Class 16

Martingales and Azuma-Hoeffding bound

Announcements

- HW7 due Friday
- HW8 out now!
  - The last one! Due Friday 3/13.
  - HW8 is a bit shorter than usual, so you can study for…
- Final exam!
  - Monday 3/16, 3:30-6:30pm, CoDa B90.
  - Practice exam posted now.
  - Cumulative, but with a focus on Classes 9-17.
    - Probabilistic Method, Markov Chains, Martingales

Recap

- $\{Z_t\}$ is a **martingale** with respect to $\{X_t\}$ if for all t:
  - Z_t is a function of $X_0, \dots, X_t$
  - $E[|Z_t|] < \infty$
  - $E[\, Z_t \mid X_0, X_1, \dots, X_{t-1}] = Z_{t-1}$
- **Doob Martingale**: $Z_t = E[\, A \mid X_0, \dots, X_t \,]$

Azuma-Hoeffding

- Let $\{Z_t\}$ be a martingale w.r.t. $\{X_t\}$ and suppose there are constants $c_1, \ldots, c_n$ so that for all $i \leq n$, $|Z_i - Z_{i-1}| \leq c_i$.

- For any $\lambda > 0$,

$$\Pr[\, |Z_n - Z_0| \geq \lambda] \leq 2 \exp\left(\frac{-\lambda^2}{2 \sum_i c_i^2}\right)$$

Questions?

Definition of Martingale? Azuma-Hoeffding? Quiz?

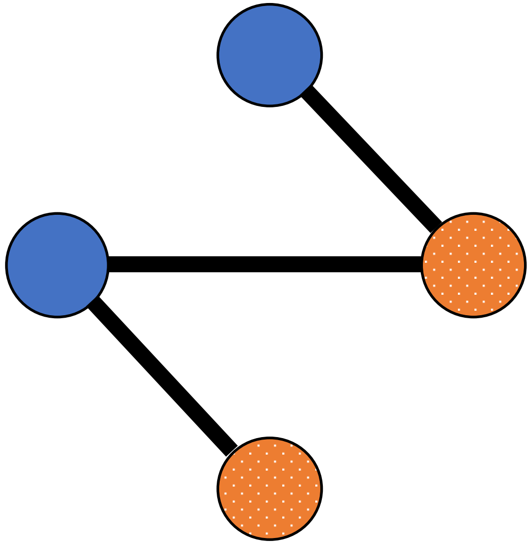
Today: Azuma-Hoeffding in Action

- Example 1: Chromatic number of random graphs
- Example 2: Gambling

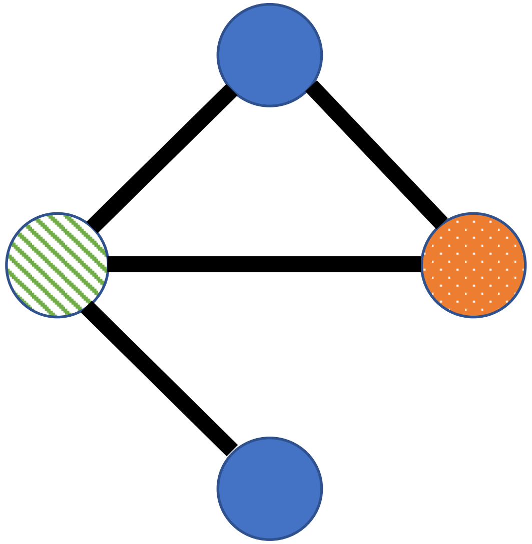
Chromatic number

(We saw this example in the mini-lectures)

- Let $G \sim G(n,p)$
- Let $A = \chi(G)$ be the minimum number of colors needed to properly color G.



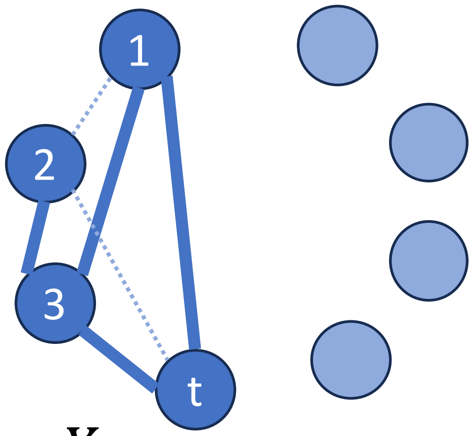
$\chi(G) = 2$



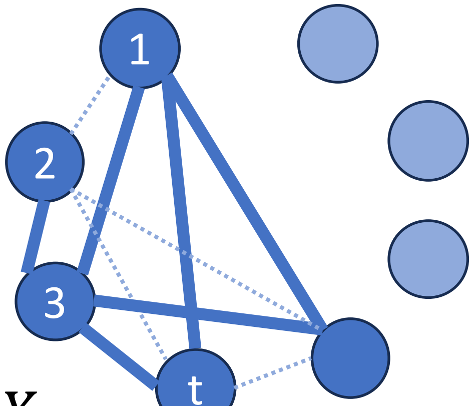
$\chi(G) = 3$

Vertex exposure martingale

- Let $G \sim G(n,p)$
- Let $A = \chi(G)$
- $X_t =$ status of edges between vertex t and vertices $1,2,\ldots,t-1$
- $Z_t = \mathbb{E}[A \mid X_1, X_2, \ldots, X_t]$



$X_1, \ldots, X_t$

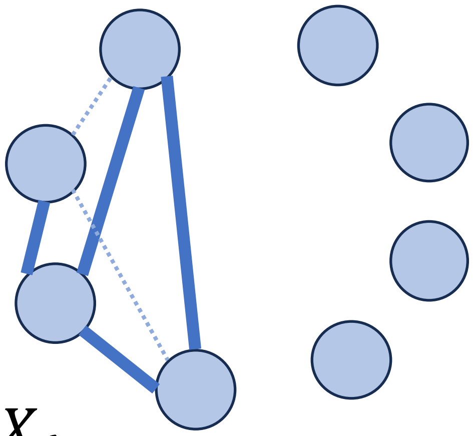


$X_1, \ldots X_{t+1}$

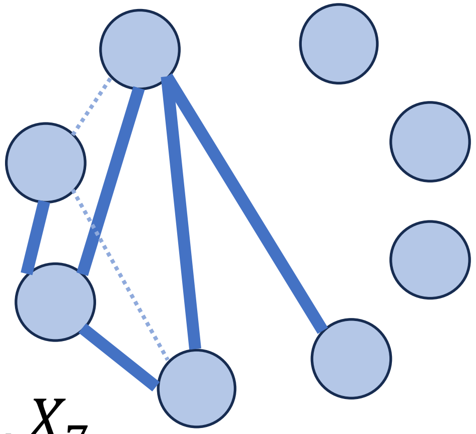
Note: this is slightly different than the lecture notes! Both work fine for this example, this one is maybe more standard.

Edge exposure martingale

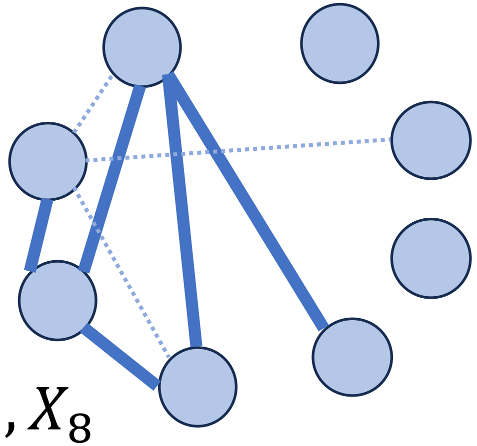
- Let $G \sim G(n,p)$
- Let $A = \chi(G)$
- X_t = status of edge t, for $t = 1, \dots, \binom{n}{2}$
- $Z_t = \mathbb{E}[A \mid X_1, X_2, \dots, X_t]$



$X_1, \dots, X_6$



$X_1, \dots, X_7$



$X_1, \dots, X_8$

Group work:

Apply Azuma-Hoeffding both ways!

1. Use Azuma-Hoeffding with the vertex exposure martingale to bound

$$\Pr[\ |A \ - E[A]| > c\sqrt{n}] \leq 2\exp\left(-\frac{c^2}{2}\right)$$

2. What happens with the edge exposure martingale?
3. (bonus) what can you say about E[A]?

1. Vertex exposure martingale

- Need to bound $|Z_t - Z_{t-1}| \leq$ ___**1**___

Intuitively, if you change the neighborhood of a single vertex, the chromatic number can change by at most 1:

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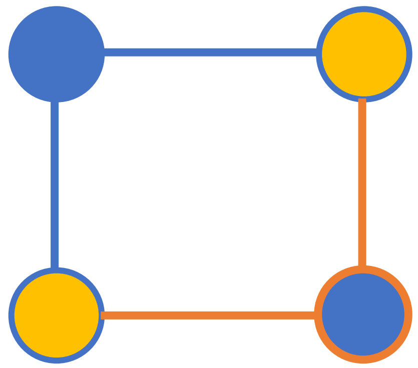
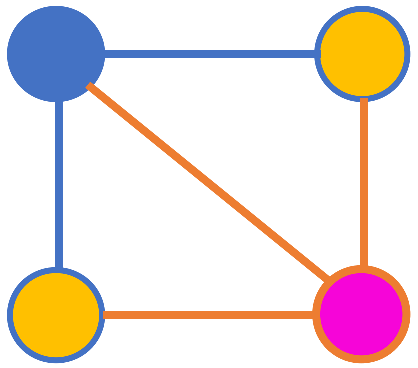


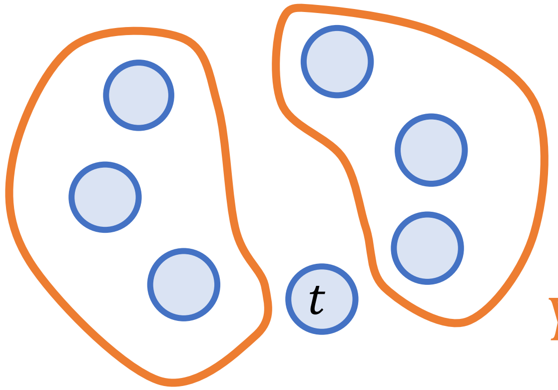
vs



Formally…

Want to show $|Z_t - Z_{t-1}| \leq 1$

$Y = X_1, \dots, X_{t-1}$



$Y' = X_{t+1}, \dots, X_n$

- Consider $E[\, A \mid Y, X_t = x \,]$ for any special x

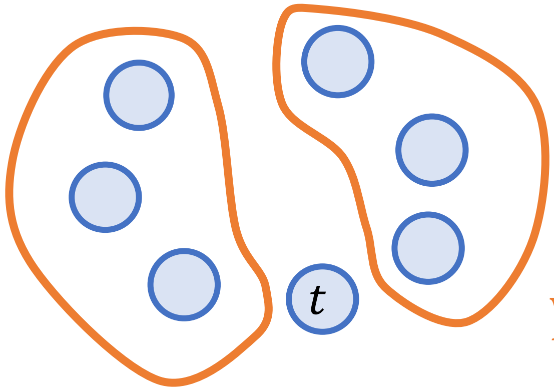
$$E[\, A \mid Y, X_t = x \,] = \sum_{y'} E[\, A \mid Y, X_t = x, Y' = y'] \, \Pr[\, Y' = y' \mid Y, X_t = x \,]$$

Can drop this by independence

Formally…

Want to show $|Z_t - Z_{t-1}| \leq 1$

$Y = X_1, \dots, X_{t-1}$



$Y' = X_{t+1}, \dots, X_n$

- Consider $E[\,A \mid Y, X_t = x\,]$ for any special x

$$E[\,A \mid Y, X_t = x\,] = \sum_{y'} E[\,A \mid Y, X_t = x, Y' = y'\,]\,\Pr[\,Y' = y'\,]$$

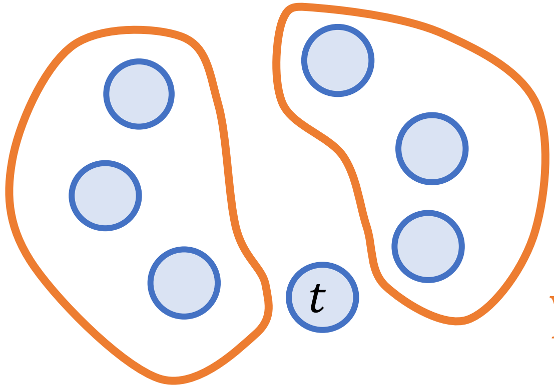
- Similarly,

$$E[\,A \mid Y, X_t\,] = \sum_{y'} E[\,A \mid Y, X_t, Y' = y'\,]\,\Pr[\,Y' = y'\,]$$

Formally…

Want to show $|Z_t - Z_{t-1}| \leq 1$

$Y = X_1, \ldots, X_{t-1}$



$Y' = X_{t+1}, \ldots, X_n$

$$E[\,A \mid Y, X_t = x\,] = \sum_{y'} E[\,A \mid Y, X_t = x, Y' = y'\,]\,\Pr[\,Y' = y'\,]$$

$$E[\,A \mid Y, X_t\,] = \sum_{y'} E[\,A \mid Y, X_t, Y' = y'\,]\,\Pr[\,Y' = y'\,]$$

$$|E[\,A \mid Y, X_t\,] - E[\,A \mid Y\,]\,|$$ (This is $|Z_t - Z_{t-1}|$)

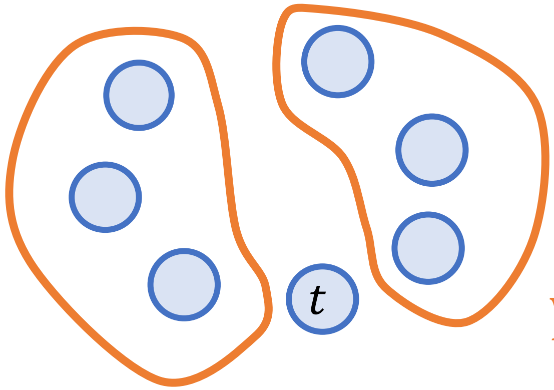
$$= |\textstyle\sum_x (E[\,A \mid Y, X_t\,] - E[\,A \mid Y, X_t = x\,])\Pr[X_t = x]|$$

Win if we can show that this is ≤ 1 (in magnitude)

Formally…

Want to show $|Z_t - Z_{t-1}| \leq 1$

$Y = X_1, \dots, X_{t-1}$



$Y' = X_{t+1}, \dots, X_n$

$$E[\, A \mid Y, X_t = x \,] = \sum_{y'} E[\, A \mid Y, X_t = x, Y' = y'] \,\Pr[Y' = y']$$

$$E[\, A \mid Y, X_t \,] = \sum_{y'} E[\, A \mid Y, X_t, Y' = y'] \,\Pr[Y' = y']$$

$$\sum_{x} \left(E[\, A \mid Y, X_t \,] - E[\, A \mid Y, X_t = x \,] \right) \Pr[X_t = x]$$

Win if we can show that this is ≤ 1 (in magnitude)

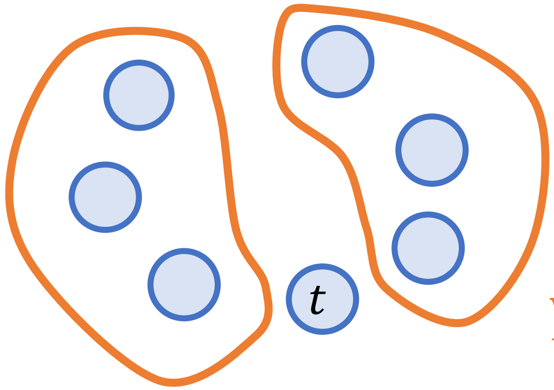
$$\sum_{y'} \left(E[\, A \mid Y, X_t, Y' = y'] - E[\, A \mid Y, X_t = x, Y' = y'] \right) \Pr[Y' = y]$$

Win if we can show that this is ≤ 1 (in magnitude)

Formally…

Want to show $|Z_t - Z_{t-1}| \leq 1$

$Y = X_1, \ldots, X_{t-1}$



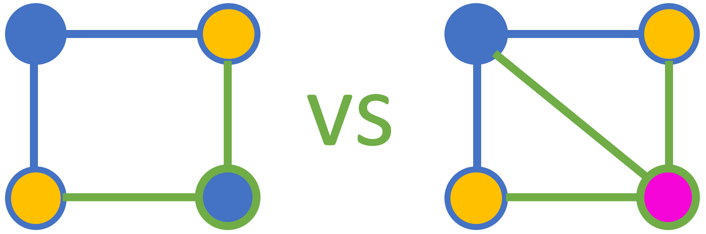
$Y' = X_{t+1}, \ldots, X_n$

$$\sum_{y'} (E[\,A \mid \boldsymbol{Y}, \boldsymbol{X_t}, Y' = y'] - E[\,A \mid \boldsymbol{Y}, X_t = x, Y' = y'])\Pr[Y' = y]$$

Win if we can show that this is ≤ 1 (in magnitude)

Formally, this is a function of $\boldsymbol{Y}$ and $\boldsymbol{X_t}$

For any values of those variables we plug in, this |difference| is at most 1…



Here, $X_t = x$

Here, $X_t =$ (whatever we plug in for $\boldsymbol{X_t}$)

1. Vertex exposure martingale

- Applying Azuma-Hoeffding:

$$\Pr[\,|Z_n - Z_0| > c\sqrt{n}\,] \leq 2\exp\left(-\frac{c^2 n}{2\sum_i c_i^2}\right)$$

$c_i = 1$

$Z_0 = E[A]$

$Z_n = E[A|X_1, \ldots, X_n] = A$

$$\Pr[\,|A - E[A]| > c\sqrt{n}\,] \leq 2\exp\left(-\frac{c^2 n}{2n}\right)$$

Want:

$$\leq 2\exp(-c^2)$$

Aside: Method of bounded differences

- We've (essentially) just proved:

- Suppose that $A = A(X_1, \dots, X_n)$ has the property that
  $|A(x_1, \dots, x_{t-1}, \boldsymbol{x_t}, x_{t+1}, \dots, x_n) - A(x_1, \dots, x_{t-1}, \boldsymbol{x_t}', x_{t+1}, \dots, x_n)| \le c_t$
  for all possible values of $x_1, \dots, x_{t-1}, \boldsymbol{x_t}, \boldsymbol{x_t}', x_{t+1}, \dots, x_n$.
- Suppose also that $X_1, \dots, X_n$ are independent.
- Then $\Pr[|A - E[A]| > \lambda] \le 2\exp\left(-\frac{\lambda^2}{2\sum_i c_i^2}\right)$

2. Edge exposure martingale

- Need to bound $|Z_t - Z_{t-1}| \le$ ___**1**___

Intuitively, if you change a single edge, the chromatic number can change by at most 1:

2. Edge exposure martingale

- Need to bound $|Z_t - Z_{t-1}| \leq$ ___**1**___

Intuitively, if you change a single edge, the chromatic number can change by at most 1:

2. Edge exposure martingale

- Applying Azuma-Hoeffding:

$$\Pr[\,|Z_n - Z_0| > c\sqrt{n}\,] \le 2\exp\left(-\frac{c^2 n}{2\sum_i c_i^2}\right)$$

Do we get…

(A) The same? $2\exp(-c^2)$
(B) Better? (smaller?)
(C) Worse? (bigger?)

What do you get with the edge exposure martingale? $\Pr[|A - \mathbb{E}A| > c\sqrt{n}] \leq$____?

$2\exp(-c^2)$ (the same)

0%

Better (a smaller upper bound)

0%

Worse (a larger upper bound)

0%

Not sure

0%

2. Edge exposure martingale

- Applying Azuma-Hoeffding:

$$\Pr[\,|Z_n - Z_0| > c\sqrt{n}\,] \leq 2\exp\left(-\frac{c^2 n}{2\sum_i {c_i}^2}\right)$$

$$\Pr[\,|A - E[A]| > c\sqrt{n}\,] \leq 2\exp\left(-\frac{c^2 n}{2\binom{n}{2}}\right) \approx 2\exp\left(-\frac{c^2}{n}\right)$$

Worse!

Next example: gambling

- A fun game:
  - At each step t, you can bet $b_t \in [0, B]$ and guess "heads or tails"
  - Flip a fair coin. If you were right, you win b_t. Otherwise you lose b_t.

- Notice that b_t can depend on everything that's happened so far.
- Let Z_t be the amount of money you have at time t.
- It's okay for $Z_t < 0$ (the casino knows you're good for it…)

- Quick question: **Is Z_t a sum of independent random variables?**

No! The amount we bet can depend on previous rounds.

Group Work

- At each step t, you can bet $b_t \in [0, B]$ and guess "heads or tails"
- Flip a fair coin. If you were right, you win b_t. Otherwise you lose b_t.
- $Z_t =$ amount of money at time t

1. Suppose your betting strategy is deterministic.

   Prove $\Pr[\,|Z_n| \geq cB\sqrt{n}\,] \leq$ ______________

2. Does the proof above work if your betting strategy is randomized?

   If not, make it work.

1a. Setting up a martingale

- Z_t = amount of $$ at time t
- $X_t =$ Outcome of coin flip at time t

- This is a martingale because…

1. Z_t is a function of $X_0, \ldots, X_t$?

   If my betting strategy is deterministic, then $X_0, \ldots, X_t$ determines everything that's happened so far, including how much I bet.

2. $E[|Z_t|] < \infty$?

   For all t, $|Z_t| \leq Bt$

3. $E[\, Z_t \,| X_0, \ldots, X_{t-1} \,] = Z_{t-1}$?

   Either I win b_t or lose b_t, with equal probability.

*Depending on B, n, and your definition of "too much"

1b. Using Azuma-Hoeffding

- $|Z_t - Z_{t-1}| \leq$ ___ B ___
- Azuma-Hoeffding:
- $\Pr[\,|Z_n - Z_0| \geq \lambda] \leq 2\exp\left(-\frac{\lambda^2}{\sum_i {c_i}^2}\right)$

$\lambda = cB\sqrt{n}$

$Z_0 = 0$

$c_i = B$

$$\Pr[\,|Z_n| \geq cB\sqrt{n}] \leq 2\exp\left(-\frac{c^2B^2n}{nB^2}\right) = 2\exp(-c^2)$$

Take-away: no matter what betting strategy you use, it's very unlikely you'll win (or lose) **too** much money at this game*.

With a randomized betting strategy?

- Z_t = amount of \$\$ at time t
- $X_t =$ Outcome of coin flip at time t, **and** the bet you decide to place at time $t+1$

- This is a martingale because…

1. Z_t is a function of $X_0, \ldots, X_t$?
2. $E[|Z_t|] < \infty$?
3. $E[\, Z_t \,| X_0, \ldots, X_{t-1} \,] = Z_{t-1}$?

- We don't have to secretly look into the future to define X_t (since b_{t+1} doesn't depend on $X_{>t}$)
- So we don't have to look into the future to define Z_t either.

Recap

- We got some practice applying Azuma-Hoeffding.
- Moral of the story I:
  - Azuma's inequality can bound sums of random variables that aren't necessarily independent!
- Moral of the story II:
  - One useful case is when you want to establish concentration for a random variable A that doesn't depend too much on any of the underlying variables X_i.
- Moral of the story III:
  - Sometimes there's more than one relevant martingale, and one might be better than the other.

Next time

- More Martingales, and the **Martingale Stopping Theorem**!