Lecture 9: Data Efficient Reinforcement Learning

Emma Brunskill

CS234 Reinforcement Learning

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- Many slides from or derived from David Silver, Examples new and proof new.

Refresh Your Understanding

Select all that are true

- RLHF and DPO both learn an explicit representation of a reward model from preference data
- Both are constrained to be at most as good as the best examples in the pairwise preference data
- DPO does not use a reference policy
- None of the above
- Not Sure

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Class Structure

- Last time: Midterm
- **This time: Data Efficient Reinforcement Learning – Bandits**
- Next time: Data Efficient Reinforcement Learning

Recall RL Involves

- Generalization
- Optimization
- Delayed outcomes
- Exploration

Evaluation Criteria

- How do we evaluate how "good" an algorithm is?
- If converges?
- If converges to optimal policy?
- How quickly reaches optimal policy?
- Mistakes make along the way?
- Will introduce different measures to evaluate RL algorithms

Settings, Frameworks & Approaches

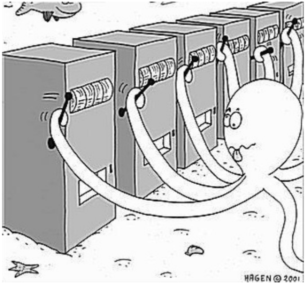
- Over next couple lectures will consider 2 settings, multiple frameworks, and approaches
- Settings: Bandits (single decisions), MDPs
- Frameworks: evaluation criteria for formally assessing the quality of a RL algorithm
- Approaches: Classes of algorithms for achieving particular evaluation criteria in a certain set
- Note: We will see that some approaches can achieve multiple frameworks in multiple settings

Today

- Setting: Introduction to multi-armed bandits & Approach: greedy methods
- Framework: Regret
- Approach: ϵ-greedy methods
- Approach: Optimism under uncertainty
- Framework: Bayesian regret
- Approach: Probability matching / Thompson sampling

Multiarmed Bandits

- Multi-armed bandit is a tuple of $(\mathcal{A}, \mathcal{R})$
- $\mathcal{A}$: known set of m actions (arms)
- $\mathcal{R}^a(r) = \mathbb{P}[r \mid a]$ is an unknown probability distribution over rewards
- At each step t the agent selects an action $a_t \in \mathcal{A}$
- The environment generates a reward $r_t \sim \mathcal{R}^{a_t}$
- Goal: Maximize cumulative reward $\sum_{\tau=1}^{t} r_\tau$



Toy Example: Ways to Treat Broken Toes

- Consider deciding how to best treat patients with broken toes
- Imagine have 3 possible options: (1) surgery (2) buddy taping the broken toe with another toe, (3) do nothing
- Outcome measure / reward is binary variable: whether the toe has healed (+1) or not healed (0) after 6 weeks, as assessed by x-ray

Note: This is a made up example. This is not the actual expected efficacies of the various treatment options for a broken toe

L11N2 Check Your Understanding: Bandit Toes

- Consider deciding how to best treat patients with broken toes
- Imagine have 3 common options: (1) surgery (2) buddy taping the broken toe with another toe (3) doing nothing
- Outcome measure is binary variable: whether the toe has healed (+1) or not (0) after 6 weeks, as assessed by x-ray
- Model as a multi-armed bandit with 3 arms, where each arm is a Bernoulli variable with an unknown parameter θ_i
- Select all that are true
  1. Pulling an arm / taking an action corresponds to whether the toe has healed or not
  2. A multi-armed bandit is a better fit to this problem than a MDP because treating each patient involves multiple decisions
  3. After treating a patient, if $\theta_i \neq 0$ and $\theta_i \neq 1$ $\forall i$ sometimes a patient's toe will heal and sometimes it may not
  4. Not sure

L11N2 Check Your Understanding: Bandit Toes Solution

- Consider deciding how to best treat patients with broken toes
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  4. Not sure

Greedy Algorithm

- We consider algorithms that estimate $\hat{Q}_t(a) \approx Q(a) = \mathbb{E}[R(a)]$
- Estimate the value of each action by Monte-Carlo evaluation

$$\hat{Q}_t(a) = \frac{1}{N_t(a)} \sum_{i=1}^{t} r_i \mathbb{1}(a_i = a)$$

- The **greedy** algorithm selects the action with highest value

$$a_t^* = \arg\max_{a \in \mathcal{A}} \hat{Q}_{t-1}(a)$$

Toy Example: Ways to Treat Broken Toes

- Imagine true (unknown) Bernoulli reward parameters for each arm (action) are
  - surgery: $Q(a^1) = \theta_1 = .95$
  - buddy taping: $Q(a^2) = \theta_2 = .9$
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Toy Example: Ways to Treat Broken Toes, Greedy

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- Greedy
  1. Sample each arm once
     - Take action a^1 ($r \sim$Bernoulli(0.95)), get 0, $\hat{Q}(a^1) = 0$
     - Take action a^2 ($r \sim$Bernoulli(0.90)), get $+1$, $\hat{Q}(a^2) = 1$
     - Take action a^3 ($r \sim$Bernoulli(0.1)), get 0, $\hat{Q}(a^3) = 0$
  2. What is the probability of greedy selecting each arm next? Assume ties are split uniformly.

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  2. Will the greedy algorithm ever find the best arm in this case?

Greedy Algorithm

- We consider algorithms that estimate $\hat{Q}_t(a) \approx Q(a) = \mathbb{E}\left[R(a)\right]$
- Estimate the value of each action by Monte-Carlo evaluation

$$\hat{Q}_t(a) = \frac{1}{N_t(a)} \sum_{t=1}^{T} r_t \mathbb{1}(a_t = a)$$

- The **greedy** algorithm selects the action with highest value

$$a_t^* = \arg\max_{a \in \mathcal{A}} \hat{Q}_{t-1}(a)$$

- **Greedy can lock onto suboptimal action, forever**

Today

- Setting: Introduction to multi-armed bandits & Approach: greedy methods
- **Framework: Regret**
- Approach: ϵ-greedy methods
- Approach: Optimism under uncertainty
- Framework: Bayesian regret
- Approach: Probability matching / Thompson sampling

Assessing the Performance of Algorithms

- How do we evaluate the quality of a RL (or bandit) algorithm?
- So far: computational complexity, convergence, convergence to a fixed point, & empirical performance performance
- Today: introduce a formal measure of how well a RL/bandit algorithm will do in any environment, compared to optimal

Regret

- **Action-value** is the mean reward for action a

$$Q(a) = \mathbb{E}[r \mid a]$$

- **Optimal value** V^*

$$V^* = Q(a^*) = \max_{a \in \mathcal{A}} Q(a)$$

- **Regret** is the opportunity loss for one step, where the expectation is taken over the decision policy used to select a_t

$$l_t = \mathbb{E}[V^* - Q(a_t)]$$

Regret

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- **Optimal value** V^*

$$V^* = Q(a^*) = \max_{a \in \mathcal{A}} Q(a)$$

- **Regret** is the opportunity loss for one step, where the expectation is taken over the decision policy used to select a_t

$$l_t = \mathbb{E}[V^* - Q(a_t)]$$

- **Total Regret** is the total opportunity loss

$$L_t = \mathbb{E}[\sum_{\tau=1}^{t} V^* - Q(a_\tau)]$$

- Maximize cumulative reward $\iff$ minimize total regret

Evaluating Regret

- **Count** $N_t(a)$ is number of times action a has been selected at time step t
- **Gap** Δ_a is the difference in value between action a and optimal action a^*, $\Delta_i = V^* - Q(a_i)$
- Regret is a function of gaps and counts

$$
\begin{aligned}
L_t &= \mathbb{E}\left[\sum_{\tau=1}^{t} V^* - Q(a_\tau)\right] \\
&= \sum_{a \in \mathcal{A}} \mathbb{E}[N_t(a)](V^* - Q(a)) \\
&= \sum_{a \in \mathcal{A}} \mathbb{E}[N_t(a)]\Delta_a
\end{aligned}
$$

- A good algorithm ensures small counts for large gap,s but gaps are not known

Toy Example: Ways to Treat Broken Toes, Optimism, Assessing Regret of Greedy

- True (unknown) Bernoulli reward parameters for each arm (action) are
  - surgery: $Q(a^1) = \theta_1 = .95$
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  - doing nothing: $Q(a^3) = \theta_3 = .1$
- Greedy

Action	Optimal Action	Observed Reward	Regret
a^1	a^1	0	
a^2	a^1	1	
a^3	a^1	0	
a^2	a^1	1	
a^2	a^1	0	

Toy Example: Ways to Treat Broken Toes, Optimism, Assessing Regret of Greedy

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- Greedy

Action	Optimal Action	Observed Reward	Regret
a^1	a^1	0	0
a^2	a^1	1	0.05
a^3	a^1	0	0.85
a^2	a^1	1	0.05
a^2	a^1	0	0.05

- Regret for greedy methods can be **linear** in the number of decisions made (timestep)

Toy Example: Ways to Treat Broken Toes, Optimism, Assessing Regret of Greedy

- Greedy

Action	Optimal Action	Observed Reward	Regret
a^1	a^1	0	0
a^2	a^1	1	0.05
a^3	a^1	0	0.85
a^2	a^1	1	0.05
a^2	a^1	0	0.05

- **Note: in real settings we cannot evaluate the regret because it requires knowledge of the expected reward of the true best action.**
- Instead we can prove an upper bound on the potential regret of an algorithm in **any bandit** problem

Today

- Setting: Introduction to multi-armed bandits & Approach: greedy methods
- Framework: Regret
- **Approach: ϵ-greedy methods**
- Approach: Optimism under uncertainty
- Framework: Bayesian regret
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ϵ-Greedy Algorithm

- The ϵ-**greedy** algorithm proceeds as follows:
  - With probability $1 - \epsilon$ select $a_t = \arg\max_{a \in \mathcal{A}} \hat{Q}_t(a)$
  - With probability ϵ select a random action
- Always will be making a sub-optimal decision ϵ fraction of the time
- Already used this in prior homeworks

Toy Example: Ways to Treat Broken Toes, ϵ-**Greedy**

- Imagine true (unknown) Bernoulli reward parameters for each arm (action) are
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  - doing nothing: $Q(a^3) = \theta_3 = .1$
- ϵ-greedy
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     - Take action a^3 ($r \sim$Bernoulli(0.1)), get 0, $\hat{Q}(a^3) = 0$
  2. Let $\epsilon = 0.1$
  3. What is the probability ϵ-greedy will pull each arm next? Assume ties are split uniformly.

Toy Example: Ways to Treat Broken Toes, Optimism, Assessing Regret of Greedy

- True (unknown) Bernoulli reward parameters for each arm (action) are
  - surgery: $Q(a^1) = \theta_1 = .95$
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- UCB1 (Auer, Cesa-Bianchi, Fischer 2002)

Action	Optimal Action	Regret
a^1	a^1	
a^2	a^1	
a^3	a^1	
a^1	a^1	
a^2	a^1	

- Will ϵ-greedy ever select a^3 again? If ϵ is fixed, how many times will each arm be selected?

Recall: Bandit Regret

- **Count** $N_t(a)$ is expected number of selections for action a
- **Gap** Δ_a is the difference in value between action a and optimal action a^*, $\Delta_i = V^* - Q(a_i)$
- Regret is a function of gaps and counts

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\begin{aligned}
L_t &= \mathbb{E}\left[\sum_{\tau=1}^{t} V^* - Q(a_\tau)\right] \\
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- A good algorithm ensures small counts for large gap, but gaps are not known

L11N3 Check Your Understanding: ϵ-greedy Bandit Regret

- **Count** $N_t(a)$ is expected number of selections for action a
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- Regret is a function of gaps and counts

$$L_t = \sum_{a \in \mathcal{A}} \mathbb{E}[N_t(a)]\Delta_a$$

- Informally an algorithm has linear regret if it takes a non-optimal action a constant fraction of the time
- Assume $\exists a\ s.t.\ \Delta_a > 0$
- Select all
  1. $\epsilon = 0.1$ ϵ-greedy can have linear regret
  2. $\epsilon = 0$ ϵ-greedy can have linear regret
  3. Not sure

L11N3 Check Your Understanding: ϵ-greedy Bandit Regret Answer

- **Count** $N_t(a)$ is expected number of selections for action a
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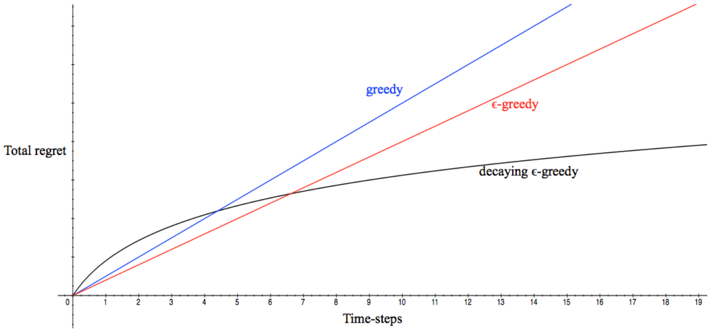
"Good": Sublinear or below regret



- **Explore forever**: have linear total regret
- **Explore never**: have linear total regret
- Is it possible to achieve sublinear (in the time steps/number of decisions made) regret?

Types of Regret bounds

- **Problem independent**: Bound how regret grows as a function of T, the total number of time steps the algorithm operates for
- **Problem dependent**: Bound regret as a function of the number of times we pull each arm and the gap between the reward for the pulled arm and a^*

Lower Bound

- Use lower bound to determine how hard this problem is
- The performance of any algorithm is determined by similarity between optimal arm and other arms
- Hard problems have similar looking arms with different means
- This is described formally by the gap Δ_a and the similarity in distributions $D_{KL}(\mathcal{R}^a \| \mathcal{R}^{a^*})$
- Theorem (Lai and Robbins): Asymptotic total regret is at least logarithmic in number of steps

$$\lim_{t \to \infty} L_t \geq \log t \sum_{a | \Delta_a > 0} \frac{\Delta_a}{D_{KL}(\mathcal{R}^a \| \mathcal{R}^{a^*})}$$

- Promising in that lower bound is sublinear

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Approach: Optimism in the Face of Uncertainty

- Choose actions that that might have a high value
- Why?
- Two outcomes:

Approach: Optimism in the Face of Uncertainty

- Choose actions that that might have a high value
- Why?
- Two outcomes:
  - Getting high reward: if the arm really has a high mean reward
  - Learn something: if the arm really has a lower mean reward, pulling it will (in expectation) reduce its average reward and the uncertainty over its value

Upper Confidence Bounds

- Estimate an upper confidence $U_t(a)$ for each action value, such that $Q(a) \leq U_t(a)$ with high probability
- This depends on the number of times $N_t(a)$ action a has been selected
- Select action maximizing Upper Confidence Bound (UCB)

$$a_t = \arg\max_{a \in \mathcal{A}} [U_t(a)]$$

Confidence Bounds for SubGaussian Variables

Corollary 5.5 [Lattimore and Szepesvari, Bandit Algorithms]. Assume that $X_i - \mu$ are independent, σ-sub-Gaussian random variables. Let $\hat{\mu} = \frac{1}{n}\sum_{t=1}^{n} X_t$. Then for any $\varepsilon \geq 0$,

$$\mathbb{P}(\hat{\mu} \geq \mu + \varepsilon) \leq \exp\left(-\frac{n\varepsilon^2}{2\sigma^2}\right), \qquad \mathbb{P}(\hat{\mu} \leq \mu - \varepsilon) \leq \exp\left(-\frac{n\varepsilon^2}{2\sigma^2}\right),$$

Confidence Bounds for SubGaussian Variables to UCB

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Therefore: For any $\delta \in (0, 1]$, with probability at least $1 - \delta$,

$$\mu \leq \hat{\mu} + \sqrt{\frac{2\sigma^2 \log(1/\delta)}{n}}.$$

UCB Bandit Regret

- This leads to the UCB1 algorithm (assume rewards are 1-sub-Gaussian ($\sigma^2 = 1$)

$$a_t = \arg\max_{a \in \mathcal{A}} \left[\hat{Q}(a) + \sqrt{\frac{2 \log \frac{1}{\delta}}{N_t(a)}} \right]$$

Toy Fake Example: Ways to Treat Broken Toes, Thompson Sampling

- True (unknown) parameters for each arm (action) are
  - surgery: $Q(a^1) = \theta_1 = .95$
  - buddy taping: $Q(a^2) = \theta_2 = .9$
  - doing nothing: $Q(a^3) = \theta_3 = .1$
- Optimism under uncertainty, UCB1 (Auer, Cesa-Bianchi, Fischer 2002)
  1. Sample each arm once

Toy Fake Example: Ways to Treat Broken Toes, Optimism

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  2. Set $t = 3$, Compute upper confidence bound on each action

$$UCB(a) = \hat{Q}(a) + \sqrt{\frac{2 \log \frac{1}{\delta}}{N_t(a)}}$$

Toy Fake Example: Ways to Treat Broken Toes, Optimism

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$$UCB(a) = \hat{Q}(a) + \sqrt{\frac{2 \log \frac{1}{\delta}}{N_t(a)}}$$

  3. $t = 3$, Select action $a_t = \arg\max_a UCB(a)$,
  4. Observe reward 1
  5. Compute upper confidence bound on each action

Toy Fake Example: Ways to Treat Broken Toes, Optimism

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  2. Set $t = 3$, Compute upper confidence bound on each action

$$UCB(a) = \hat{Q}(a) + \sqrt{\frac{2 \log \frac{1}{\delta}}{N_t(a)}}$$

  3. $t = t + 1$, Select action $a_t = \arg\max_a UCB(a)$,
  4. Observe reward 1
  5. Compute upper confidence bound on each action

Toy Example: Ways to Treat Broken Toes, Optimism, Assessing Regret

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Action	Optimal Action	Regret
a^1	a^1	
a^2	a^1	
a^3	a^1	
a^1	a^1	
a^2	a^1	

Confidence Level δ

- Subtle
- Union bound: $P(\cup E_i) \leq \sum_i P(E_i)$

Regret Bound for UCB Multi-armed Bandit Sketch

Regret Bound for UCB Multi-armed Bandit Sketch

Regret Bound for UCB Multi-armed Bandit Sketch

Regret Bound for UCB (Sketch)

Let $\Delta_i = \mu^\star - \mu_i$, where $\mu_i = \mathbb{E}[a_i]$ and $\mu^\star$ is the mean reward of the optimal arm. Let $N_i(n)$ denote the number of times arm i is pulled up to time n.

$$R_n = \sum_{i:\Delta_i>0} \Delta_i \, \mathbb{E}[N_i(n)].$$

Define the "good event" [the true mean is always below its UCB]:

$$G_i := \left\{ \mu_i \leq \min_t \mathrm{UCB}_i(t,\delta) \right\},$$

Recall

$$\mathrm{UCB}_i(t,\delta) = \hat{\mu}_i(t-1) + \sqrt{2\log(1/\delta)/N_{t-1}(i)}.$$

Let

$$U_i := 2\log(1/\delta)/\Delta_i^2.$$

Claim. If G_i holds, then arm $i \neq a^\star$ is played at most U_i times.
Proof (by contradiction). Assume G_i holds and that $N_t(i) > U_i$ for some t. Then

$$\mathrm{UCB}_i(t,\delta) \leq \mu_i + \sqrt{2\log(1/\delta)/U_i} = \mu_i + \Delta_i = \mu^\star.$$

Also, by definition of G_i,

$$\mu^\star < \mathrm{UCB}_{a^\star}(t,\delta),$$

so arm i cannot be selected at time t, a contradiction. Hence $N_t(i) \leq U_i$ when G_i holds.
Therefore,

$$\mathbb{E}[N_i(n)] \leq U_i + \Pr(G_i^c)\, n.$$

Choosing $\delta = 1/n^2$ and applying a union bound ($Pr(G_i^c) \leq \sum_n \delta = n\delta = \frac{1}{n}$) yields

$$R_n \lesssim \sum_{i:\Delta_i>0} \frac{4\log n}{\Delta_i} + \sum_i \Delta_i.$$

Optional Check Your Understanding

- An alternative would be to always select the arm with the highest lower bound
- Does this ensure low regret?
- Consider a two arm case for simplicity

Today

- Setting: Introduction to multi-armed bandits & Approach: greedy methods
- Framework: Regret
- Approach: ϵ-greedy methods
- Approach: Optimism under uncertainty
- Note: bandits are a simpler place to see these ideas, but these ideas will extend to MDPs
- Next time: more fast learning