Class 13

Markov Chains I

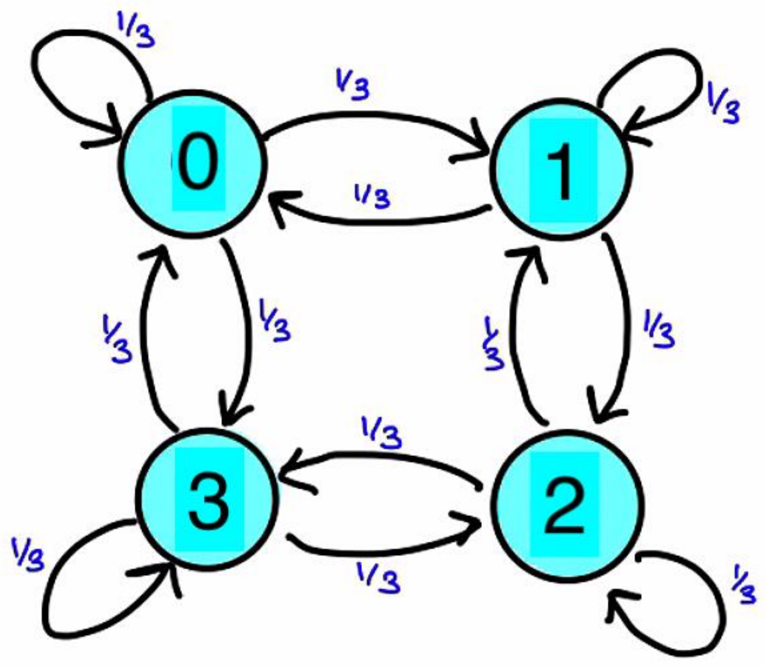
Warm-Up

- What is the transition matrix for this Markov Chain?

- How would you compute

$\Pr[\, X_t = 2 \mid X_0 = 0 \,]$ for t=1, 2, …, 100?

- As $t \to \infty$, what do you think this probability should tend to?



Announcements

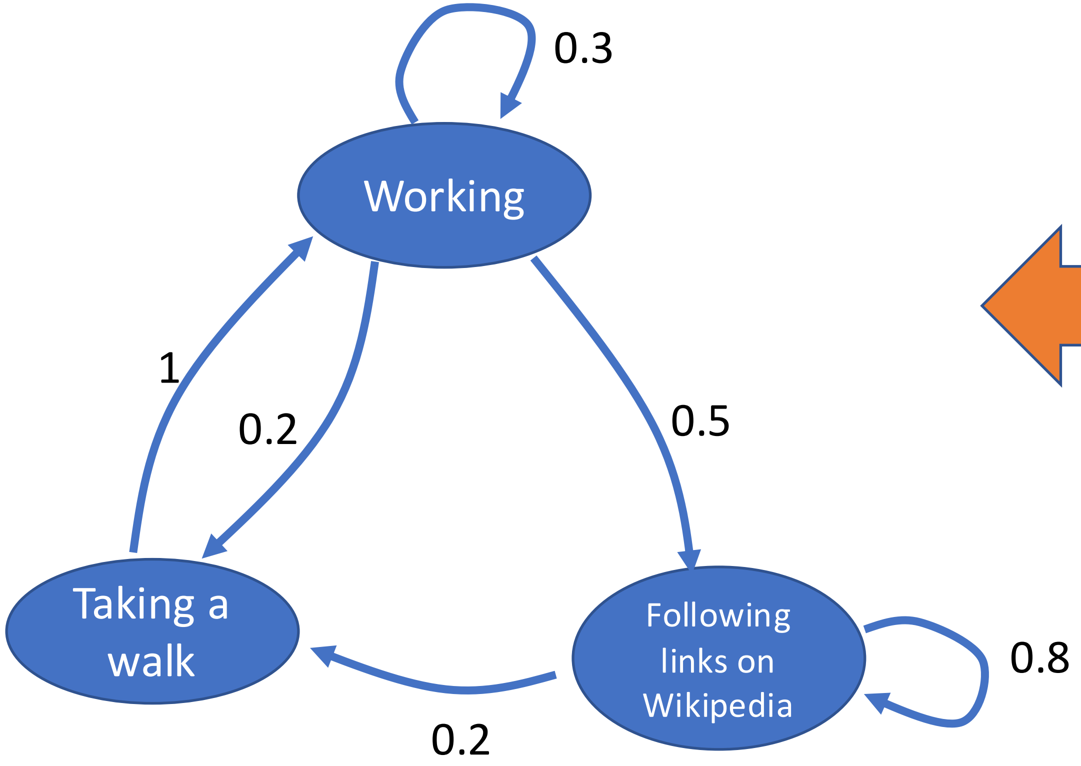
- HW6 due Friday!

- Midterm will be graded real soon now – thanks for your patience!

Recap

(Time homogeneous, finite) Markov Chains!

Memorylessness: $\Pr[\, X_t = a \mid X_0, \ldots, X_{t-1}] = \Pr[\, X_t = a \mid X_{t-1}]$



	Working	Walking	Wikipedia
Working	0.3	0.2	0.5
Walking	1	0	0
Wikipedia	0	0.2	0.8

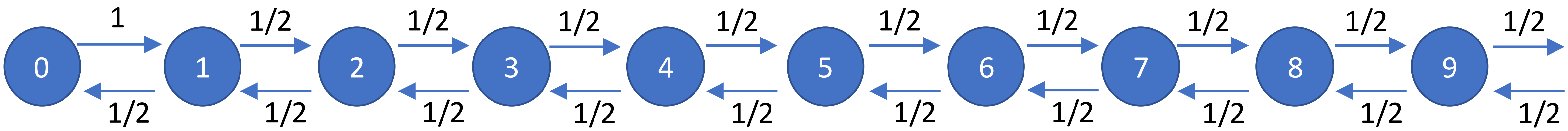
e.g., $\varphi = (x_1 \vee x_2) \wedge (\overline{x_1} \vee x_3) \wedge \cdots \wedge (x_5 \vee \overline{x_7})$

Recap

Randomized algorithm for 2SAT!

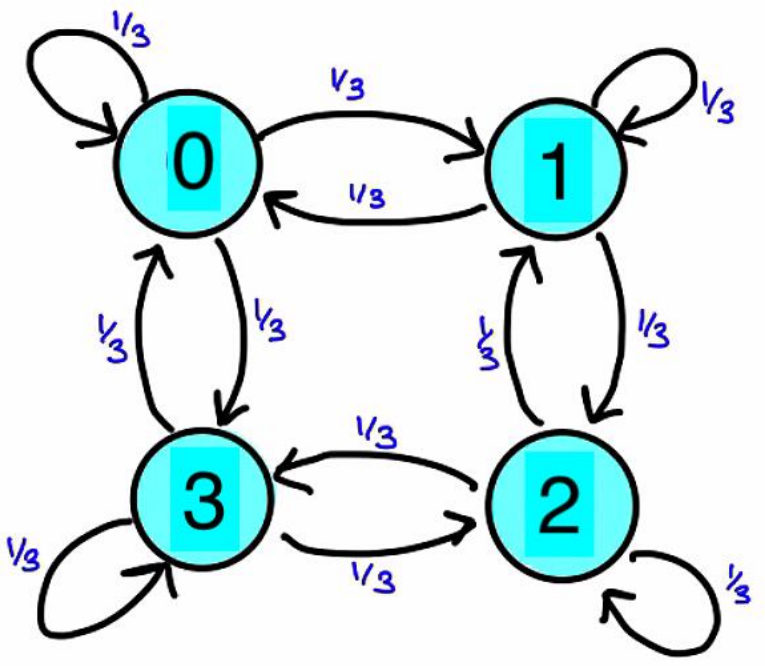
- Algorithm:
  - While not done:
    - Find an unsatisfied clause, flip one of the variables at random.

- Analysis: This Markov chain hits (n) in no more than $100n^2$ steps whp:



Questions?

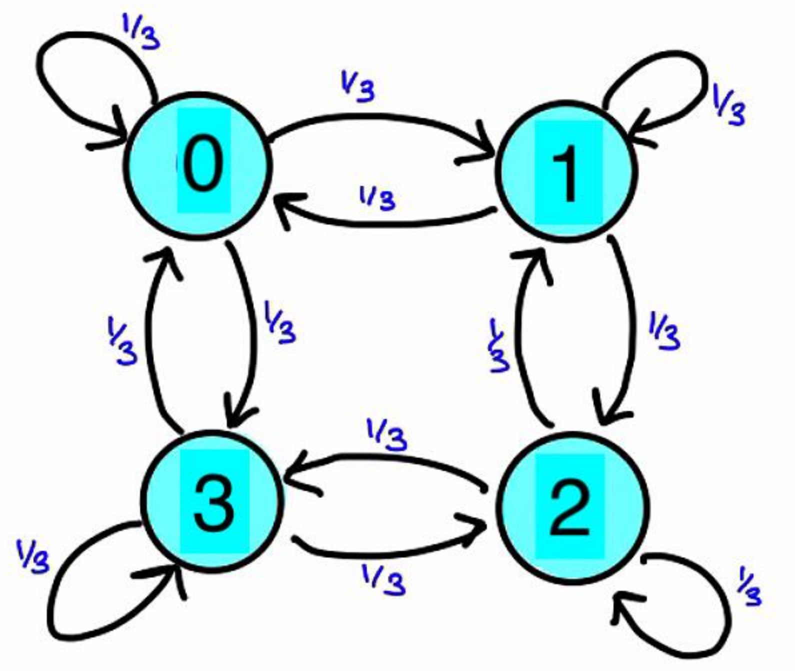
Mini-lectures, quiz, warm-up?



Warm-Up

- Transition matrix?

$$P = \begin{pmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \end{pmatrix}$$



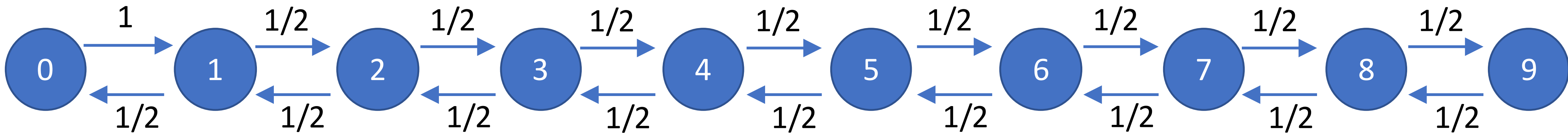
- $\Pr[\, X_t = 2 \mid X_0 = 0 \,]$ for t=1, 2, …, 100?

$$(1,0,0,0) \cdot P^t \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

- As $t \to \infty$, what do you think this prob. should go to? $\frac{1}{4}$?

In the mini-lectures

- We saw one way to analyze a chain that looked like this:

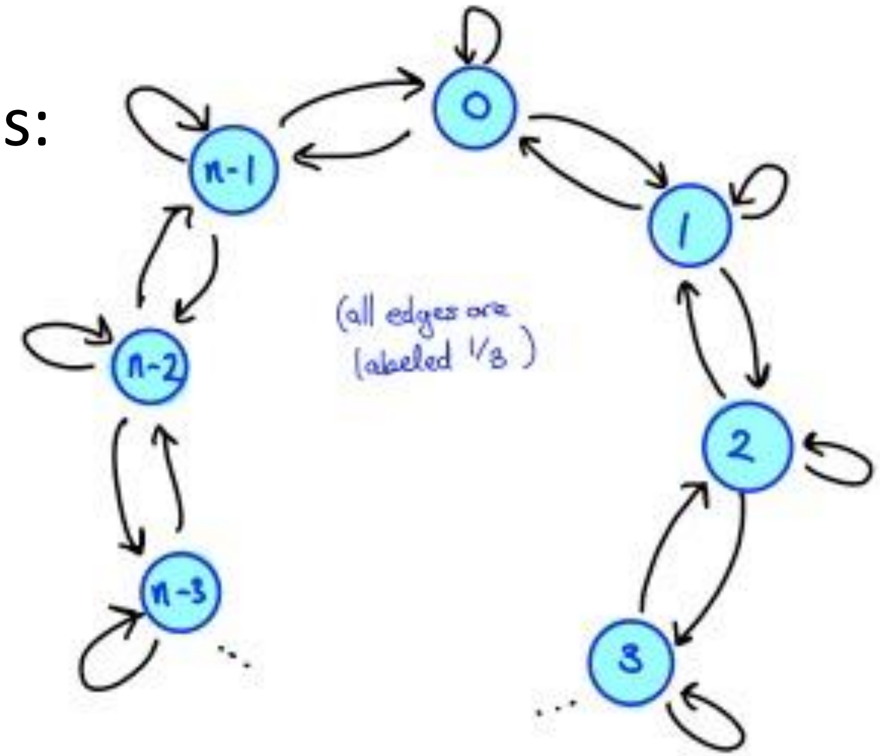


- Whp, it took $\Theta(n^2)$ steps to reach n.

Today

- We will see another way to use the transition matrix to analyze (certain nice) Markov chains.

- We'll analyze a Markov chain that looks like this:



- What should happen if the Markov chain runs for long enough?

- What's your intuition about how long "long enough" is?

How long do you think it will take for the Markov chain to get "close" to uniform?

$O(n)$ steps

0%

$O(n^2)$ steps

0%

$O(n^c)$ steps, for some $c \geq 3$

0%

$\exp(n)$ steps

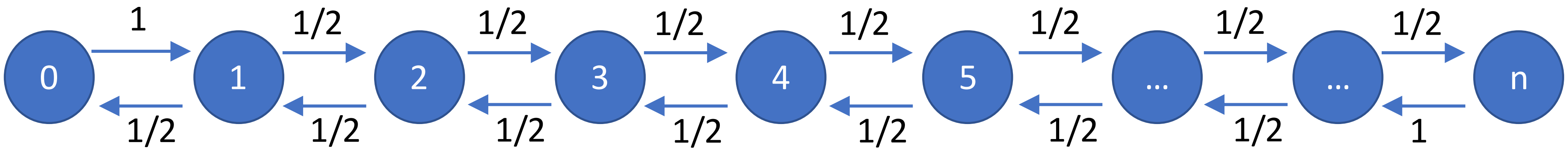
0%

No idea

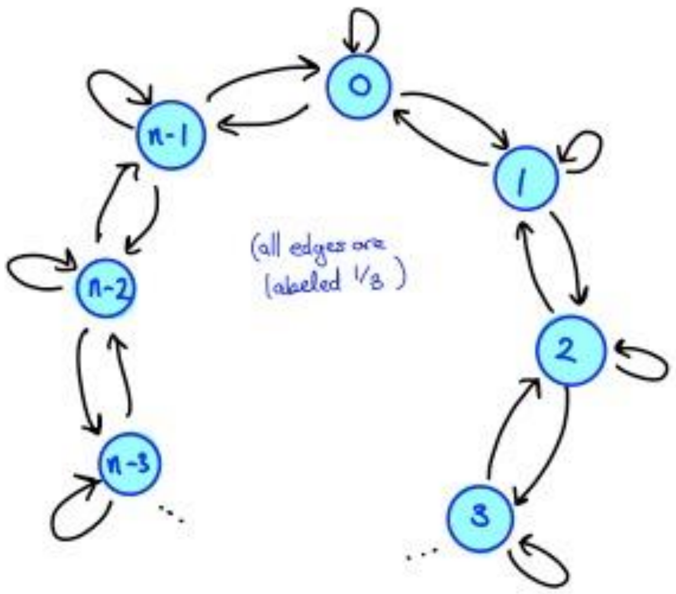
0%

Intuition?

- Maybe $O(n^2)$?



- This ↑ took $\Theta(n^2)$ to get to n…
- Same logic should apply to the number of steps you need to "wrap around" the cycle once…
- You can't be close to uniform until you can at least get around the cycle…



Today…

- We’ll see a different way to analyze this: **spectral methods!**

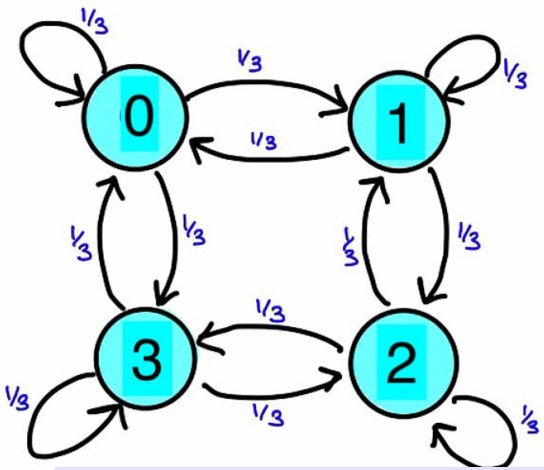
First group work!

More details on handout!

- Show that:

$$F = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

$$P = F \cdot \begin{pmatrix} 1 & & & \\ & 1/3 & & \\ & & -1/3 & \\ & & & 1/3 \end{pmatrix} \cdot F^*.$$



$$P = \begin{pmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \end{pmatrix}$$

  - Hint: show that the columns of F are eigenvectors of P. What are the eigenvalues?
- Use this to find a nicer way of computing $\Pr[\, X_t = 2 \mid X_0 = 0 \,]$

Show $P = F \cdot \begin{pmatrix} 1 & & & \\ & 1/3 & & \\ & & -1/3 & \\ & & & 1/3 \end{pmatrix} \cdot F^*.$

$$F = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

$$\begin{pmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix} = \frac{1}{3} \cdot \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}$$

$$\begin{pmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \frac{-1}{3} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix} = \frac{1}{3} \cdot \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}$$

Show $P = F \cdot \begin{pmatrix} 1 & & & \\ & 1/3 & & \\ & & -1/3 & \\ & & & 1/3 \end{pmatrix} \cdot F^*.$

$$F = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

$$\begin{pmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ & 1/3 & & \\ & & -1/3 & \\ & & & 1/3 \end{pmatrix}$$

$$P \quad \cdot \quad F \quad = \quad F \quad \cdot \quad \Lambda$$

$$P = F\Lambda F^*$$

A nicer way of computing $\Pr[\, X_t = 2 \mid X_0 = 0 \,]$

$$F = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

$$P = F \cdot \begin{pmatrix} 1 & & & \\ & 1/3 & & \\ & & -1/3 & \\ & & & 1/3 \end{pmatrix} \cdot F^*.$$

$$(1,0,0,0) \cdot P^t \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{aligned} P^t &= (F \Lambda F^*)^t \\ &= F\Lambda F^* F \Lambda F^* F \Lambda F^* F \Lambda F^* \cdots F \Lambda F^* F \Lambda F^* \\ &= F \Lambda^t F^* \quad = F \begin{pmatrix} 1 & & & \\ & (1/3)^t & & \\ & & (-1/3)^t & \\ & & & (1/3)^t \end{pmatrix} F^* \end{aligned}$$

A nicer way of computing $\Pr[\, X_t = 2 \mid X_0 = 0 \,]$

$$F = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

$$P = F \cdot \begin{pmatrix} 1 & & & \\ & 1/3 & & \\ & & -1/3 & \\ & & & 1/3 \end{pmatrix} \cdot F^*.$$

$$(1,0,0,0) \cdot F \begin{pmatrix} 1 & & & \\ & (1/3)^t & & \\ & & (-1/3)^t & \\ & & & (1/3)^t \end{pmatrix} F^* \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{4} \cdot (1,1,1,1) \begin{pmatrix} 1 & & & \\ & (1/3)^t & & \\ & & (-1/3)^t & \\ & & & (1/3)^t \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{4} - \frac{1}{2}\left(\frac{1}{3}\right)^t + \frac{1}{4}\left(-\frac{1}{3}\right)^t \quad = \frac{1}{4} + O(3^{-t})$$

So indeed this converges to $\frac{1}{4}$!
And very quickly!

Aside: Spectral analysis

- Say that a transition matrix P is symmetric.
- Write $P = V\Lambda V^*$ where V is Hermitian and Λ is diagonal with real values on the diagonal.
- $P^t = V\Lambda^t V^*$

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Aside: Spectral analysis

- $P^t = V \Lambda^t V^*$
- What can we say about $\Pr[\, X_t = j \mid X_0 = i] = e_i^T P^t e_j$?

$$= (e_i^T V) \begin{pmatrix} \lambda_1^t & & & \\ & \lambda_2^t & & \\ & & \ddots & \\ & & & \lambda_n^t \end{pmatrix} (V^* e_j)$$

- What is λ_1? (The largest eigenvalue of P?)

$$\begin{pmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \end{pmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda_1 = 1$, because every row/column is a probability distribution.

- Say that a transition matrix P is symmetric.
- Write $P = V\Lambda V^*$ where V is Hermitian and Λ is diagonal with real values on the diagonal.

Aside: Spectral analysis

- $P^t = V\Lambda^t V^*$
- What can we say about $\Pr[\, X_t = j \mid X_0 = i] = e_i^T P^t e_j$?

$$= \left(e_i^T V\right)\begin{pmatrix} 1 & & & \\ & \lambda_2^t & & \\ & & \ddots & \\ & & & \lambda_n^t \end{pmatrix}\left(V^* e_j\right)$$

All of these have $|\lambda_i| \leq 1$

$\Rightarrow$ As $t \to \infty$, all these get small*

*Assuming that $|\lambda_i| < 1$ …

Aside: Spectral analysis

- Say that a transition matrix P is symmetric.
- Write $P = V\Lambda V^*$ where V is Hermitian and Λ is diagonal with real values on the diagonal.

- $P^t = V\Lambda^t V^*$
- What can we say about $\Pr[\, X_t = j \mid X_0 = i] = e_i^T P^t e_j$?

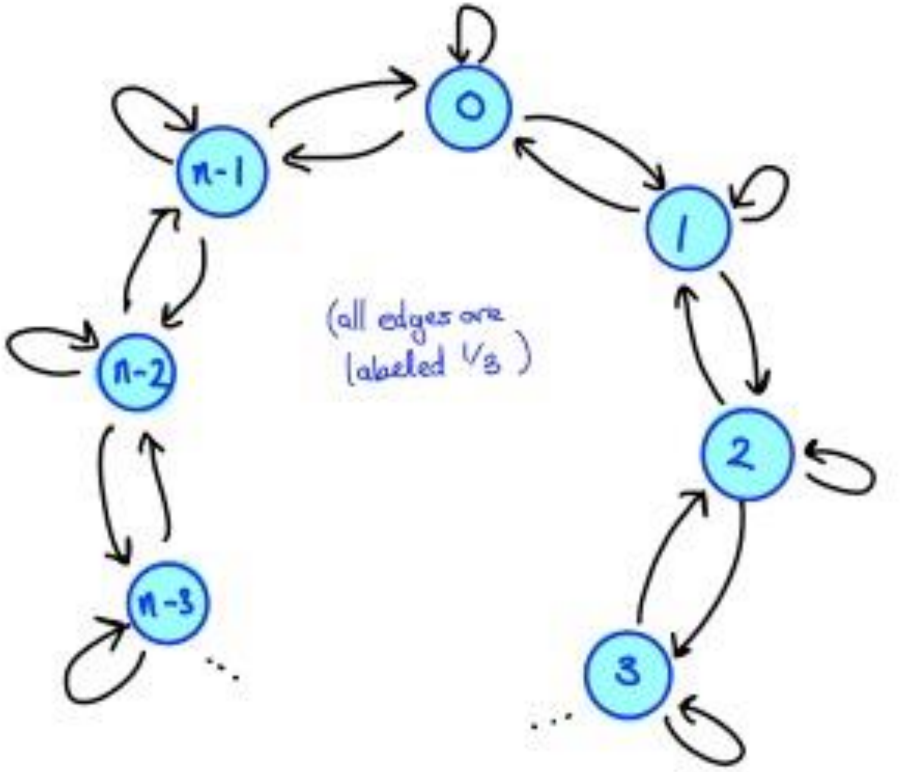
$$= (e_i^T V) \begin{pmatrix} 1 & & & \\ & \text{tiny} & & \\ & & \ddots & \\ & & & \text{tiny} \end{pmatrix} (V^* e_j)$$

$$= v_{1,i} \cdot v_{1,j} + \text{tiny} \quad \approx v_{1,i} \cdot v_{1,j}$$

So we can compute this! (Up to some small error term that we can hopefully quantify!)

Back to group work!
Do the same thing for a large cycle.

- We'll do Part 1 together since the math is a bit gross.



Part 1: Decomposing the transition matrix

Let P be the transition matrix.

Claim: $P = F_n \Lambda F_n^*$, where:

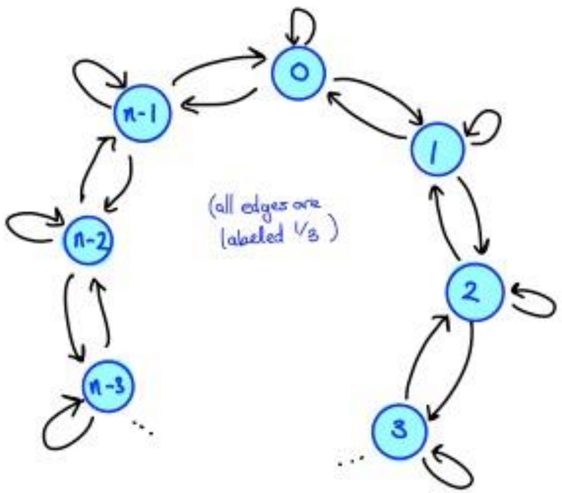
$$P = \frac{1}{3}\begin{pmatrix} 1 & 1 & & & & & 1 \\ 1 & 1 & 1 & & & & \\ & 1 & 1 & 1 & & & \\ & & 1 & \ddots & & & \\ & & & \ddots & & & \\ & & & & \ddots & 1 & \\ & & & & 1 & 1 & 1 \\ & & & & & 1 & 1 & 1 \\ 1 & & & & & & 1 & 1 \end{pmatrix}$$

$$F_n = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & & \\ 1 & \omega^3 & \omega^6 & \ddots & & \\ 1 & \vdots & & & \ddots & \\ 1 & \omega^{n-1} & & & & \omega^{(n-1)^2} \end{pmatrix}$$

$$\omega = e^{-\frac{2\pi i}{n}}$$

$$F_n[j,k] = \frac{1}{\sqrt{n}}\omega^{jk}$$

Λ is diagonal and $\Lambda_{j,j} = \frac{1 + 2\cos\left(\frac{2\pi j}{n}\right)}{3}$



$\omega = e^{-\frac{2\pi i}{n}}$

FACT: $e^{ix} + e^{-ix} = 2\cos(x)$

$\Rightarrow \omega^j + \omega^{-j} = 2\cos\left(\frac{2\pi j}{n}\right)$

Part 1: $P = F_n \Lambda F_n^*$ … why?

- As before, it's enough to show that each column of F_n is an eigenvector of P, with corresponding eigenvalue $\lambda_j = \frac{1 + 2\cos\left(\frac{2\pi j}{n}\right)}{3}$

$$\frac{1}{3}\begin{pmatrix} 1 & 1 & & & & & & 1 \\ 1 & 1 & 1 & & & & & \\ & 1 & 1 & 1 & & & & \\ & & 1 & \ddots & & & & \\ & & & & \ddots & & & \\ & & & & \ddots & 1 & & \\ & & & & 1 & 1 & 1 & \\ & & & & & 1 & 1 & 1 \\ 1 & & & & & & 1 & 1 \end{pmatrix} \frac{1}{\sqrt{n}}\begin{pmatrix} 1 \\ \omega^j \\ \omega^{2j} \\ \vdots \\ \omega^{kj} \\ \vdots \\ \vdots \\ \omega^{(n-1)j} \end{pmatrix} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

What goes in the k^{th} position?

$$\frac{1}{3\sqrt{n}}\left(\omega^{(k-1)j} + \omega^{kj} + \omega^{(k+1)j}\right)$$

$$\frac{\omega^{kj}}{3\sqrt{n}}\left(\omega^{-j} + 1 + \omega^{j}\right)$$

$$\frac{\omega^{kj}}{\sqrt{n}}\left(\frac{1 + 2\cos\left(\frac{2\pi j}{n}\right)}{3}\right)$$

Rest of the group work! See handout for more details and hints

2. Come up with an expression for $\Pr[X_t = 0 \mid X_0 = 0]$
3. Convince yourself that $\Pr[X_t = 0 \mid X_0 = 0] \to \frac{1}{n}$ as $t \to \infty$
4. (If time) How large does t have to be before

$$\Pr[X_t = 0 \mid X_0 = 0] = \frac{1+o(1)}{n}\ ?$$

$$P = F_n \Lambda F_n^*$$

$$F_n = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & & \\ 1 & \omega^3 & \omega^6 & \ddots & & \\ 1 & \vdots & & & \ddots & \\ 1 & \omega^{n-1} & & & & \omega^{(n-1)^2} \end{pmatrix}$$

Λ is diagonal and $\Lambda_{j,j} = \frac{1+2\cos\left(\frac{2\pi j}{n}\right)}{3}$

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Part 2: $\Pr[X_t = 0 \mid X_0 = 0]$

$$\Pr[X_t = 0 \mid X_0 = 0] = e_0^T F_n \Lambda^t F_n^* e_0$$

$$\frac{1}{\sqrt{n}}(1,1,1,\ldots,1) \begin{pmatrix} \Lambda_{0,0}^t & & & & \\ & \Lambda_{1,1}^t & & & \\ & & \ddots & & \\ & & & \Lambda_{n-2,n-2}^t & \\ & & & & \Lambda_{n-1,n-1}^t \end{pmatrix} \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$= \frac{1}{n}\sum_{j=0}^{n-1} \left(\frac{1 + 2\cos\left(\frac{2\pi j}{n}\right)}{3} \right)^t$$

Part 3: What happens as t gets big?

$$\Pr[X_t = 0 \mid X_0 = 0] = \frac{1}{n}\sum_{j=0}^{n-1}\left(\frac{1 + 2\cos\left(\frac{2\pi j}{n}\right)}{3}\right)^t$$

$$\rightarrow \frac{1}{n} \cdot 1$$

When $j = 0$: $\left(\frac{1 + 2\cos\left(\frac{2\pi j}{n}\right)}{3}\right)^t = \left(\frac{1+2}{3}\right)^t = 1$

When $j = 1, \dots, n-1$: $\left(\frac{1+2\cdot(<1)}{3}\right)^t = \left(\frac{1+(<2)}{3}\right)^t \rightarrow 0$ as $t \rightarrow \infty$

Part 4: How big does t need to be?

Aka, how big does t need to be before $\left(\frac{1+2\cos\left(\frac{2\pi j}{n}\right)}{3}\right)^t$ is really small for all $1 \le j \le n-1$?

$$\left(\frac{1+2\cos\left(\frac{2\pi j}{n}\right)}{3}\right)^t \approx \left(\frac{1}{3}\cdot\left(1+2\left(1-\frac{1}{2}\left(\frac{2\pi j}{n}\right)^2\right)\right)\right)^t$$

$$= \left(1-\frac{1}{3}\left(\frac{2\pi j}{n}\right)^2\right)^t$$

$$\approx \exp\left(-\frac{4\pi^2 j^2}{3}\cdot\frac{t}{n^2}\right)$$

Note: This approximation is only good for small j! But if j is large enough, then $\frac{1+2\cos\left(\frac{2\pi j}{n}\right)}{3}$ is bounded away from 1, and so $[\text{that}]^t$ is very small anyway.

If this is going to be small for all j, we need $t \gg n^2$

This matches our intuition from the beginning! ☺

Take-aways

- If we can diagonalize the transition matrix, it can make analyzing a Markov chain easier.
  - This is called “spectral analysis.”
- If the second eigenvalue of a (symmetric) transition matrix is bounded away from 1, then the Markov chain “mixes” quickly.
  - We’ll see what it means to “mix quickly” more formally next time!

Next time

- More Markov chains!
- **Heads up**, there’s slightly more video content for next time, budget time appropriately!