally high variance estimator
  - Reducing variance can require a lot of data
  - In cases where data is very hard or expensive to acquire, or the stakes are high, MC may be impractical
- Requires episodic settings
  - Episode must end before data from episode can be used to update V

Monte Carlo (MC) Policy Evaluation Summary

- Aim: estimate $V^\pi(s)$ given episodes generated under policy π
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from π
- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$ under policy π
- $V^\pi(s) = \mathbb{E}_\pi[G_t | s_t = s]$
- Simple: Estimates expectation by empirical average (given episodes sampled from policy of interest)
- Updates V estimate using **sample** of return to approximate the expectation
- Does not assume Markov process
- Converges to true value under some (generally mild) assumptions
- **Note:** Sometimes is preferred over dynamic programming for policy evaluation *even if know the true dynamics model and reward*

This Lecture: Policy Evaluation

- Estimating the expected return of a particular policy if don't have access to true MDP models
- Monte Carlo policy evaluation
- **Temporal Difference (TD)**
- Certainty Equivalence with dynamic programming
- Batch policy evaluation

Temporal Difference Learning

- "If one had to identify one idea as central and novel to reinforcement learning, it would undoubtedly be temporal-difference (TD) learning." – Sutton and Barto 2017
- Combination of Monte Carlo & dynamic programming methods
- Model-free
- Can be used in episodic or infinite-horizon non-episodic settings
- Immediately updates estimate of V after each (s, a, r, s') tuple

Temporal Difference Learning for Estimating V

- Aim: estimate $V^\pi(s)$ given episodes generated under policy π
- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$ in MDP M under policy π
- $V^\pi(s) = \mathbb{E}_\pi[G_t | s_t = s]$
- Recall Bellman operator (if know MDP models)

$$B^\pi V(s) = r(s, \pi(s)) + \gamma \sum_{s' \in S} p(s'|s, \pi(s)) V(s')$$

- In incremental every-visit MC, update estimate using 1 sample of return (for the current ith episode)

$$V^\pi(s) = V^\pi(s) + \alpha(G_{i,t} - V^\pi(s))$$

- **Idea**: have an estimate of V^π, use to estimate expected return

$$V^\pi(s) = V^\pi(s) + \alpha([r_t + \gamma V^\pi(s_{t+1})] - V^\pi(s))$$

Temporal Difference [$TD(0)$] Learning

- Aim: estimate $V^\pi(s)$ given episodes generated under policy π
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from π
- TD(0) learning / 1-step TD learning: update estimate towards target

$$V^\pi(s_t) = V^\pi(s_t) + \alpha(\underbrace{[r_t + \gamma V^\pi(s_{t+1})]}_{\text{TD target}} - V^\pi(s_t))$$

- TD(0) error:

$$\delta_t = r_t + \gamma V^\pi(s_{t+1}) - V^\pi(s_t)$$

- Can immediately update value estimate after (s, a, r, s') tuple
- Don't need episodic setting

Temporal Difference [$TD(0)$] Learning Algorithm

Input: α
Initialize $V^\pi(s) = 0, \forall s \in S$
Loop

- Sample **tuple** (s_t, a_t, r_t, s_{t+1}) $\sim \pi(s_t)$
- $V^\pi(s_t) = V^\pi(s_t) + \alpha(\underbrace{[r_t + \gamma V^\pi(s_{t+1})]}_{\text{TD target}} - V^\pi(s_t))$

Worked Example TD Learning

Input: α
Initialize $V^\pi(s) = 0$, $\forall s \in S$
Loop

- Sample **tuple** (s_t, a_t, r_t, s_{t+1})
- $V^\pi(s_t) = V^\pi(s_t) + \alpha(\underbrace{[r_t + \gamma V^\pi(s_{t+1})]}_{\text{TD target}} - V^\pi(s_t))$

Example Mars rover: R = [1 0 0 0 0 0 +10] for any action

- $\pi(s) = a_1$ $\forall s$, $\gamma = 1$. any action from s_1 and s_7 terminates episode
- Trajectory = (s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1, terminal)

Worked Example TD Learning

Input: α
Initialize $V^\pi(s) = 0, \forall s \in S$
Loop

- Sample **tuple** (s_t, a_t, r_t, s_{t+1})
- $V^\pi(s_t) = V^\pi(s_t) + \alpha(\underbrace{[r_t + \gamma V^\pi(s_{t+1})]}_{\text{TD target}} - V^\pi(s_t))$

Example:

- Mars rover: R = [1 0 0 0 0 0 +10] for any action
- $\pi(s) = a_1\ \forall s$, $\gamma = 1$. any action from s_1 and s_7 terminates episode
- Trajectory = (s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1, terminal)
- TD estimate of all states (init at 0) with $\alpha = 1$, $\gamma < 1$ at end of this episode?

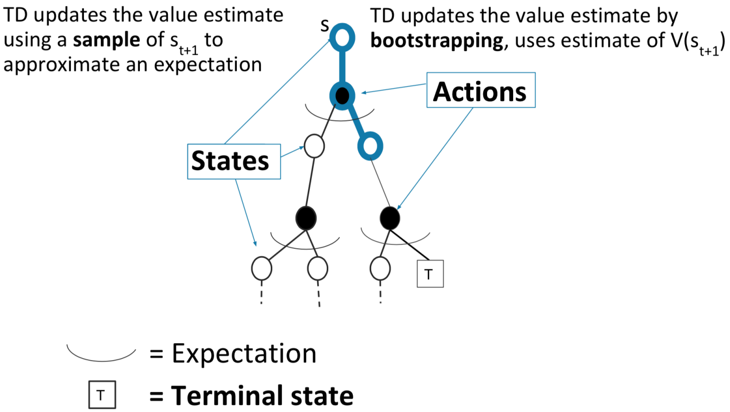
  V = [1 0 0 0 0 0 0]

- First visit MC estimate of V of each state? $[1\ \gamma\ \gamma^2\ 0\ 0\ 0\ 0]$

Temporal Difference (TD) Policy Evaluation

$$V^\pi(s_t) = r(s_t, \pi(s_t)) + \gamma \sum_{s_{t+1}} P(s_{t+1}|s_t, \pi(s_t)) V^\pi(s_{t+1})$$

$$V^\pi(s_t) = V^\pi(s_t) + \alpha([r_t + \gamma V^\pi(s_{t+1})] - V^\pi(s_t))$$



Check Your Understanding L3N2: Polleverywhere Poll Temporal Difference [$TD(0)$] Learning Algorithm

Input: α
Initialize $V^\pi(s) = 0$, $\forall s \in S$
Loop

- Sample **tuple** (s_t, a_t, r_t, s_{t+1})
- $V^\pi(s_t) = V^\pi(s_t) + \alpha(\underbrace{[r_t + \gamma V^\pi(s_{t+1})]}_{\text{TD target}} - V^\pi(s_t))$

Select all that are true

1. If $\alpha = 0$ TD will weigh the TD target more than the past V estimate
2. If $\alpha = 1$ TD will update the V estimate to the TD target
3. If $\alpha = 1$ TD in MDPs where the policy goes through states with multiple possible next states, V may oscillate forever
4. There exist deterministic MDPs where $\alpha = 1$ TD will converge

Check Your Understanding L3N2: Polleverywhere Poll Temporal Difference [$TD(0)$] Learning Algorithm

Input: α
Initialize $V^\pi(s) = 0$, $\forall s \in S$
Loop

- Sample **tuple** (s_t, a_t, r_t, s_{t+1})
- $V^\pi(s_t) = V^\pi(s_t) + \alpha(\underbrace{[r_t + \gamma V^\pi(s_{t+1})]}_{\text{TD target}} - V^\pi(s_t))$

Answers. If $\alpha = 1$ TD will update to the TD target. If $\alpha = 1$ TD in MDPs where the policy goes through states with multiple possible next states, V may oscillate forever. There exist deterministic MDPs where $\alpha = 1$ TD will converge.

Summary: Temporal Difference Learning

- Combination of Monte Carlo & dynamic programming methods
- Model-free
- **Bootstraps and samples**
- Can be used in episodic or infinite-horizon non-episodic settings
- Immediately updates estimate of V after each (s, a, r, s') tuple
- Biased estimator (early on will be influenced by initialization, and won't be unibased estimator)
- Generally lower variance than Monte Carlo policy evaluation
- Consistent estimator if learning rate α satisfies same conditions specified for incremental MC policy evaluation to converge
- **Note: algorithm I introduced is TD(0). In general can have approaches that interpolate between TD(0) and Monte Carlo approach**

This Lecture: Policy Evaluation

- Estimating the expected return of a particular policy if don't have access to true MDP models
- Monte Carlo policy evaluation
- Temporal Difference (TD)
- **Certainty Equivalence with dynamic programming**
- Batch policy evaluation

Certainty Equivalence V^π MLE MDP Model Estimates

- Model-based option for policy evaluation without true models
- After each (s_i, a_i, r_i, s_{i+1}) tuple
  - Recompute maximum likelihood MDP model for (s, a)

$$\hat{P}(s'|s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{i} \mathbb{1}(s_k = s, a_k = a, s_{k+1} = s')$$

$$\hat{r}(s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{i} \mathbb{1}(s_k = s, a_k = a) r_k$$

  - Compute V^π using MLE MDP [2] (using any dynamic programming method from lecture 2))
- Optional worked example at end of slides for Mars rover domain.

[2] Requires initializing for all (s, a) pairs

Certainty Equivalence V^π MLE MDP Model Estimates

- Model-based option for policy evaluation without true models
- After each (s, a, r, s') tuple
  - Recompute maximum likelihood MDP model for (s, a)

$$\hat{P}(s'|s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{T_k - 1} 1(s_{k,t} = s, a_{k,t} = a, s_{k,t+1} = s')$$

$$\hat{r}(s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{T_k - 1} 1(s_{k,t} = s, a_{k,t} = a) r_{t,k}$$

  - Compute V^π using MLE MDP
- Cost: Updating MLE model and MDP planning at each update ($O(|S|^3)$ for analytic matrix solution, $O(|S|^2|A|)$ for iterative methods)
- Very data efficient and very computationally expensive
- Consistent (will converge to right estimate for Markov models)
- Can also easily be used for off-policy evaluation (which we will shortly define and discuss)

Optional Worked Example MC On Policy Evaluation Answers

Initialize $N(s) = 0$, $G(s) = 0 \ \forall s \in S$
Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}, a_{i,T_i}, r_{i,T_i}$
- $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$
- For each time step t until T_i (the end of the episode i)
    - If this is the **first** time t that state s is visited in episode i
        - Increment counter of total first visits: $N(s) = N(s) + 1$
        - Increment total return $G(s) = G(s) + G_{i,t}$
        - Update estimate $V^\pi(s) = G(s)/N(s)$

- Mars rover: R = [1 0 0 0 0 0 +10] for any action
- Trajectory = (s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1, terminal)
- Let $\gamma < 1$. Compare the first visit & every visit MC estimates of s_2.
  First visit: $V^{MC}(s_2) = \gamma^2$, Every visit: $V^{MC}(s_2) = \frac{\gamma^2 + \gamma}{2}$

Optional Check Your Understanding L3: Incremental MC (State if each is True or False)

First or Every Visit MC

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}, a_{i,T_i}, r_{i,T_i}$
- $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$
  - For all s, for **first or every** time t that state s is visited in episode i
    - $N(s) = N(s) + 1$, $G(s) = G(s) + G_{i,t}$
    - Update estimate $V^\pi(s) = G(s)/N(s)$

Incremental MC

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}, a_{i,T_i}, r_{i,T_i}$
- $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$
- for $t = 1 : T_i$ where T_i is the length of the i-th episode
  - $V^\pi(s_{it}) = V^\pi(s_{it}) + \alpha(G_{i,t} - V^\pi(s_{it}))$

1. Incremental MC with $\alpha = 1$ is the same as first visit MC
2. Incremental MC with $\alpha = \frac{1}{N(s_{it})}$ is the same as every visit MC
3. Incremental MC with $\alpha > \frac{1}{N(s_{it})}$ could be helpful in non-stationary domains

Optional Check Your Understanding L3 Incremental MC Answers

First or Every Visit MC

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}, a_{i,T_i}, r_{i,T_i}$
- $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$
    - For all s, for **first or every** time t that state s is visited in episode i
        - $N(s) = N(s) + 1, \; G(s) = G(s) + G_{i,t}$
        - Update estimate $V^\pi(s) = G(s)/N(s)$

Incremental MC

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}, a_{i,T_i}, r_{i,T_i}$
- for $t = 1 : T_i$ where T_i is the length of the i-th episode
    - $V^\pi(s_{it}) = V^\pi(s_{it}) + \alpha(G_{i,t} - V^\pi(s_{it}))$

1. Incremental MC with $\alpha = 1$ is the same as first visit MC
   false
2. Incremental MC with $\alpha = \frac{1}{N(s_{it})}$ is the same as every visit MC
   true
3. Incremental MC with $\alpha > \frac{1}{N(s_{it})}$ could help in non-stationary domains
   true

Optional Check Your Understanding L3 Incremental MC (State if each is True or False)

First or Every Visit MC

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}, a_{i,T_i}, r_{i,T_i}$
- $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$
  - For all s, for **first or every** time t that state s is visited in episode i
    - $N(s) = N(s) + 1$, $G(s) = G(s) + G_{i,t}$
    - Update estimate $V^\pi(s) = G(s)/N(s)$

Incremental MC

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}, a_{i,T_i}, r_{i,T_i}$
- $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$
- for $t = 1 : T_i$ where T_i is the length of the i-th episode
  - $V^\pi(s_{it}) = V^\pi(s_{it}) + \alpha(G_{i,t} - V^\pi(s_{it}))$

1. Incremental MC with $\alpha = 1$ is the same as first visit MC
2. Incremental MC with $\alpha = \frac{1}{N(s_{it})}$ is the same as every visit MC
3. Incremental MC with $\alpha > \frac{1}{N(s_{it})}$ could be helpful in non-stationary domains

Check Your Understanding L3N1: Polleverywhere Poll Incremental MC Answers

First or Every Visit MC

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}, a_{i,T_i}, r_{i,T_i}$
- $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$
    - For all s, for **first or every** time t that state s is visited in episode i
        - $N(s) = N(s) + 1$, $G(s) = G(s) + G_{i,t}$
        - Update estimate $V^\pi(s) = G(s)/N(s)$

Incremental MC

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}, a_{i,T_i}, r_{i,T_i}$
- for $t = 1 : T_i$ where T_i is the length of the i-th episode
    - $V^\pi(s_{it}) = V^\pi(s_{it}) + \alpha(G_{i,t} - V^\pi(s_{it}))$

1. Incremental MC with $\alpha = 1$ is the same as first visit MC
   false
2. Incremental MC with $\alpha = \frac{1}{N(s_{it})}$ is the same as every visit MC
   true
3. Incremental MC with $\alpha > \frac{1}{N(s_{it})}$ could help in non-stationary domains
   true

Certainty Equivalence V^π MLE MDP Worked Example

s_1	s_2	s_3	s_4	s_5	s_6	s_7
$R(s_1) = +1$ Okay Field Site	$R(s_2) = 0$	$R(s_3) = 0$	$R(s_4) = 0$ 	$R(s_5) = 0$	$R(s_6) = 0$	$R(s_7) = +10$ Fantastic Field Site

- Mars rover: R $= [\ 1\ 0\ 0\ 0\ 0\ 0\ +10]$ for any action
- $\pi(s) = a_1\ \forall s$, $\gamma = 1$. any action from s_1 and s_7 terminates episode
- Trajectory $= (s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1, \text{terminal})$
- First visit MC estimate of V of each state? $[1\ \gamma\ \gamma^2\ 0\ 0\ 0\ 0]$
- TD estimate of all states (init at 0) with $\alpha = 1$ is $[1\ 0\ 0\ 0\ 0\ 0\ 0]$
- Optional exercise: What is the certainty equivalent estimate?

Certainty Equivalence V^π MLE MDP Worked Ex Solution

s_1	s_2	s_3	s_4	s_5	s_6	s_7
$R(s_1) = +1$ Okay Field Site	$R(s_2) = 0$	$R(s_3) = 0$	$R(s_4) = 0$ 	$R(s_5) = 0$	$R(s_6) = 0$	$R(s_7) = +10$ Fantastic Field Site

- Mars rover: R = [1 0 0 0 0 0 +10] for any action
- $\pi(s) = a_1 \ \forall s$, $\gamma = 1$. any action from s_1 and s_7 terminates episode
- Trajectory = $(s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1, \text{terminal})$
- First visit MC estimate of V of each state? $[1\ \gamma\ \gamma^2\ 0\ 0\ 0\ 0]$
- TD estimate of all states (init at 0) with $\alpha = 1$ is $[1\ 0\ 0\ 0\ 0\ 0\ 0]$
- Optional exercise: What is the certainty equivalent estimate?
- $\hat{r} = [1\ 0\ 0\ 0\ 0\ 0\ 0]$, $\hat{p}(terminate|s_1, a_1) = \hat{p}(s_2|s_3, a_1) = 1$
  $\hat{p}(s_2|s_2, a_1) = 0.5 = \hat{p}(s_1|s_2, a_1)$

$$V = [0\ \frac{\gamma * 0.5}{1 - 0.5\gamma}\ \frac{\gamma^2 * 0.5}{1 - 0.5\gamma}\ 0\ 0\ 0\ 0]$$

Typo: V(s1) should = 1
see next slide for derivation

Certainty Equivalence V^{π} MLE MDP Worked Ex Solution

- Trajectory = (s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1, terminal)
- $\hat{r} = [1\ 0\ 0\ 0\ 0\ 0\ 0]$, $\hat{p}(terminate|s_1, a_1) = \hat{p}(s_2|s_3, a_1) = 1$
  $\hat{p}(s_2|s_2, a_1) = 0.5 = \hat{p}(s_1|s_2, a_1)$
- Recall $V = R + \gamma PV$, which implies $(I - \gamma P)^V = R$ or $V = (I - \gamma P)^{-1} R$
- Doing this only for states $s1$ $s2$ $s3$ $s_{terminal}$

$$(I - \gamma P) = \begin{pmatrix} 1 & 0 & 0 & -\gamma \\ -\frac{\gamma}{2} & 1 - \frac{\gamma}{2} & 0 & 0 \\ 0 & -\gamma & 1 & 0 \\ 0 & 0 & 0 & 1-\gamma \end{pmatrix}, (I - \gamma P)^{-1} = \begin{pmatrix} 1 & 0 & 0 & -\frac{\gamma}{\gamma - 1} \\ \frac{\gamma}{2-\gamma} & \frac{2}{2-\gamma} & 0 & \frac{\gamma^2}{(\gamma-1)(\gamma-2)} \\ \frac{\gamma^2}{2-\gamma} & \frac{2\gamma}{2-\gamma} & 1 & \frac{\gamma^3}{(\gamma-1)(\gamma-2)} \\ 0 & 0 & 0 & -\frac{1}{\gamma-1} \end{pmatrix}.$$

$$V = [1\ \frac{\gamma}{2-\gamma}\ \frac{\gamma^2}{2-\gamma}\ 0\ 0\ 0]$$

Recall: Dynamic Programming for Policy Evaluation

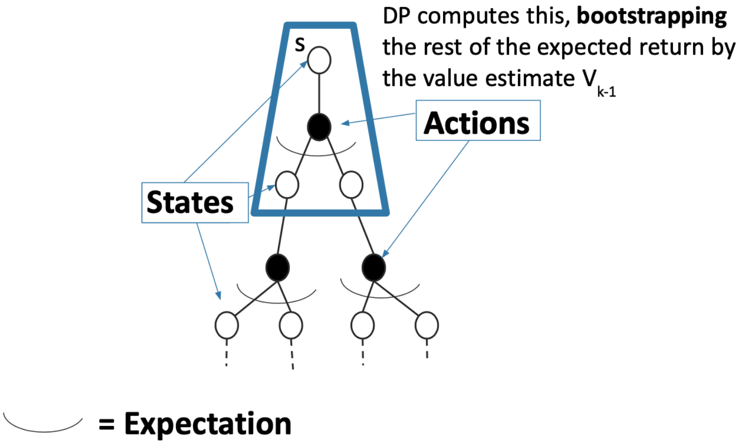
- If we knew dynamics and reward model, we can do policy evaluation
- Initialize $V_0^\pi(s) = 0$ for all s
- For $k = 1$ until convergence
  - For all s in S

$$V_k^\pi(s) = r(s, \pi(s)) + \gamma \sum_{s' \in S} p(s'|s, \pi(s)) V_{k-1}^\pi(s')$$

- $V_k^\pi(s)$ is exactly the k-horizon value of state s under policy π
- $V_k^\pi(s)$ is an **estimate of the infinite horizon** value of state s under policy π

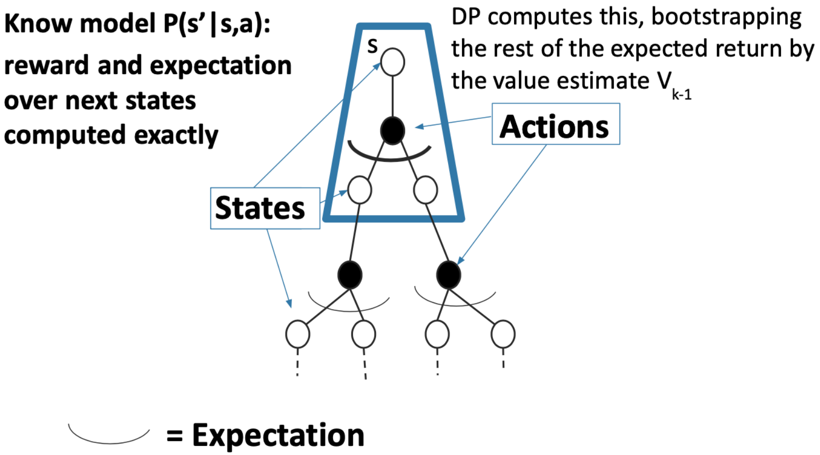
$$V^\pi(s) = \mathbb{E}_\pi[G_t|s_t = s] \approx \mathbb{E}_\pi[r_t + \gamma V_{k-1}|s_t = s]$$

Dynamic Programming Policy Evaluation $V^\pi(s) \leftarrow \mathbb{E}_\pi[r_t + \gamma V_{k-1} | s_t = s]$



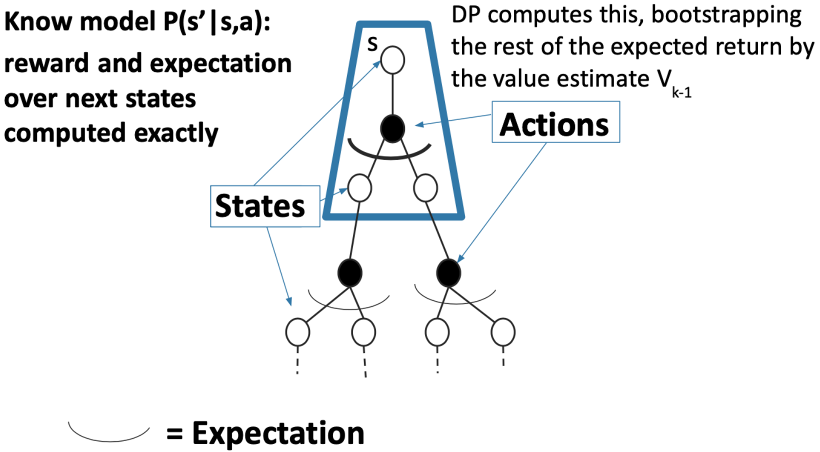
- Bootstrapping: Update for V uses an estimate

Dynamic Programming Policy Evaluation $V^\pi(s) \leftarrow \mathbb{E}_\pi[r_t + \gamma V_{k-1} | s_t = s]$



- Bootstrapping: Update for V uses an estimate

What about when we don't know the models?



Bias/Variance of Model-free Policy Evaluation Algorithms

- Return G_t is an unbiased estimate of $V^\pi(s_t)$
- TD target $[r_t + \gamma V^\pi(s_{t+1})]$ is a biased estimate of $V^\pi(s_t)$
- But often much lower variance than a single return G_t
- Return function of multi-step sequence of random actions, states & rewards
- TD target only has one random action, reward and next state
- MC
    - Unbiased (for first visit)
    - High variance
    - Consistent (converges to true) even with function approximation
- TD
    - Some bias
    - Lower variance
    - TD(0) converges to true value with tabular representation
    - TD(0) does not always converge with function approximation

s_1	s_2	s_3	s_4	s_5	s_6	s_7
$R(s_1) = +1$ *Okay Field Site*	$R(s_2) = 0$	$R(s_3) = 0$	$R(s_4) = 0$ 	$R(s_5) = 0$	$R(s_6) = 0$	$R(s_7) = +10$ *Fantastic Field Site*

- Mars rover: R $= [\ 1\ 0\ 0\ 0\ 0\ 0\ +10]$ for any action
- $\pi(s) = a_1\ \forall s$, $\gamma = 1$. any action from s_1 and s_7 terminates episode
- Trajectory $= (s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1,$ terminal$)$
- First visit MC estimate of V of each state? $[1\ 1\ 1\ 0\ 0\ 0\ 0]$
- TD estimate of all states (init at 0) with $\alpha = 1$ is $[1\ 0\ 0\ 0\ 0\ 0\ 0]$
- TD(0) only uses a data point (s, a, r, s') once
- Monte Carlo takes entire return from s to end of episode