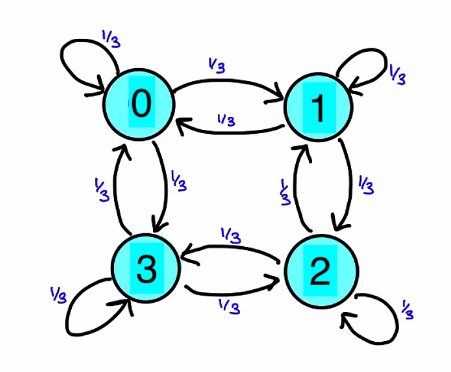
CS265, Winter 2026

Class 13: Agenda and Questions

1 Warm-Up

Consider the Markov chain given by:



Group Work

1. What is the transition matrix for this Markov chain?
2. Suppose that you start in state 0. What is the probability that you are in state 2 after one step? Two steps? Three steps? 100 steps? (Don't actually compute this, just say how you would).
3. As $t \to \infty$, what do you think is $\lim_{t\to\infty} \Pr[X_t = 2 | X_0 = 0]$? (No need for a formal proof here, just use your intuition).

2 Questions/Lecture Recap

Any questions from the minilectures and/or the quiz? (Markov chains and a randomized algorithm for 2SAT)

3 Spectral Analysis of Markov Chains

Next, we'll see how we can use linear algebra to help us out in computing things like $\Pr[X_t = 2 | X_0 = 0]$ for general t. We'll focus on example in the warm-up, but as we go, keep in mind what you think the general principle should be.

Group Work

1. Let

$$F = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

   where $i = \sqrt{-1}$. (You may recognize F as the 4×4 discrete Fourier matrix, so $F_{jk} = \frac{1}{2}e^{-2\pi ijk/4}$.) Notice that F is a Hermitian matrix, which means that $F^*F = FF^* = I$, where F^* denotes the Hermitian conjugate (e.g., take the transpose and change all of the i's to $-i$'s).

   Convince yourself that

$$P = F \cdot \begin{pmatrix} 1 & & & \\ & 1/3 & & \\ & & -1/3 & \\ & & & 1/3 \end{pmatrix} \cdot F^*,$$

   where P is the transition matrix from the warm-up.

   ***Hint**: Check that the columns of F are eigenvectors for P. What are their eigenvalues?*

   ***Note:** If your linear algebra is rusty and you trust me, just remind yourself what an eigenvector actually is. The main point here is that you should understand the conclusion of Question 1 so that you can use it in Question 2.*

2. In the warm-up, you came up with an expression for $\Pr[X_t = 2|X_0 = 0]$, which seemed pretty obnoxious to compute. Use the previous part to find a closed-form expression for this. Recall that our intuition was that $\Pr[X_t = 2|X_0 = 0] \to \frac{1}{4}$ as $t \to \infty$. Quantify this by coming up with a statement like

$$\Pr[X_t = 2|X_0 = 0] = \frac{1}{4} \pm O(\_\_\_)$$

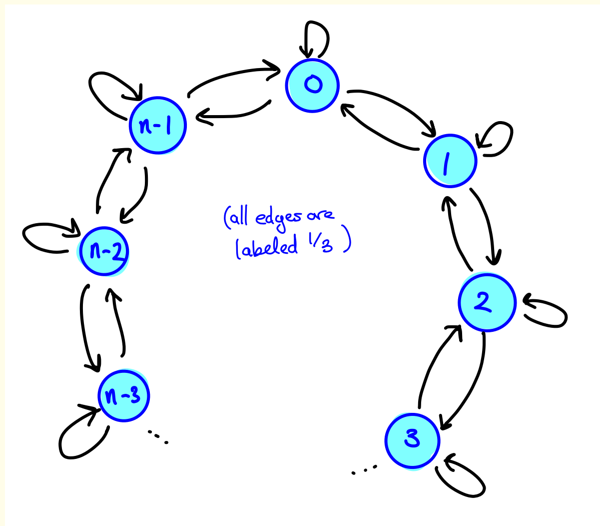
   where the thing in the $O(\_\_\_)$ term should depend on t. What is the best bound you can get?

Before we move on to larger cycles, let's take a minute to reflect on what just went on. *[Insert a bit of lecture about spectral analysis. The point is that if we have a symmetric Markov chain, we can always write the transition matrix as $P = V\Lambda V^*$ for a Hermitian matrix V and a diagonal matrix Λ with real values on the diagonals. Then we can write $P^t = V\Lambda^t V^*$, and as long as the second-largest eigenvalue is strictly less than 1, eventually Λ^t will look like $\mathrm{diag}(1, tiny, tiny, \ldots, tiny)$. This means that we can compute transition probabilities after t steps up to very small error terms.]*

In this next part, you'll generalize what you saw above to larger cycles.

Group Work

1. Consider the analogous Markov chain to the 4-state one that you saw before, except that it has n states. That is, it looks like this:



   Let $P \in \mathbb{R}^{n \times n}$ be the transition matrix for this Markov chain. We will see together in class that:
$$P = F_n \Lambda F_n^*,$$
   where Λ is a diagonal matrix whose j'th entry is
$$\Lambda_{j,j} = \frac{1 + 2\cos(2\pi j/n)}{3},$$
   where $j = 0, \ldots, n-1$. (Importantly, j is zero-indexed here!) Above, F_n is the $n \times n$ DFT, so
$$(F_n)_{j,k} = \frac{1}{\sqrt{n}} e^{-2\pi ijk/n}.$$
   (There is no question here, we'll do this bit together.)

2. Come up with an expression for $\Pr[X_t = 0 | X_0 = 0]$. You should get a kind of nasty sum involving some cosines, that's okay.

3. Convince yourself that as $t \to \infty$, $\Pr[X_t = 0 | X_0 = 0] \to 1/n$.

...another part on next page

4. **Bonus, if time:** Try to think about how *fast* this convergence is. That is, how large does t have to be before $\Pr[X_t = 0 | X_0 = 0] = \frac{1+o(1)}{n}$? (Don't try to come up with a formal proof, just some back-of-the-envelope calculations).

   Also, how does this compare to what we saw in the mini-lectures about the walk on the line?

   Hint: *You may find the Taylor expansion* $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$ *of* $\cos(x)$ *about zero helpful. In particular, when* x *is small,* $\cos(x) \approx 1 - \frac{x^2}{2}$. *You may also want to use the approximation* $1 - x \approx e^{-x}$ *for small* x *liberally.*