# Class 15

Mixing Times, Coupling, and Strong Stationary Stopping Times

# Announcements

- HW7 due Friday!

# Recap: Mixing Times

Distribution of $X_t$,
conditioned on $X_0 = s$

$$\Delta(t) = \max_s \|\pi - P_s^t\|$$

$$\tau_{mix} = \min\left\{t : \Delta(t) < \frac{1}{2e}\right\}$$

# Recap: Coupling

- Motivation:

$$\Delta(t) \leq \max_{s,s'} \left\| P_s^t - P_{s'}^t \right\| \leq 2\Delta(t)$$

- Def of Coupling:
  - $X_t, Y_t$ both individually follow the Markov chain (but may be correlated!)
  - If $X_t = Y_t$, then $X_{t+1} = Y_{t+1}$

- Why we care:

$$\Delta(t) \leq \max_{s,s'} \Pr[X_t \neq Y_t \mid X_0 = s, Y_0 = s']$$

Aka, $\tau_{mix} \leq$ time for $\{X_t\}$ and $\{Y_t\}$ to meet with probability at least $\frac{1}{2e}$.

# Questions?

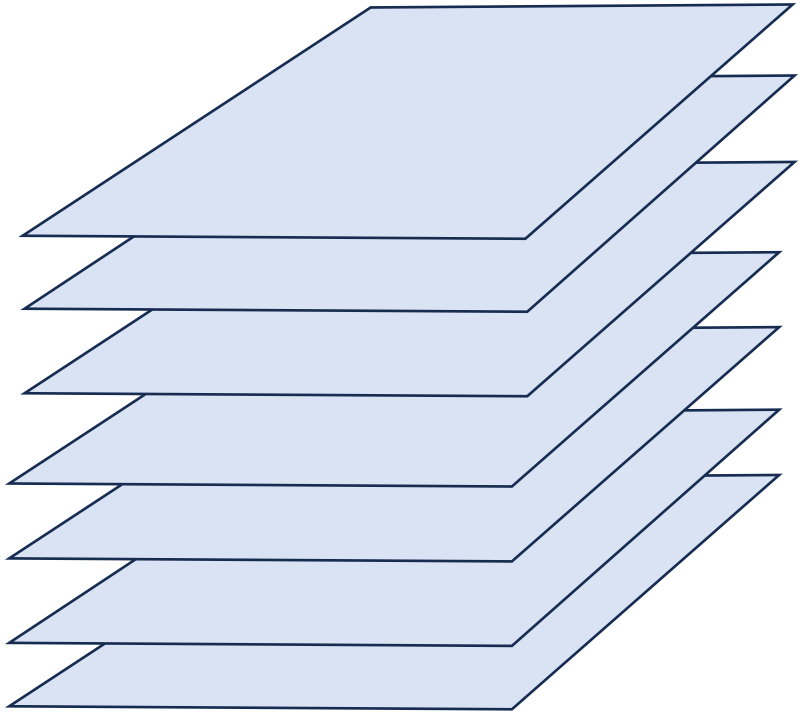
Mixing times, coupling, quiz?

# Plan for today: Shuffling!

- Practice with coupling to bound mixing times
- New technique: strong stationary stopping times

# Today: Shuffling!

- Shuffling procedure (for n cards):
  - Pick a random card.
  - Move it to the top of the deck.



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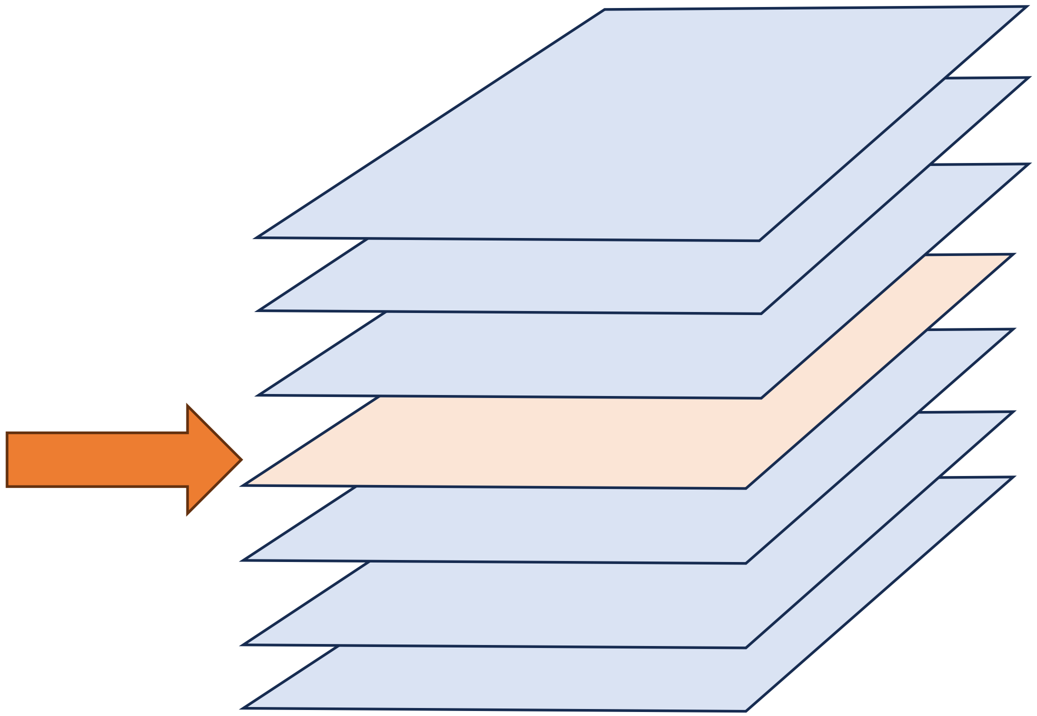

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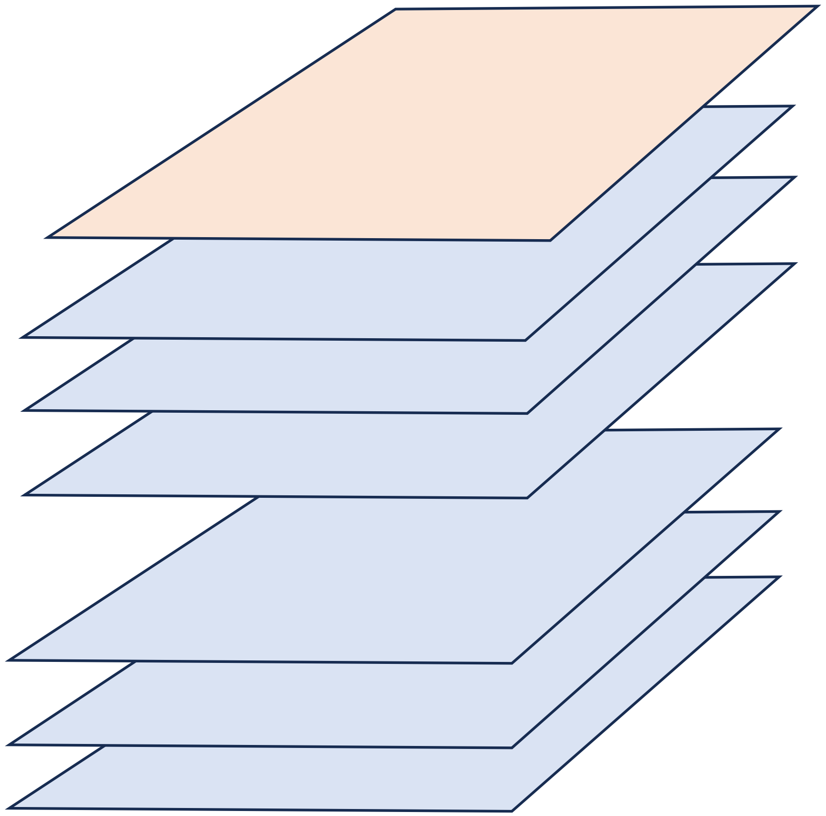


# First Group Work: practice with coupling

More details and hints on handout!

- Shuffling procedure (for n cards):
  - Pick a random card.
  - Move it to the top of the deck.

1. Is this irreducible and aperiodic? What's the stationary dist?
2. Come up with a good coupling.
3. How long until your coupling couples whp?
4. Bound the mixing time.

# Q1

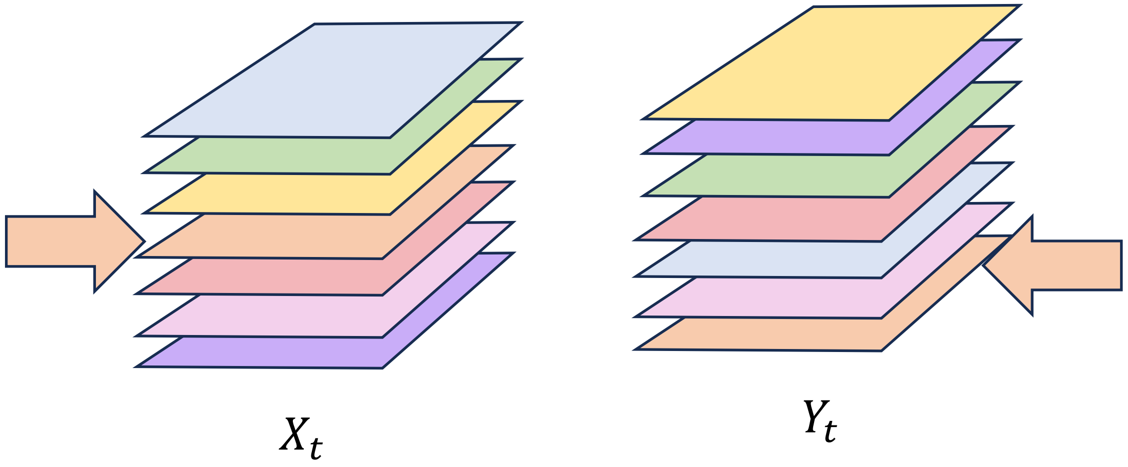
- This is aperiodic, irreducible:
  - Irreducible since you can get to any ordering by picking cards to move in the order that you want.
  - Aperiodic since there are self-loops.
- The stationary distribution is ___Uniform!___
  - If you start with a uniformly random deck, and move a random card to the top, it's still uniformly random.

# Q2: A good coupling?

# A coupling

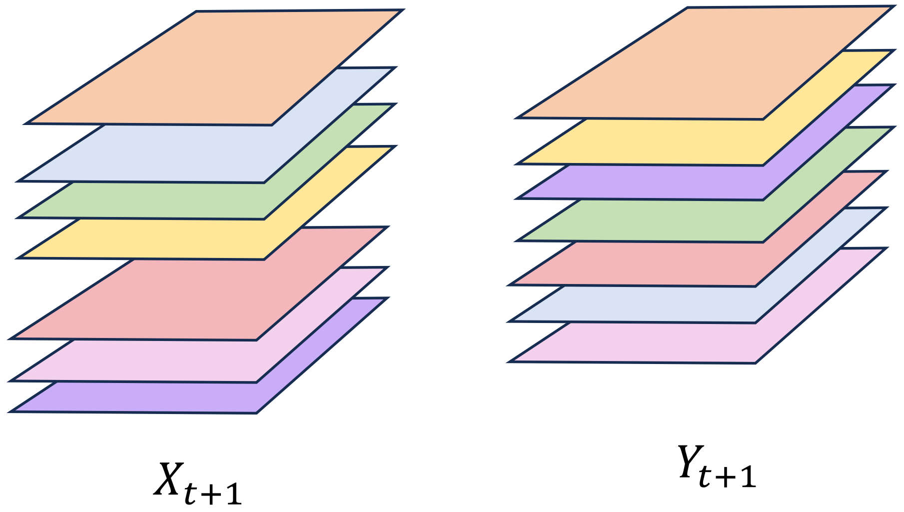
- Want to define $(X_0, Y_0), (X_1, Y_1), \ldots$

- Choose the **same** card in both decks (both the "X" and the "Y" deck).
  - A.k.a., if we move the Ace of Spades in the X deck, also move the Ace of Spades in the Y deck.



# A coupling

- Want to define $(X_0, Y_0), (X_1, Y_1), \ldots$
- Choose the **same** card in both decks (both the "X" and the "Y" deck).
  - A.k.a., if we move the Ace of Spades in the X deck, also move the Ace of Spades in the Y deck.
- This is a valid coupling:
  - Looking at just one deck, it looks like we are always choosing a uniformly random card and moving it to the top.
  - Once the two decks are the same, they will always stay the same.



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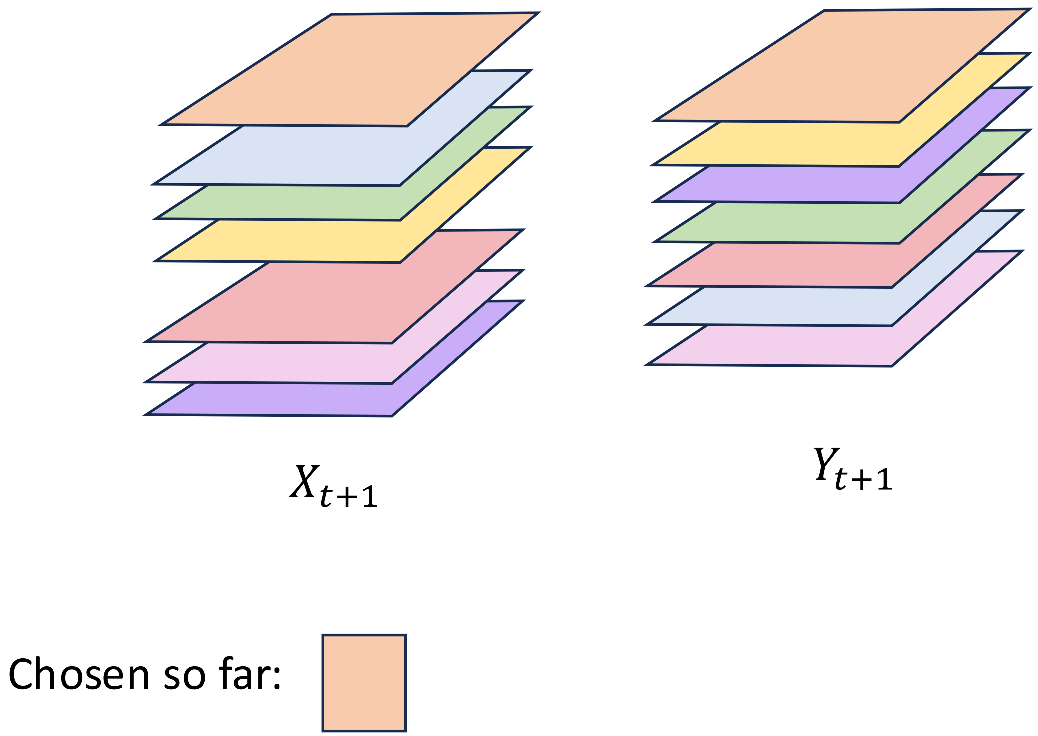

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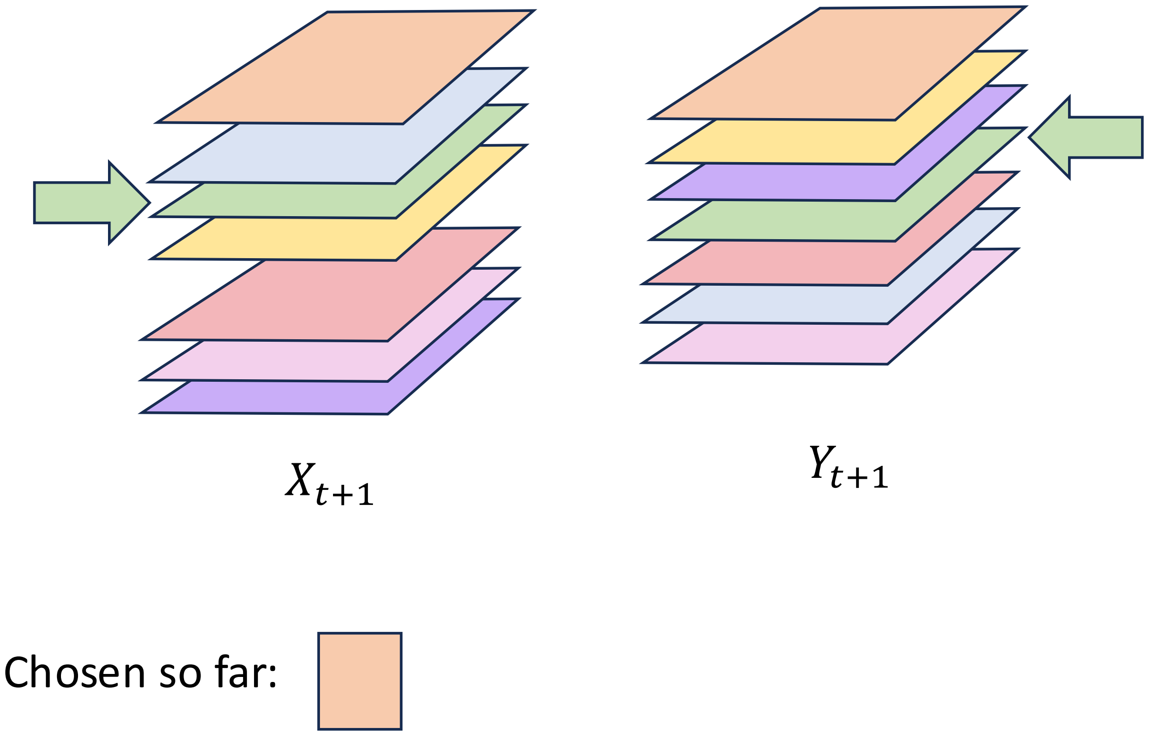

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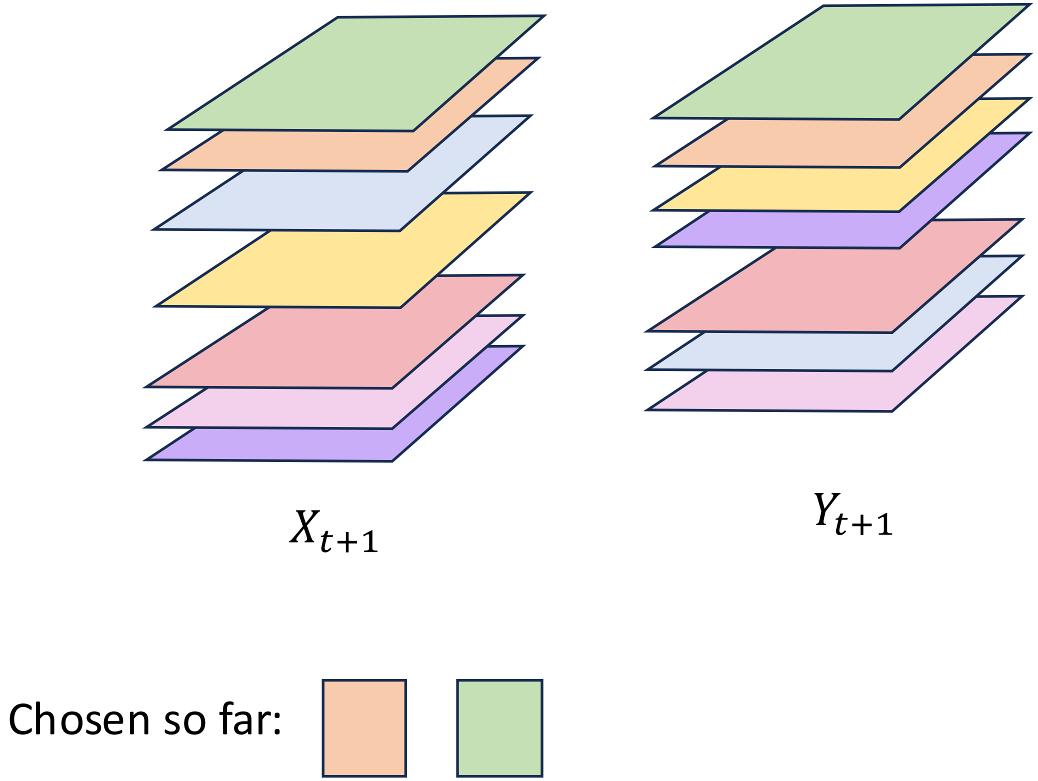

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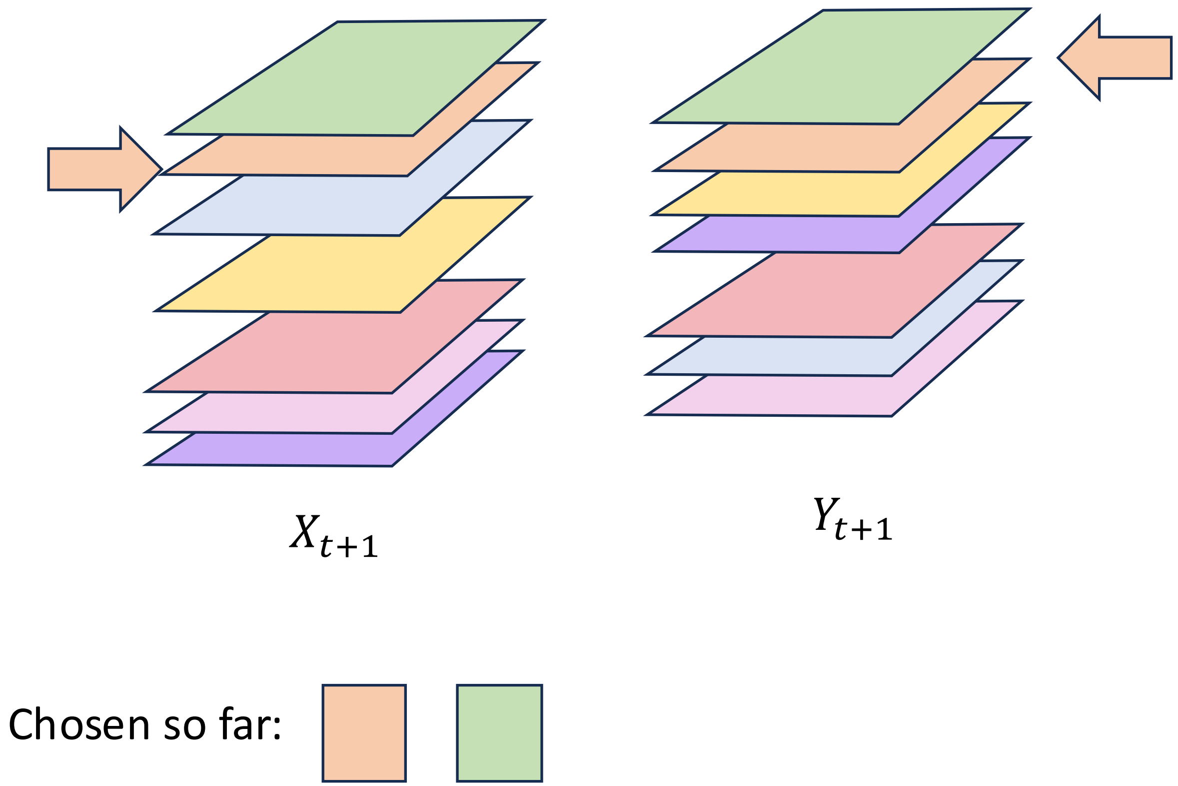

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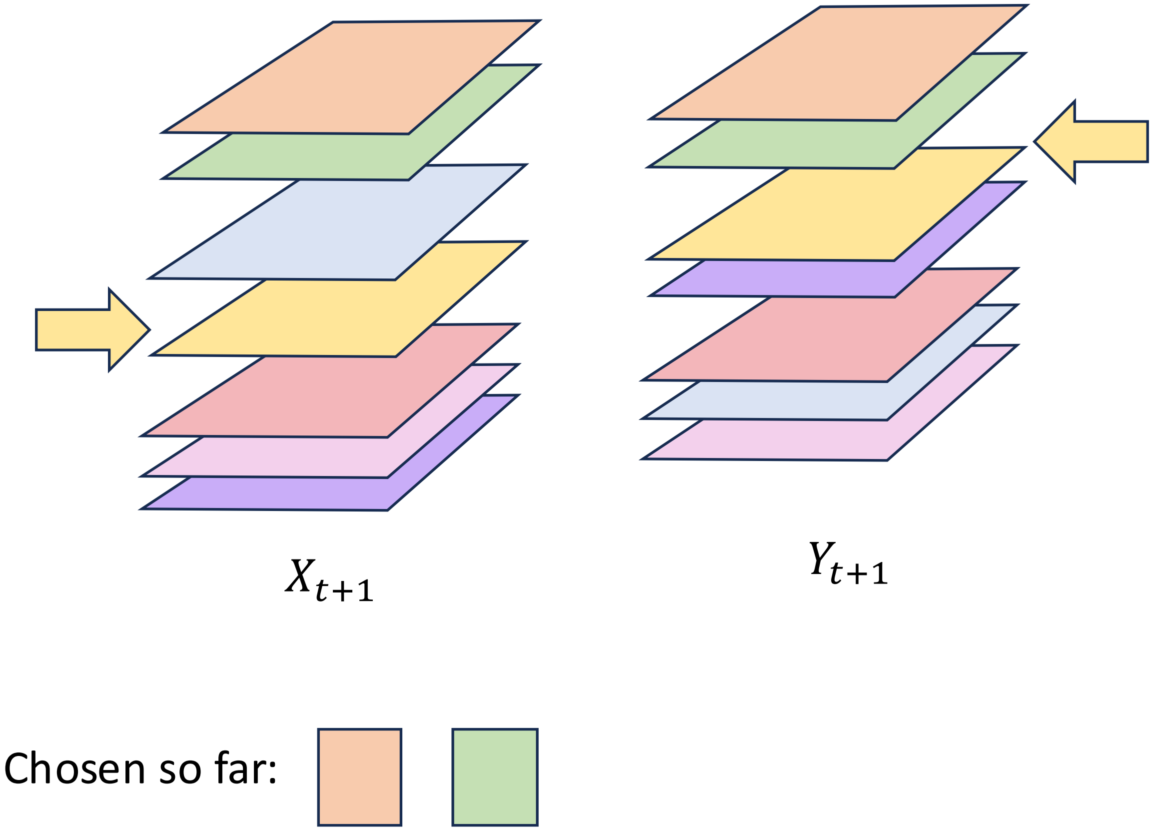

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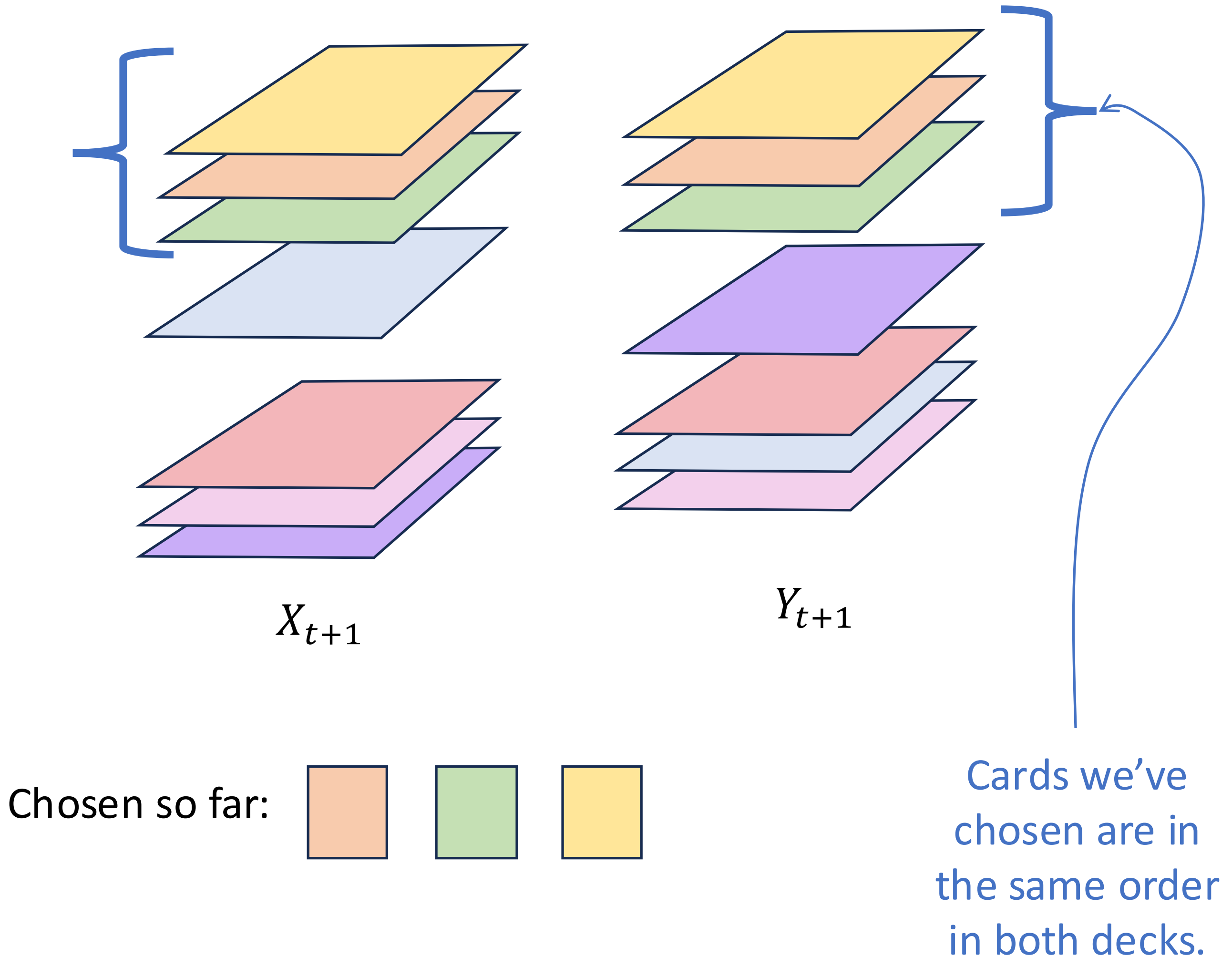


# Q3. How long until the chains couple?

- Observation: The chains have coupled once every card has been chosen.
- How long until every card has been chosen?
  - This is coupon collecting! About $n \log n$.

Formally:

$$\max_{s,s'} \Pr\left[T_{s,s'} \geq 2n \log n\right] \leq \Pr[\text{ time to choose all n cards} \geq 2n \log n]$$

$$= o(1)$$

Time until the two chains (started at $s$ and $s'$) couple

# Q4. Translate this to a bound on mixing time

$$\Delta(2n\log n) \le \max_{s,s'} \Pr\left[T_{s,s'} \ge 2n\log n\right] = o(1)$$

$\Delta(t) = \max_s || \pi - P_s^t ||$

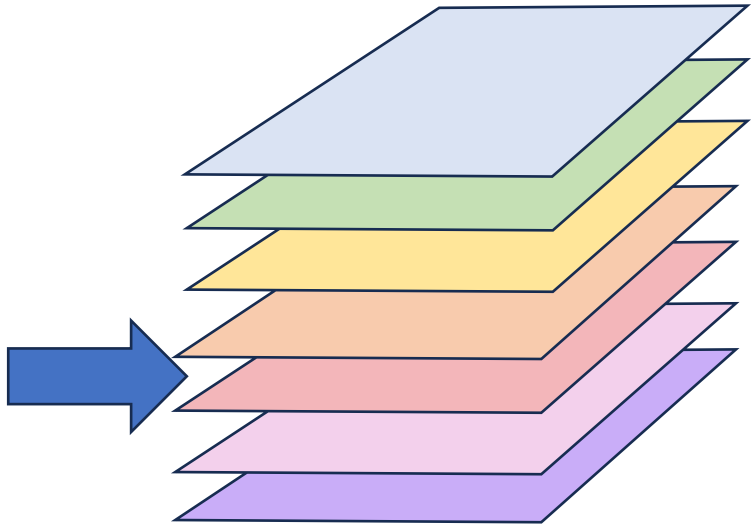
$P_s^t$ = $X_t$ started at $s$

$$\Rightarrow \tau_{mix} \le 2n\log n$$

as long as n is big enough that the $o(1)$ term is at most 1/(2e).

# Group Work II: More shuffling!

- Another shuffling procedure:
  - Choose the top card.
  - Put it somewhere random.



$X_t$

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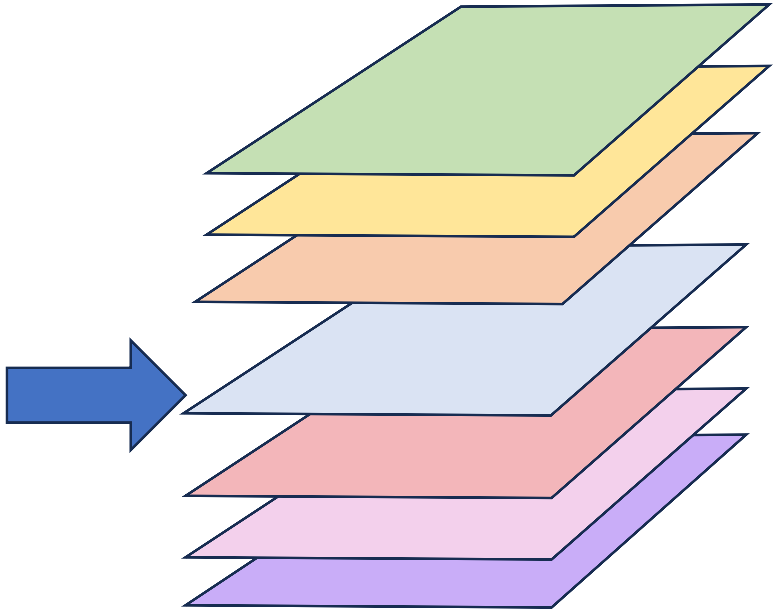

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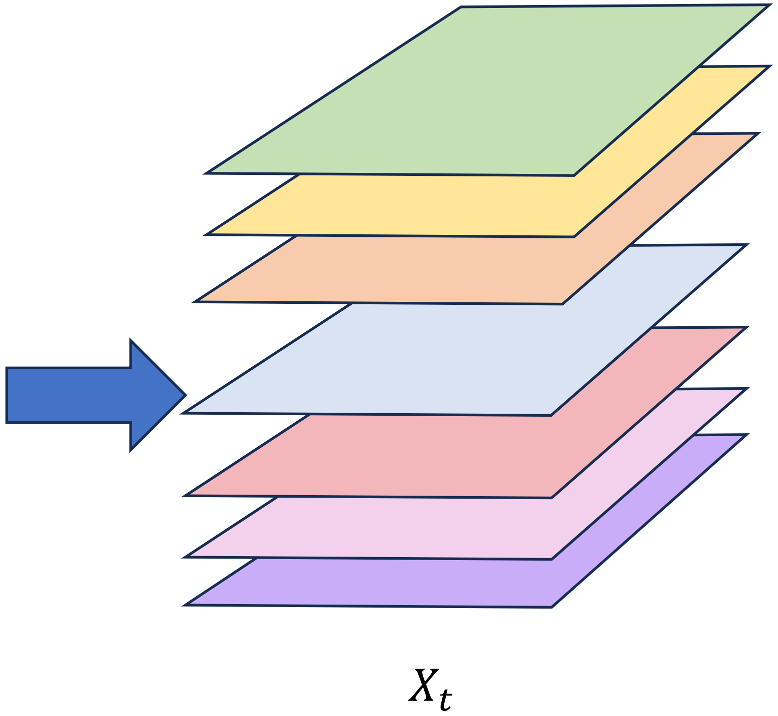


- We'll analyze this one with a different technique:

Strong stationary stopping times!

# Group work!

- Another shuffling procedure:
  - Choose the top card.
  - Put it somewhere random.

1. Convince yourself that this chain is aperiodic and irreducible, with uniform stationary distribution.
2. Let T be the first time the original bottom card is placed randomly somewhere. Show that the deck is completely uniform at any time $t > T$.
3. What is E[T]?
4. Bound the mixing time.

More details and hints on handout!

# Q1

- This is irreducible and aperiodic.
  - Actually, this was a quiz question last time! It's aperiodic since there are self-loops, and irreducible since you can get any deck you want by building it on the bottom of the deck.
- The stationary distribution is uniform.
  - If you take a uniformly random deck and put the top card somewhere random, it's still a uniformly random deck.

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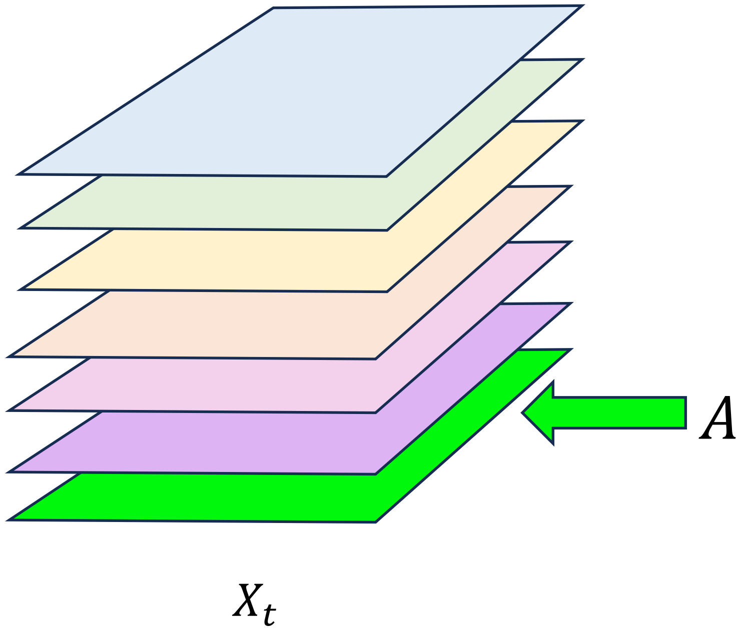
- When $t \geq T$, $X_t$ is uniform.



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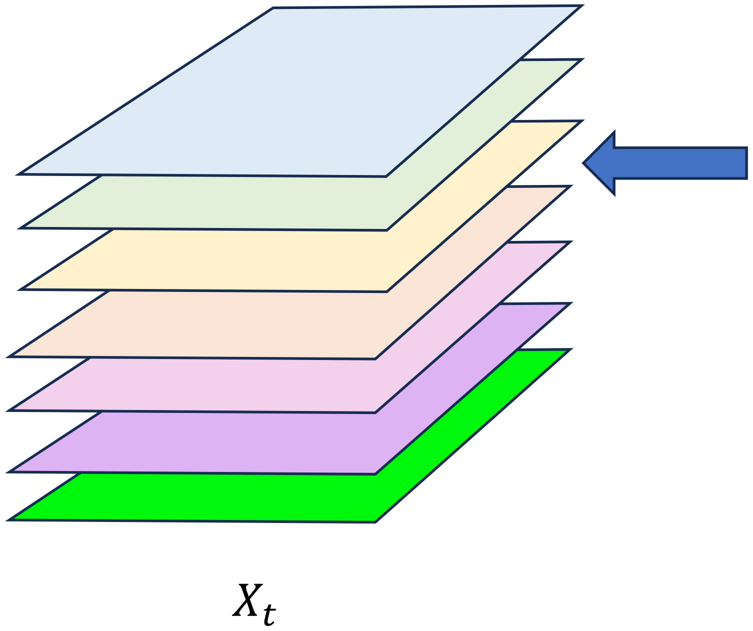

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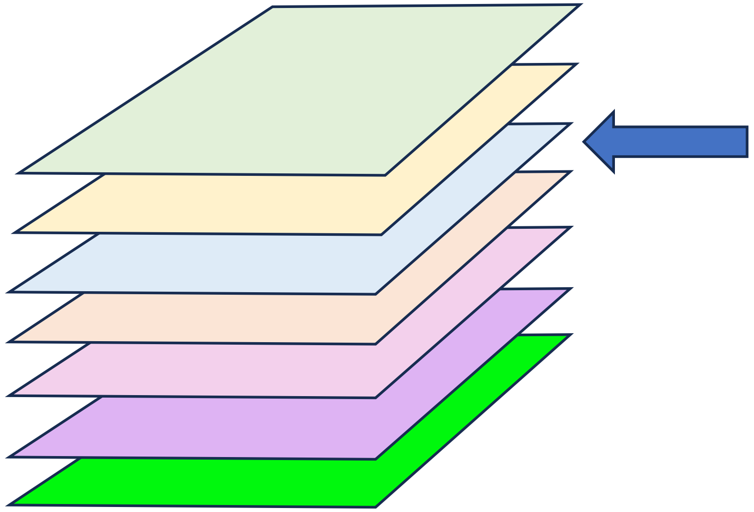

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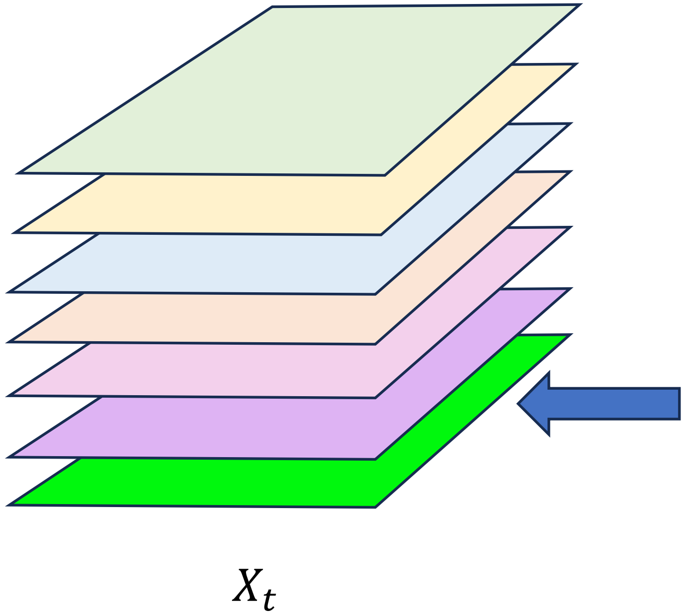

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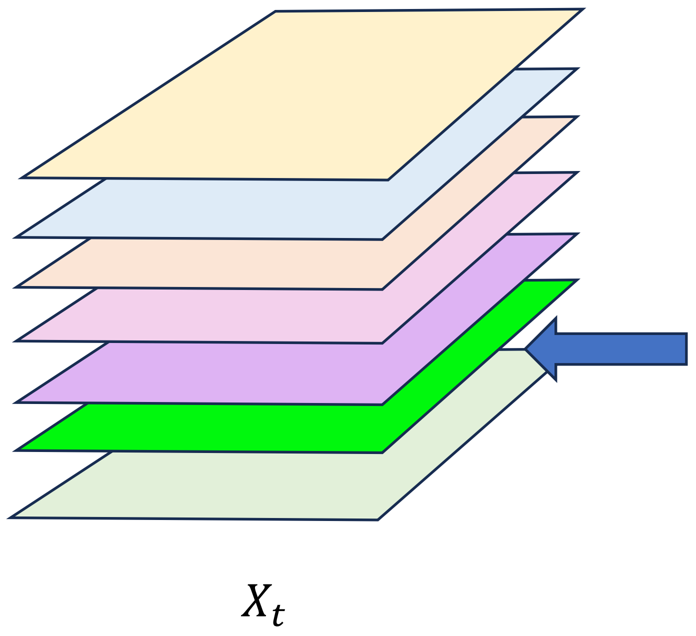

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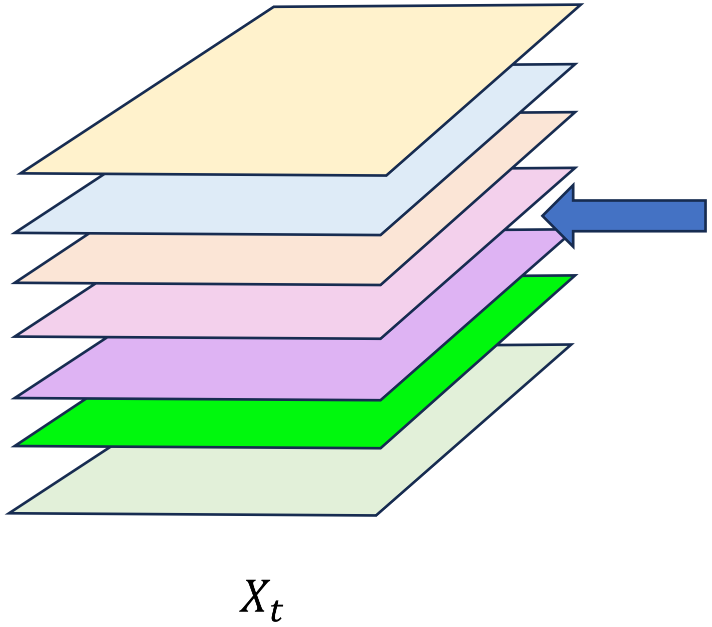

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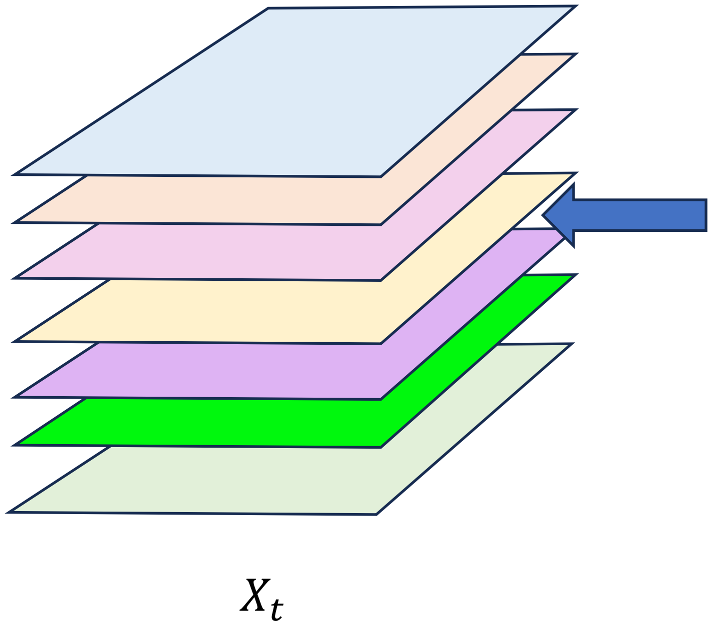

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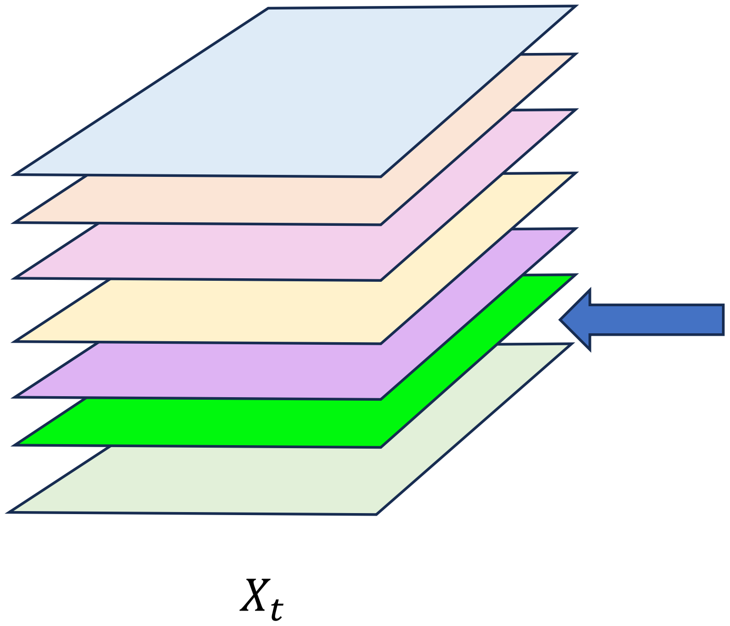

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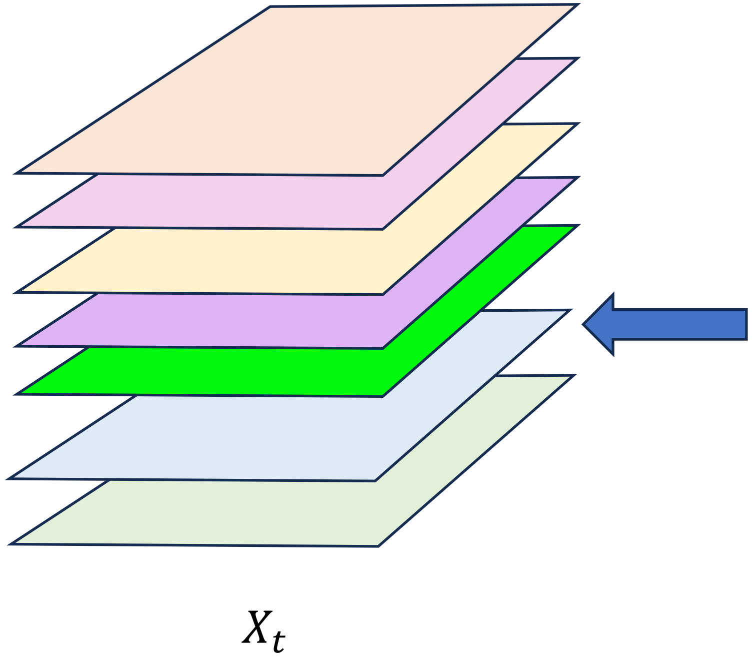

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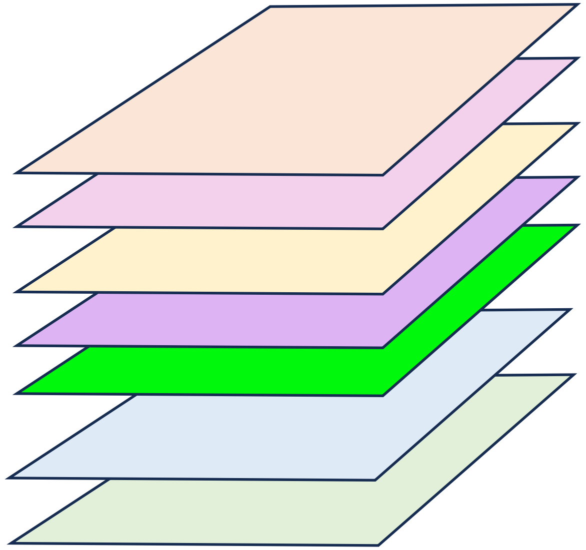
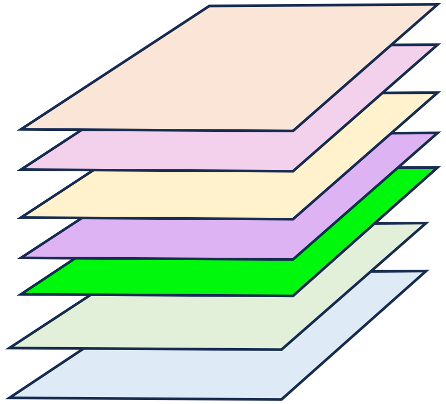


$X_t$

With equal probability we would have had:



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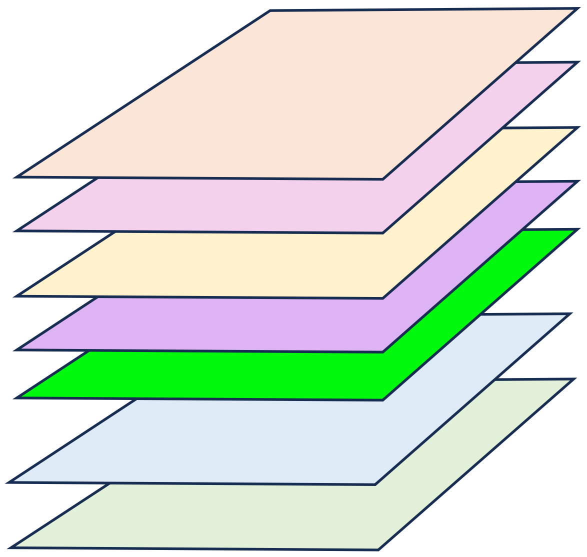

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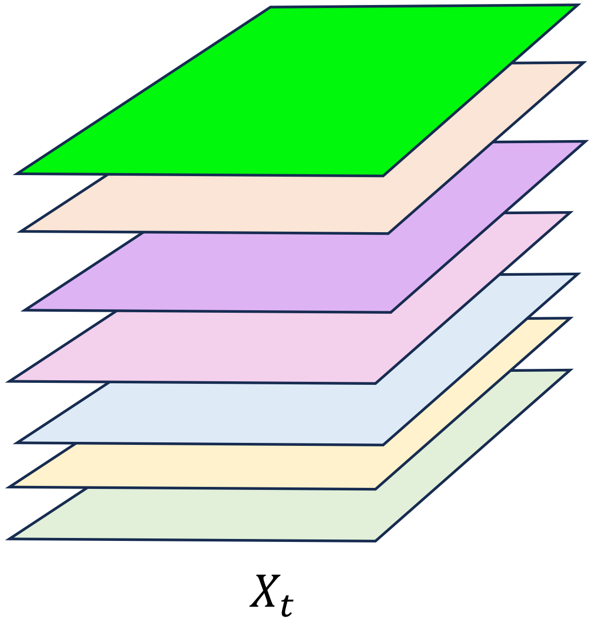

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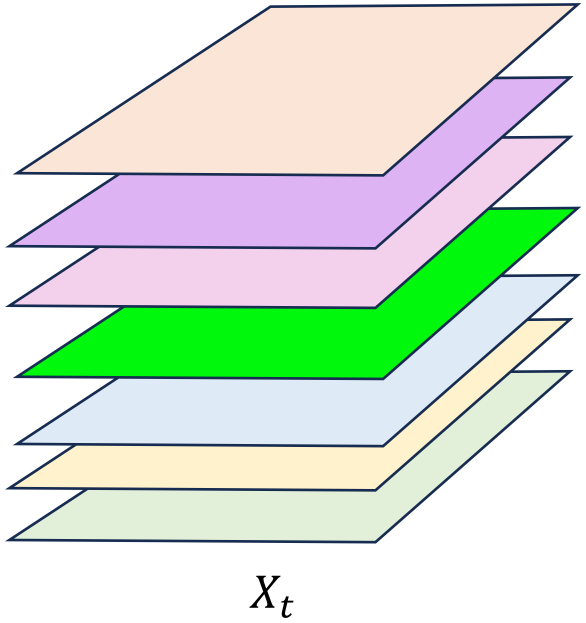
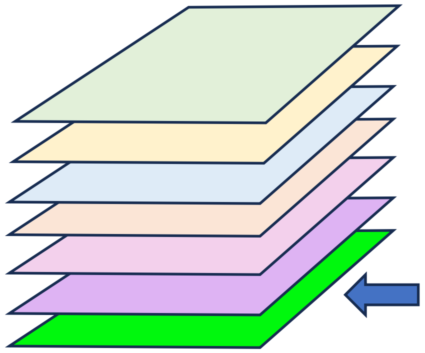


# Q3: What is $\mathbb{E}[T]$?

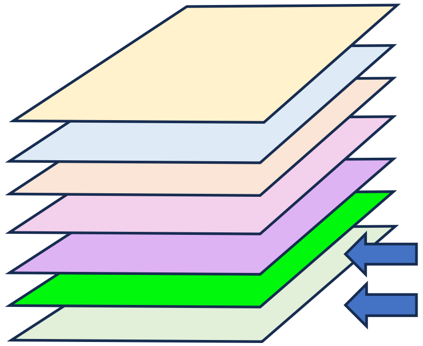
$$\mathbb{E}[T] = \mathbb{E}\left[\text{Time to move from last to second-to-last}\right] + \mathbb{E}\left[\text{Time to move from second-to-last to third-to-last}\right] + \cdots + \mathbb{E}\left[\text{Time to move from } 2^{nd} \text{ to } 1^{st}\right] + \mathbb{E}\left[\text{time to place A once it's } 1^{st}\right]$$

$$n \qquad \frac{n}{2} \qquad \frac{n}{n-1} \qquad 1$$



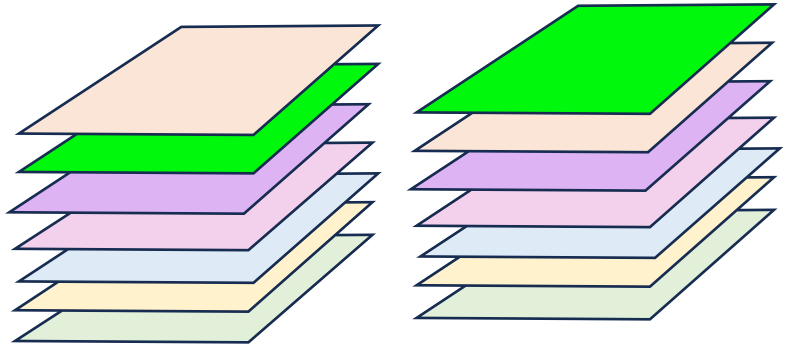
Have to put top card here…

…Probability is $\frac{1}{n}$



Have two options…

…Probability is $\frac{2}{n}$



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$$n \qquad \frac{n}{2} \qquad \frac{n}{n-1} \qquad 1$$

$$\mathbb{E}[T] = \sum_{j=1}^{n} \frac{n}{j} = \Theta(n \log n)$$

# Q4: Bound the mixing time by $O(n \log n)$

- Markov's inequality: $\Pr[\, T \geq 2e \cdot \mathbb{E}[T]] \leq \frac{1}{2e}$
- Choose $t \geq 2e\, \mathbb{E}[T] = \Theta(n \log n)$
- **Hint:** $P_s^t = p \cdot \sigma + (1 - p) \cdot \pi$, where $p = \Pr[\, T > t\, ]$

- $\Delta(t) = \max_s \left\| \pi - P_s^t \right\|_{TV} = \| \pi - \pi(1-p) - p\sigma \|_{TV}$

$$= \| p\pi - p\sigma \|_{TV}$$

$$= p \| \pi - \sigma \|_{TV} \leq p \leq \frac{1}{2e}$$

# Strong Stationary Stopping Times

- A random variable T is a **strong stationary stopping time** if:
  1. The event that $T = t$ depends only on $X_0, \dots, X_t$
  2. For all states $s$, $\Pr[X_t = s \mid T \geq t] = \pi(s)$

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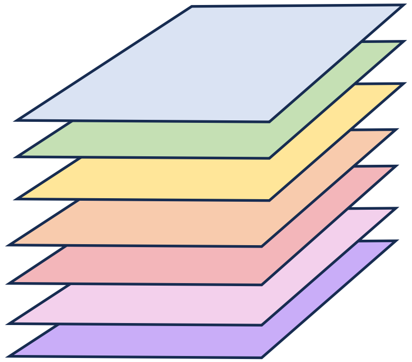
  2. For all states $s$, $\Pr[X_t = s \mid T \geq t] = \pi(s)$

We essentially just proved:

- **Theorem:** For any strong stationary stopping time, $\Delta(t) \leq P[T > t]$.

# Bonus group work if time!

- Shuffling procedure:
  - Assign each card “L” or “R” independently, with probability 1/2
  - Put all ”L” cards to the left, preserving their relative order
  - Put all “R” cards to the right, preserving their relative order
  - Put the “L” stack on top of the “R” stack.



This is the **inverse** of a standard riffle shuffle!

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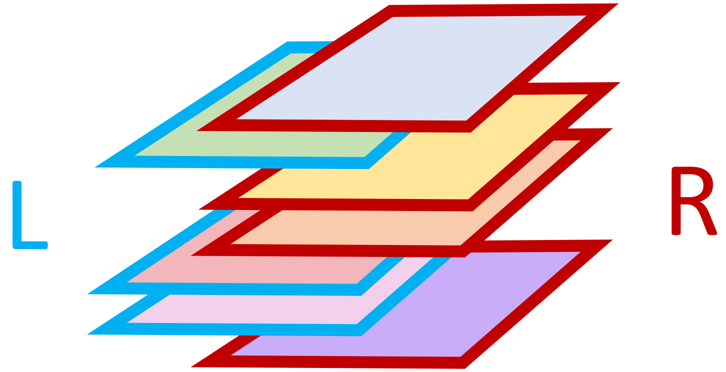


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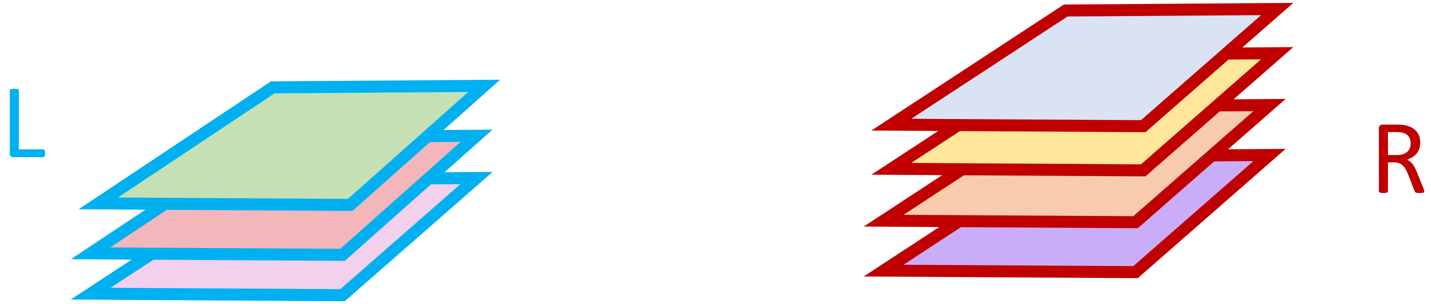
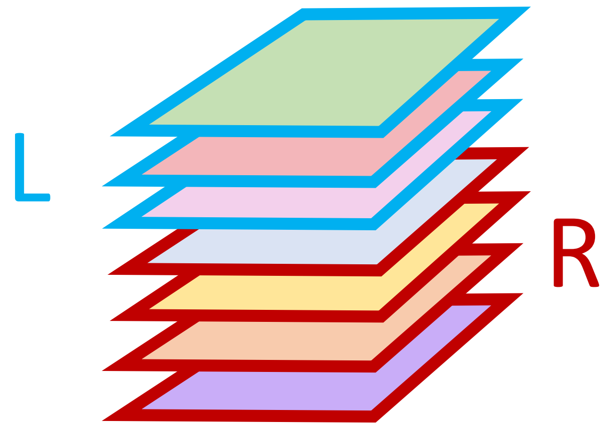
This is the **inverse** of a standard riffle shuffle!




# Bonus group work if time!

- Shuffling procedure:
  - Assign each card “L” or “R” independently, with probability 1/2
  - Put all ”L” cards to the left, preserving their relative order
  - Put all “R” cards to the right, preserving their relative order
  - Put the “L” stack on top of the “R” stack.

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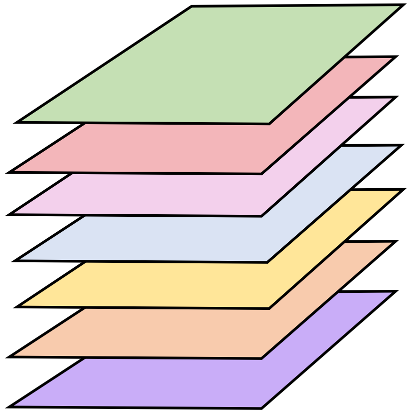

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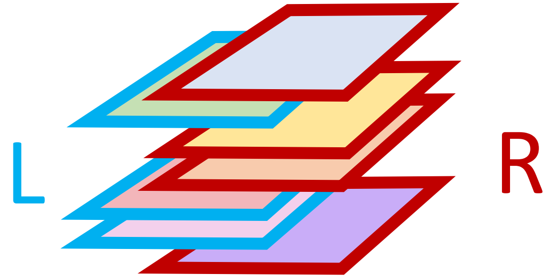
This is the **inverse** of a standard riffle shuffle!

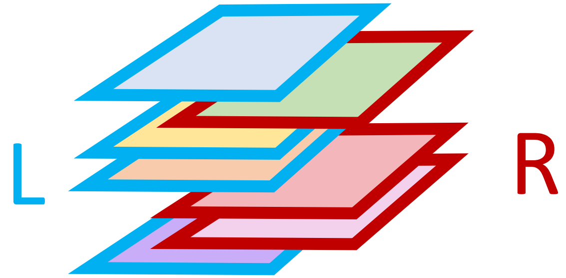
**Group work:**
Use the method of strong stationary stopping times to show that the mixing time of this shuffle is $O(\log n)$.

# Solution

- View the shuffling procedure this way:
  - Assign each card a random string in $\{0,1\}^t$
  - Assign L,R to 0,1 at random, independently for each of t steps
  - This tells you which way each card goes in each of t steps
- If two cards have the same string, they maintain their relative order (since they always end up on the same "side").
- But if two cards have different strings, their relative order is random.
- This generalizes to more than two cards.




is just as likely as




So if two cards ever end up in different "sides", their order relative to each other is random.

Let $T$ be the first time that all the cards have different strings.

$T$ is a strong stationary stopping time!

You can tell that $T$ has happened at time $T$

Once $T$ has happened, all of the cards are in random order

# Solution

- Let $T$ be the first time that all the cards have different strings.

- $\Pr[\text{ card x and card y have the same string }] = 1/2^t$
- $\Pr[\text{ any two cards have the same string }] \leq n^2 \cdot 2^{-t}$
- Choose $t \geq 3 \log n$ (say) and this is $\leq 1/n$.

$$\Delta(3 \log n) \leq P[T > 3 \log n] \leq \frac{1}{n} \leq \frac{1}{2e} \quad \text{for large enough n}$$

$$\Rightarrow \tau_{mix} \leq 3 \log n$$

# Recap

- Two ways to bound mixing times:
  1. Coupling
  2. Strong Stationary Mixing Times

- (Plus spectral techniques, which we saw last week)

# Next time

- Martingales!