Finding Similar Items: Locality Sensitive Hashing
New thread: High dim. data

High dim. data
- Locality sensitive hashing
- Clustering
- Dimensionality reduction

Graph data
- PageRank, SimRank
- Network analysis
- Spam detection

Infinite data
- Filtering data streams
- Web advertising
- Queries on streams

Machine learning
- SVM
- Decision trees
- Perceptron, kNN

Apps
- Recommender systems
- Association rules
- Duplicate document detection
Given a query image patch, find similar images
Collect billions of images
Determine feature vector for each image (4k dim)
Given a query Q, find nearest neighbors FAST
How does it work?

Nearest neighbor query in the embedding space
Application: Visual Search

Visually similar results

- Nike
- V
- "zizi repetto"
- kris van assche sneakers
- This COS top from the men's section ticks all the right...
- stan smith outfits - Buscar con Google
- Glorious Ladies
Many problems can be expressed as finding “similar” sets:
- Find near-neighbors in high-dimensional space
- **Examples:**
  - Pages with similar words
    - For duplicate detection, classification by topic
  - Customers who purchased similar products
    - Products with similar customer sets
  - Images with similar features
    - Image completion
  - Recommendations and search
Problem for today’s lecture

- **Given:** High dimensional data points $x_1, x_2, \ldots$  
  - **For example:** Image is a long vector of pixel colors
- **And some distance function** $d(x_1, x_2)$  
  - which quantifies the “distance” between $x_1$ and $x_2$
- **Goal:** Find all pairs of data points $(x_i, x_j)$ that are within distance threshold $d(x_i, x_j) \leq s$
- **Note:** Naïve solution would take $O(N^2)$  
  where $N$ is the number of data points
- **MAGIC:** This can be done in $O(N)$!! How??
LSH: The Bigfoot of CS

- LSH is really a family of related techniques
- In general, one throws items into buckets using several different “hash functions”
- You examine only those pairs of items that share a bucket for at least one of these hashings

**Upside:** Designed correctly, only a small fraction of pairs are ever examined

**Downside:** There are *false negatives* – pairs of similar items that never even get considered
Motivating Application: Finding Similar Documents
Suppose we need to find near-duplicate documents among $N = 1$ million documents

Naïvely, we would have to compute **pairwise similarities** for every pair of docs

- $N(N - 1)/2 \approx 5 \times 10^{11}$ comparisons
- At $10^5$ secs/day and $10^6$ comparisons/sec, it would take **5 days**
- For $N = 10$ million, it takes more than a year...

Similarly, we have a dataset of 10m images, quickly find the image most similar to query image $Q$
3 Essential Steps for Similar Docs

1. **Shingling**: Converts a document into a set representation (Boolean vector)

2. **Min-Hashing**: Convert large sets to short signatures, while preserving similarity

3. **Locality-Sensitive Hashing**: Focus on pairs of signatures likely to be from similar documents
   - **Candidate pairs!**
The set of strings of length $k$ that appear in the document.

**Signatures:** short integer vectors that represent the sets, and reflect their similarity.

**Candidate pairs:** those pairs of signatures that we need to test for similarity.
Step 1: **Shingling**
Convert a document into a set

The set of strings of length $k$ that appear in the document
Documents as High-Dim Data

Step 1: **Shingling:** Converts a document into a set

- A *k-shingle* (or *k-gram*) for a document is a sequence of *k* tokens that appears in the doc
  - Tokens can be characters, words or something else, depending on the application
  - Assume tokens = characters for examples
- To *compress long shingles*, we can *hash* them to (say) 4 bytes
- Represent a document by the set of hash values of its *k*-shingles
Example: $k=2$; document $D_1 = abcab$
Set of 2-shingles: $S(D_1) = \{ab, bc, ca\}$
Hash the shingles: $h(D_1) = \{1, 5, 7\}$

$k = 8, 9, \text{ or } 10$ is often used in practice

Benefits of shingles:
- Documents that are intuitively similar will have many shingles in common
- Changing a word only affects $k$-shingles within distance $k-1$ from the word
Document $D_i$ is represented by a set of its $k$-shingles $C_i = S(D_i)$

A natural similarity measure is the Jaccard similarity:

$$sim(D_1, D_2) = \frac{|C_1 \cap C_2|}{|C_1 \cup C_2|}$$

Jaccard distance: $d(C_1, C_2) = 1 - \frac{|C_1 \cap C_2|}{|C_1 \cup C_2|}$

3 in intersection.
8 in union.
Jaccard similarity
$= \frac{3}{8}$
Encode sets using 0/1 (bit, Boolean) vectors

- **Rows** = elements (shingles)
- **Columns** = sets (documents)
  - 1 in row \( e \) and column \( s \) if and only if \( e \) is a member of \( s \)
  - Column similarity is the Jaccard similarity of the corresponding sets (rows with value 1)

- **Typical matrix is sparse!**

- **Each document is a column:**
  - **Example:** \( \text{sim}(C_1, C_2) = ? \)
    - Size of intersection = 3; size of union = 6, Jaccard similarity (not distance) = 3/6
    - \( d(C_1, C_2) = 1 - (\text{Jaccard similarity}) = 3/6 \)

We don’t really construct the matrix; just imagine it exists.
So far:
- Documents → Sets of shingles
- Represent sets as Boolean vectors in a matrix

Next goal: Find similar columns while computing small signatures
- Similarity of columns == similarity of signatures

Warnings:
- Comparing all pairs takes too much time: Job for LSH
  - These methods can produce false negatives, and even false positives (if the optional check is not made)
Step 2: **Min-Hashing**: Convert large sets to short signatures, while preserving similarity.

Shingling: The set of strings of length $k$ that appear in the document.

**Signatures**: short integer vectors that represent the sets, and reflect their similarity.
**Hashing Columns (Signatures)**

- **Key idea:** “hash” each column $C$ to a small **signature** $h(C)$, such that:
  - $\text{sim}(C_1, C_2)$ is the same as the “similarity” of signatures $h(C_1)$ and $h(C_2)$

- **Goal:** Find a hash function $h(\cdot)$ such that:
  - If $\text{sim}(C_1, C_2)$ is high, then with high prob. $h(C_1) = h(C_2)$
  - If $\text{sim}(C_1, C_2)$ is low, then with high prob. $h(C_1) \neq h(C_2)$

- **Idea:** Hash docs into buckets. Expect that “most” pairs of near duplicate docs hash into the same bucket!
Goal: Find a hash function $h(\cdot)$ such that:
- if $\text{sim}(C_1, C_2)$ is high, then with high prob. $h(C_1) = h(C_2)$
- if $\text{sim}(C_1, C_2)$ is low, then with high prob. $h(C_1) \neq h(C_2)$

Clearly, the hash function depends on the similarity metric:
- Not all similarity metrics have a suitable hash function

There is a suitable hash function for the Jaccard similarity: It is called Min-Hashing
Permute the rows of the Boolean matrix using some permutation $\pi$
- Thought experiment – not real

Define **minhash function** for this permutation $\pi$, $h_\pi(C) =$ the number of the first (in the permuted order) row in which column $C$ has value 1.
- Denoted this as: $h_\pi(C) = \min_\pi \pi(C)$

Apply, to all columns, several randomly chosen permutations $\pi$ to create a **signature** for each column

**Result is a signature matrix**: Columns = sets, Rows = minhash values for each permutation $\pi$
Min-Hashing Example

Input matrix (Shingles x Documents)

Permutation $\pi$

Signature matrix $M$

$h_\pi(C) = \min_\pi \pi(C)$

$2^{nd}$ element of the permutation (row 1) is the first to map to a 1

$h_2(3)=1$ (permutation 2, column 3)

$4^{th}$ element of the permutation (row 1) is the first to map to a 1
Students sometimes ask whether the minhash value should be the original number of the row, or the number in the permuted order (as we did in our example)

**Answer:** It doesn’t matter

- We only need to be consistent, and assure that two columns get the same value if and only if their first 1’s in the permuted order are in the same row.
Choose a random permutation $\pi$

Claim: $\Pr[h_\pi(C_1) = h_\pi(C_2)] = \text{sim}(C_1, C_2)$

Why?

- Let $X$ be a doc (set of shingles), $z \in X$ is a shingle
- Then: $\Pr[\pi(z) = \min(\pi(X))] = 1/|X|$
  - It is equally likely that any $z \in X$ is mapped to the $\min$ element
- Now, let $y$ be s.t. $\pi(y) = \min(\pi(C_1 \cup C_2))$
- Then either: $\pi(y) = \min(\pi(C_1))$ if $y \in C_1$, or $\pi(y) = \min(\pi(C_2))$ if $y \in C_2$
- So the prob. that both are true is the prob. $y \in C_1 \cap C_2$
- $\Pr[\min(\pi(C_1))=\min(\pi(C_2))] = |C_1 \cap C_2| / |C_1 \cup C_2| = \text{sim}(C_1, C_2)$

One of the two cols should have 1 at position $y$
Given cols C₁ and C₂, rows are classified as:

<table>
<thead>
<tr>
<th></th>
<th>C₁</th>
<th>C₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

- Define: \( a = \# \text{ rows of type A} \), etc.
- Note: \( \text{sim}(C₁, C₂) = a/(a + b + c) \)
- Then: \( \text{Pr}[h(C₁) = h(C₂)] = Sim(C₁, C₂) \)
  - Look down the permuted cols C₁ and C₂ until we see a 1
  - If it’s a type-A row, then \( h(C₁) = h(C₂) \)
  - If a type-B or type-C row, then not
Similarity for Signatures

- We know: \( \Pr[h_{\pi}(C_1) = h_{\pi}(C_2)] = sim(C_1, C_2) \)
- Now generalize to multiple hash functions

- The *similarity of two signatures* is the fraction of the hash functions in which they agree

- Thus, the expected similarity of two signatures equals the Jaccard similarity of the columns or sets that the signatures represent
  - And the longer the signatures, the smaller will be the expected error
### Min-Hashing Example

#### Input matrix (Shingles x Documents)

<table>
<thead>
<tr>
<th>Permutation $\pi$</th>
<th>2</th>
<th>4</th>
<th>3</th>
<th>1</th>
<th>6</th>
<th>5</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

#### Signature matrix $M$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>1</th>
<th>2</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

#### Similarities:

<table>
<thead>
<tr>
<th></th>
<th>1-3</th>
<th>2-4</th>
<th>1-2</th>
<th>3-4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Col/Col</td>
<td>0.75</td>
<td>0.75</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Sig/Sig</td>
<td>0.67</td>
<td>1.00</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Permuting rows even once is prohibitive

Row hashing!

- Pick $K = 100$ hash functions $h_i$
- Ordering under $h_i$ gives a random permutation $\pi$ of rows!

One-pass implementation

- For each column $c$ and hash-func. $h_i$ keep a “slot” $M(i, c)$ for the min-hash value of column $c$ and hash-func $i$
- Initialize all $M(i, c) = \infty$

Scan rows looking for 1s

- Suppose row $j$ has 1 in column $c$
- Then for each $h_i$:
  - If $h_i(j) < M(i, c)$, then $M(i, c) \leftarrow h_i(j)$

---

How to pick a random hash function $h(x)$?

Universal hashing:

$$h_{a,b}(x)=((a \cdot x + b) \mod p) \mod N$$

where:

- $a,b$ … random integers
- $p$ … prime number ($p > N$)
Step 3: **Locality Sensitive Hashing:**
Focus on pairs of signatures likely to be from similar documents.

**Candidate pairs:** those pairs of signatures that we need to test for similarity.

- **Docu-ment** → **Shingling**
  - The set of strings of length $k$ that appear in the document

- **Min-Hash-ing**
  - **Signatures:** short integer vectors that represent the sets, and reflect their similarity

- **Locality-Sensitive Hashing**
Goal: Find documents with Jaccard similarity at least $s$ (for some similarity threshold, e.g., $s=0.8$)

LSH – General idea: Use a hash function that tells whether $x$ and $y$ is a candidate pair: a pair of elements whose similarity must be evaluated

For Min-Hash matrices:
- Hash columns of signature matrix $M$ to many buckets
- Each pair of documents that hashes into the same bucket is a candidate pair
Pick a similarity threshold $s$ ($0 < s < 1$)

Columns $x$ and $y$ of $M$ are a candidate pair if their signatures agree on at least fraction $s$ of their rows:

$$M(i, x) = M(i, y)$$

for at least frac. $s$ values of $i$

- We expect documents $x$ and $y$ to have the same (Jaccard) similarity as their signatures
Big idea: Hash columns of signature matrix $M$ several times

Arrange that (only) similar columns are likely to hash to the same bucket, with high probability

Candidate pairs are those that hash to the same bucket
Partition $M$ into $b$ Bands

Signature matrix $M$

$b$ bands

$r$ rows per band

One signature
Partition $M$ into Bands

- Divide matrix $M$ into $b$ bands of $r$ rows

- For each band, hash its portion of each column to a hash table with $k$ buckets
  - Make $k$ as large as possible

- **Candidate** column pairs are those that hash to the same bucket for $\geq 1$ bands

- Tune $b$ and $r$ to catch most similar pairs, but few non-similar pairs
Columns 2 and 6 are probably identical (candidate pair)

Columns 6 and 7 are surely different.

Matrix $M$

$r$ rows

$b$ bands

Buckets
Simplifying Assumption

- There are **enough buckets** that columns are unlikely to hash to the same bucket unless they are **identical** in a particular band.

- Hereafter, we assume that “**same bucket**” means “**identical in that band**”.

- Assumption needed only to simplify analysis, not for correctness of algorithm.
Example of Bands

Assume the following case:

- Suppose 100,000 columns of $M$ (100k docs)
- Signatures of length 100, stored as integers (rows)
- Therefore, signatures take 40MB
- **Goal:** Find pairs of documents that are at least $s = 0.8$ similar
- Choose $b = 20$ bands of $r = 5$ integers/band
**C₁, C₂ are 80% Similar**

- **Find pairs of \( \geq s = 0.8 \) similarity, set \( b = 20, \ r = 5 \)**
- **Assume**: \( \text{sim}(C₁, C₂) = 0.8 \)
  - Since \( \text{sim}(C₁, C₂) \geq s \), we want \( C₁, C₂ \) to be a candidate pair: We want them to hash to at least 1 common bucket (at least one band is identical)
- **Probability \( C₁, C₂ \) identical in one particular band**: \( (0.8)^5 = 0.328 \)
- **Probability \( C₁, C₂ \) are *not* similar in all of the 20 bands**: \( (1 - 0.328)^{20} = 0.00035 \)
  - i.e., about 1/3000th of the 80%-similar column pairs are false negatives (we miss them)
- **We would find 99.965% pairs of truly similar documents**
C₁, C₂ are 30% Similar

- **Find pairs of** \( \geq s=0.8 \) similarity, set \( b=20, r=5 \)
- **Assume:** \( \text{sim}(C₁, C₂) = 0.3 \)
  - Since \( \text{sim}(C₁, C₂) < s \) we want \( C₁, C₂ \) to hash to **NO common buckets** (all bands should be different)
- **Probability C₁, C₂ identical in one particular band:** \( (0.3)^5 = 0.00243 \)
- Probability \( C₁, C₂ \) identical in at least 1 of 20 bands: \( 1 - (1 - 0.00243)^{20} = 0.0474 \)
  - In other words, approximately 4.74% pairs of docs with similarity 0.3 end up becoming candidate pairs
  - They are **false positives** since we will have to examine them (they are candidate pairs) but then it will turn out their similarity is below threshold \( s \)
LSH Involves a Tradeoff

- **Pick:**
  - The number of Min-Hashes (rows of $M$)
  - The number of bands $b$, and
  - The number of rows $r$ per band to balance false positives/negatives
    - Note, $M = b \times r$

- **Example:** If we had only 10 bands of 10 rows, the number of false positives would go down, but the number of false negatives would go up
Analysis of LSH – What We Want

Similarity \( t = \text{sim}(C_1, C_2) \) of two sets

Probability of sharing a bucket

No chance if \( t < s \)

Similarity threshold \( s \)

Probability = 1 if \( t > s \)

Say “yes” if you are below the line.
Remember:
Probability of equal hash-values = similarity

Probability of sharing a bucket

Similarity $t = \text{sim}(C_1, C_2)$ of two sets
What 1 Band of 1 Row Gives You

Probability of sharing a bucket

Similarity $t = \text{sim}(C_1, C_2)$ of two sets

Say "yes" if you are below the line.

False positives

False negatives
Say columns $C_1$ and $C_2$ have similarity $t$

Pick any band ($r$ rows)

- Prob. that all rows in band equal = $t^r$
- Prob. that some row in band unequal = $1 - t^r$

Prob. that no band identical = $(1 - t^r)^b$

Prob. that at least 1 band identical = $1 - (1 - t^r)^b$
What \( b \) Bands of \( r \) Rows Gives You

- All rows of a band are equal
- Some row of a band unequal
- At least one band identical
- No bands identical
- Probability of sharing a bucket

Similarity \( t = \text{sim}(C_1, C_2) \) of two sets

\[
1 - \left(1 - t^r \right)^b
\]
Example: $b = 20; r = 5$

- **Similarity threshold $s$**
- **Prob. that at least 1 band is identical:**

<table>
<thead>
<tr>
<th>$s$</th>
<th>$1-(1-s^r)^b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.006</td>
</tr>
<tr>
<td>0.3</td>
<td>0.047</td>
</tr>
<tr>
<td>0.4</td>
<td>0.186</td>
</tr>
<tr>
<td>0.5</td>
<td>0.470</td>
</tr>
<tr>
<td>0.6</td>
<td>0.802</td>
</tr>
<tr>
<td>0.7</td>
<td>0.975</td>
</tr>
<tr>
<td>0.8</td>
<td>0.9996</td>
</tr>
</tbody>
</table>
Picking $r$ and $b$: The S-curve

- Picking $r$ and $b$ to get the best S-curve
  - 50 hash-functions ($r=5$, $b=10$)

Yellow area: False Negative rate
Blue area: False Positive rate
- Tune $M, b, r$ to get almost all pairs with similar signatures, but eliminate most pairs that do not have similar signatures
- Check in main memory that candidate pairs really do have similar signatures
- Optional: In another pass through data, check that the remaining candidate pairs really represent similar documents
Summary: 3 Steps

- **Shingling:** Convert documents to set representation
  - We used hashing to assign each shingle an ID
- **Min-Hashing:** Convert large sets to short signatures, while preserving similarity
  - We used similarity preserving hashing to generate signatures with property \( \Pr[h_\pi(C_1) = h_\pi(C_2)] = sim(C_1, C_2) \)
  - We used hashing to get around generating random permutations
- **Locality-Sensitive Hashing:** Focus on pairs of signatures likely to be from similar documents
  - We used hashing to find **candidate pairs** of similarity \( \geq s \)