Instructions that will appear on the real exam

- **DO NOT OPEN THE EXAM UNTIL YOU ARE INSTRUCTED TO.**
- **The last page of this exam is a sheet with some useful formulas and theorem statements.** Feel free to rip it off of the exam.
- Answer all of the questions as well as you can. You have **180 minutes.**
- The exam is **non-collaborative**; you must complete it on your own. If you have any clarification questions, please ask the course staff. We cannot provide any hints or help.
- This exam is **closed-book**, except for **up to three double-sided sheets of paper** that you have prepared ahead of time. You can have anything you want written on these sheets of paper.
- **Please DO NOT separate pages of your exam** (other than the reference sheet on the last page). The course staff is not responsible for finding lost pages, and you may not get credit for a problem if it goes missing.
- There are a few pages of extra paper at the back of the exam in case you run out of room on any problem. If you use them, please clearly indicate on the relevant problem page that you have used them, and please clearly label any work on the extra pages.
- Please make sure to sign out of the roster when handing in your completed exam to the teaching team.
- **Please do not discuss the exam until after solutions are posted!**

General Advice

- If you get stuck on a question or a part, move on and come back to it later. The questions on this exam have a wide range of difficulty, and you can do well on the exam even if you don't get a few questions.
- Pay attention to the point values. Don't spend too much time on questions that are not worth a lot of points.
- There are **100** total points on this exam.

Name and SUNet ID (please print clearly):

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Honor Code

The Honor Code is an undertaking of the Stanford academic community, individually and collectively. Its purpose is to uphold a culture of academic honesty. Students will support this culture of academic honesty by neither giving nor accepting unpermitted academic aid on this examination.

This course is participating in the proctoring pilot overseen by the Academic Integrity Working Group (AIWG), therefore proctors will be present in the exam room. The purpose of this pilot is to determine the efficacy of proctoring and develop effective practices for proctoring in-person exams at Stanford.

Unpermitted Aid on this exam includes but is not limited to the following: collaboration with anyone else; reference materials other than your cheat-sheet (see below); and internet access.

Permitted aid on this exam includes a "cheat-sheet:" three double-sided sheets of paper with anything written on them, which you have prepared yourself ahead of time.

I acknowledge and attest that I will abide by the Honor Code:

[signed] ____

Exam Break Sign-out

I pledge that during my exam break:

- I will not bring any paper, electronic devices (phone, smart watch, smart glasses, etc), or aid (permitted or unpermitted) *out of or into* the exam room.
- I will not communicate with anyone other than the course instructional staff about the content of the exam.

Signature Confirming Honor Code Pledge	Exit Time	Return Time	Proctor Initial	Length (min)

If you are feeling unwell and are not able to complete the exam, please connect with the proctor to discuss options.

Good Luck!

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1. **(20 pt.)** For each of the parts below, select the *best bound* you can to correctly fill in the blank.

   (a) **(5 pt.)** Let $Y_1, Y_2, \ldots, Y_n$ be i.i.d. random variables in $\{0, 1\}$, with $\mathbb{E}[Y_i] = p$, and let $S = \sum_{i=1}^{n} Y_i$. Suppose that $\Pr[S \geq pn + t] \leq 0.1$. Then t can be as small as ______.

   (A) $O(\sqrt{\log n})$ (B) $O(\log n)$ (C) $O(\sqrt{n})$ (D) $O(n)$

   (b) **(5 pt.)** Let $Y_1, \ldots, Y_n$ be i.i.d. random variables, uniformly distributed in $[0, 1]$. Let $Y = \sqrt{\sum_{i=1}^{n} Y_i^2}$. Then

   $$\Pr[|Y - \mathbb{E}Y| \geq t] \leq \_\_\_\_\_$$

   (A) $O(1/\sqrt{t})$ (B) $O(1/t)$ (C) $\exp(-\Omega(t^2))$ (D) $\exp(-\Omega(t^2/n))$

   (c) **(5 pt.)** Let $Y_1, Y_2, \ldots, Y_k \in \{0, 1\}$ be pairwise independent random variables with $\mathbb{E}[Y_i] = 1/2$. (Recall that this means that for any $i \neq j$, Y_i and Y_j are independent). Then

   $$\Pr\left[\sum_{i=1}^{k} Y_i = 0\right] \leq \_\_\_\_\_$$

   (A) $\exp(-\Omega(k))$ (B) $O(1/k)$ (C) $O(1/\sqrt{k})$ (D) $O(1)$

   (d) **(5 pt.)** Let G be a graph with n vertices and m edges, where $m = 20n$. Then there is an independent set in G of size at least ______.

   (A) $\Omega(1)$ (B) $\Omega(\log n)$ (C) $\Omega(\sqrt{n})$ (D) $\Omega(n)$

2. **(20 pt.)** Let $G = (V, E)$ be a simple graph (that is, an unweighted, undirected graph with no self-loops and no parallel edges). Let $D \geq 1$ be an integer. Suppose that each vertex $v \in V$ is associated with a set $S(v)$ of colors of size exactly $10D$. Suppose also that for each $v \in V$ and $c \in S(v)$, there are at most D neighbors u of v so that $c \in S(u)$. (For example, if "blue" is in $S(v)$, then v has at most D neighbors that also like the color "blue").

   (a) **(10 pt.)** Prove that there is a way to properly color G so that for all $v \in V$, v's color is in $S(v)$. (Recall that a *proper* coloring is a coloring of the vertices so that no two neighboring vertices have the same color).

   [**HINT:** *Try the LLL, and consider a bad event $A_{e,c}$ for each edge e and each color c.*]

another part on next page

(b) **(5 pt.)** Give a randomized algorithm to find such a coloring, given $G = (V, E)$. Your algorithm should have expected running time polynomial time in n, the number of vertices in V, and should succeed with probability 1.

You should clearly describe your algorithm so that someone who has not taken this class (but has taken, say, a basic programming/algorithms course) should be able to code it up without thinking too hard. You do not need to explain why your algorithm is correct.

3. **(20 pt.)** Let $\{X_t\}$ be an irreducible, aperiodic, time-homogeneous Markov chain on states $\{1, \ldots, n\}$, so that for all $j \in \{1, \ldots, n\}$, $\Pr[X_{t+1} = 1 | X_t = j] \geq \varepsilon$ for some $\varepsilon > 0$.

   (a) **(5 pt.)** Let π be the stationary distribution of $\{X_t\}$. Fill in the blank with the best bound you can (do not use big-Oh notation). Justify why your answer is correct (you don't need to justify why it is the best possible).

   $$\pi(1) \geq \_\_\_\_$$

   Justification:

   (b) **(15 pt.)** Show that there is a constant $C > 0$ (which depends on neither n nor ε) so that the mixing time τ_{mix} of $\{X_t\}$ satisfies $\tau_{mix} \leq \frac{C}{\varepsilon}$.
   [**HINT:** *Set up a coupling.*]
   [**HINT:** *Depending on how you do the problem, it might be helpful that* $\ln\left(\frac{1}{1-x}\right) = x + x^2/2 + x^3/3 + x^4/4 + \cdots$ *for any* $x \in (-1, 1)$.]

4. **(20 pt.)** Let $G = (V, E)$ be a directed, weighted graph with positive edge weights and no self-loops; for $(u, v) \in E$, let $w(u, v) > 0$ denote the edge weight of (u, v). Let $W = \sum_{e \in E} w(e)$ be the total weight of all the edges.

Let $\pi : V \to \{1, \ldots, n\}$ be a bijection, and think of it as an ordering of the vertices (so, $\pi(v) = 1$ means that v is the first vertex, $\pi(w) = 2$ means that w is the second vertex, and so on). For such an ordering π, define

$$\mathrm{val}(\pi) = \sum_{(u,v) \in E} \mathbf{1}[\pi(u) < \pi(v)] w(u, v).$$

That is, $\mathrm{val}(\pi)$ is the total weight of the edges that are going "forward" in the ordering.

(a) **(5 pt.)** Prove that there exists an ordering π so that $\mathrm{val}(\pi) \geq W/2$.

(b) **(5 pt.)** Let π be a uniformly random ordering. Fix $t > 0$, and condition on the values of $P_t := (\pi^{-1}(1), \pi^{-1}(2), \ldots, \pi^{-1}(t))$. That is, we are conditioning on the choices of the first through t'th vertices. Let

$$U = V \setminus \{\pi^{-1}(1), \pi^{-1}(2), \ldots, \pi^{-1}(t)\}$$

be the vertices that have not been chosen yet. Explain why there exists a $u \in U$ so that

$$\mathbb{E}[\mathrm{val}(\pi) | P_t, \pi(u) = t + 1] \geq \mathbb{E}[\mathrm{val}(\pi) | P_t]$$

more parts on next page

(c) **(5 pt.)** With the same notation as in part (b), explain how to efficiently and *deterministically* find a $u \in U$ so that

$$\mathbb{E}[\text{val}(\pi) | P_t, \pi(u) = t + 1] \geq \mathbb{E}[\text{val}(\pi) | P_t].$$

Your answer should not use any probabilistic language, and should contain a clear description of a decision procedure that runs in time polynomial in n. You don't need to explain why your answer is correct.

[**HINT:** *Consider* $\mathbb{E}[\text{val}(\pi) | P_t, \pi(u) = t + 1] - \mathbb{E}[\text{val}(\pi) | P_t]$.]

(d) **(5 pt.)** Put together your answers from the previous parts to give a deterministic algorithm to find an ordering π so that $\text{val}(\pi) \geq W/2$. Your algorithm should run in time polynomial in n. Your algorithm should be clear enough that someone who has not taken this class can code it up without thinking too hard. You do not need to explain why it is correct.

5. **(20 pt.)** Suppose that there are n people, each of whom has a hat. Everyone takes off their hat at a party and puts them in a pile. When the party is over, people try to find their hats by grabbing random hats from the pile. More precisely, what happens is this:

- A uniformly random permutation maps the n hats to the n people.
- If a person ends up with their own hat, they take it and go home.

Suppose that X_1 people find their hats this way and leave. Now there are $n - X_1$ people left and they do it again:

- A uniformly random permutation maps the remaining $n-X_1$ hats to the remaining $n - X_1$ people.
- If a person ends up with their own hat, they take it and go home.

This goes on for R rounds until everyone has left. Let X_i be the number of people who find their hats and leave in the i'th round. (That is, we end up with $X_1, X_2, \ldots, X_R$ so that $X_R > 0$ and $\sum_{i=1}^{R} X_i = n$).

(a) **(5 pt.)** Let $Y_j = \sum_{i=1}^{j}(X_i - \mathbb{E}[X_i|X_1, \ldots, X_{i-1}])$. Show that $\{Y_j\}$ is a martingale with respect to $\{X_j\}$.

(b) **(5 pt.)** Show that for any i, $\mathbb{E}[X_i|X_1, \ldots, X_{i-1}] = 1$.

more parts on next page

Continued from hat problem on previous page

(c) **(5 pt.)** Use the Martingale stopping theorem to compute $\mathbb{E}[R]$, the expected number of rounds until everyone has found their hat. You do not (yet) need to prove that the Martingale stopping theorem applies.

(d) **(5 pt.)** Prove that the Martingale Stopping Theorem applies to R.

This is the end!

This is the end of the exam! You can use this page for extra work on any problem. **Keep this page attached** to the exam packet (whether or not you use it), and if you want extra work on this page to be graded, clearly label which question your extra work is for.

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Some useful inequalities, definitions and theorem statements

Note: We have not always stated full theorems here, just the quantitative punchlines. You are responsible for knowing when each theorem applies.

Inequalities and Series

- $1 - x \le e^{-x}$ for any x.
- $(n/k)^k \le \binom{n}{k} \le (en/k)^k$ for all $k \le n$.
- $\binom{n}{k} \le \frac{n^k}{k!}$ for all $k \le n$.
- $\sum_{i=1}^{n} 1/i = \Theta(\log n)$
- $\sum_{i=1}^{n} 1/i^c = O(1)$ for all $c > 1$.

Definitions

- $f(n) = O(g(n))$ means $\exists$ constants $c, n_0 > 0$ so that for all $n \ge n_0$, $f(n) \le cg(n)$.
- $f(n) = \Omega(g(n))$ means $\exists$ constants $c, n_0 > 0$ so that for all $n \ge n_0$, $f(n) \ge cg(n)$.
- $f(n) = o(g(n))$ means that $\frac{f(n)}{g(n)} \to 0$ as $n \to \infty$.
- $f(n) = \omega(g(n))$ means that $\frac{f(n)}{g(n)} \to \infty$ as $n \to \infty$.
- If $X \sim \text{Poi}(\lambda)$, then $\Pr[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}$.
- If $X \sim N(\mu, \sigma^2)$, then the pdf of X is $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$
- If $X \sim \text{Ber}(p)$, then $X \in \{0, 1\}$ and $\Pr[X = 1] = p$.

Concentration Inequalities

- Markov's inequality: For a non-negative random variable X, $\Pr[X > t] \le \frac{\mathbb{E}X}{t}$.
- Chebyshev's inequality: For any random variable X, $\Pr[|X - \mathbb{E}X| > t] \le \frac{\text{Var}(X)}{t^2}$.
- A few Chernoff bounds: For independent $X_i \in \{0, 1\}$, if $X = \sum_{i=1}^{n} X_i$, then:
  - For $\delta > 0$, $\Pr[X \ge (1+\delta)\mathbb{E}[X]] \le \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^{\mathbb{E}[X]}$. If $\delta \in (0, 1]$ this is $\le \exp(-\delta^2\mathbb{E}[X]/3)$.
  - For $\delta \in (0, 1]$, $\Pr[X \le (1-\delta)\mathbb{E}[X]] \le \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mathbb{E}[X]}$. If $\delta \in (0, 1]$, this is $\le \exp(-\delta^2\mathbb{E}[X]/2)$.
  - For $c \ge 6$, $\Pr[X \ge c\mathbb{E}X] \le 2^{-c\mathbb{E}X}$.
  - Hoeffding's inequality: if $X_i \in [a_i, b_i]$, $\Pr\left[|X - \mathbb{E}X| \ge t\right] \le \exp\left(\frac{-2t^2}{\sum_i (a_i - b_i)^2}\right)$.

- Tail bound for Poisson random variables: If $X \sim \mathrm{Poi}(\lambda)$, then for any $c > 0$, $\Pr[|X - \lambda| \geq c] \leq 2\exp\left(\frac{-c^2}{2(c+\lambda)}\right)$.
- Azuma-Hoeffding Inequality: Let $\{Z_t\}$ be a martingale with respect to $\{X_t\}$, and suppose $|Z_i - Z_{i-1}| \leq c_i$ for all $i \leq n$. For any $\lambda > 0$, $\Pr[|Z_n - Z_0| \geq \lambda] \leq 2\exp\left(\frac{-\lambda^2}{2\sum_{i=1}^n c_i^2}\right)$.
- Method of bounded differences: if the X_t are independent and Z_t is the Doob martingale for some quantity A, you can replace the condition $|Z_i - Z_{i-1}| \leq c_i$ above with "A doesn't change by more than c_i when you change X_i."

Probabilistic Method

- Second moment method: for real-valued X, $\Pr[X = 0] \leq \frac{\mathrm{Var}[X]}{(\mathbb{E}[X])^2}$.
- LLL: Let $A_1, \ldots, A_n$ be events so that $\Pr[A_i] \leq p$ for all i, and where each A_i is mutually independent of all but d other events. Then:
  - If $pd \leq 1/4$, then $\Pr[\bigcap_i \overline{A_i}] > 0$
  - If $p(d+1) \leq 1/e$, then $\Pr[\bigcap_i \overline{A_i}] > 0$.
- Algorithmic LLL: If $p(d+1) \leq 1/e$, then you can find an assignment as above with $O(|\mathcal{A}|/(d+1))$ re-randomizations.

Markov Chains / Martingales

- Fundamental theorem of Markov chains: Let $\{X_t\}$ be an irreducible aperiodic Markov chain over a finite state space with transition matrix P. Then there is a unique stationary distribution π so that $\Pr[X_t = i | X_0 = j] \to \pi_i$ for all states i, j. Further, $1/\pi_i$ is the expected return time of state i, and $\pi P = \pi$.
- Let $\{X_t\}$ be a finite irreducible aperiodic Markov chain with a coupling $\{(X_t, Y_t)\}$. Then $\Delta(t) \leq \max_{s,s'} \Pr[X_t \neq Y_t | X_0 = s, Y_0 = s']$.
- The mixing time τ_{mix} of a Markov chain is defined as $\min\left\{t : \Delta(t) \leq \frac{1}{2e}\right\}$.
- Let $\{X_t\}$ be a finite irreducible aperiodic Markov chain and let T be a strong stationary stopping time. Then $\Delta(t) \leq \Pr[T > t]$.
- The *Doob Martingale* for a quantity A is $Z_t = \mathbb{E}[A | X_0, \ldots, X_t]$. Theorem: it is a martingale.
- Martingale stopping theorem: Let $\{Z_t\}$ be a martingale with respct to $\{X_t\}$. Let T be a stopping time for $\{X_t\}$. Then $\mathbb{E}[Z_T] = \mathbb{E}[Z_0]$ if at least one of the following holds:
  1. There is a constant c s.t. $|Z_i| \leq c$ for all i.
  2. There is a constant c s.t. $T < c$ with probability 1.
  3. $\mathbb{E}[T] < \infty$ and there is a constant c s.t. for all i, $\mathbb{E}[|Z_{i+1} - Z_i| | X_0, \ldots, X_i] < c$.