

Class 16: Agenda/Questions

1 Questions?

Any questions from the minilectures and/or the quiz? (Martingales, the Doob martingale, Azuma-Hoeffding)

2 Practice with Azuma-Hoeffding: Chromatic numbers

Group Work

Let $G \sim G_{n,p}$ be a Erdos-Renyi random graph (so there are n vertices, and each edge is present independently with probability p). Let $A = \chi(G)$ be the chromatic number of G . That is, A is the minimum number of colors necessary to properly color G (ie color the nodes of the graph such that no pair of neighboring nodes are assigned the same color).

1. Consider the Doob *vertex exposure* martingale. That is:
 - For $i \in \{1, \dots, n\}$, let X_i denote the the status of the edges between vertex i and vertices $\{1, \dots, i - 1\}$.
 - $Z_i = \mathbb{E}[A|X_1, \dots, X_i]$

[Note: this is a slightly different definition of the vertex exposure martingale than was in the lecture notes. Both work fine for this example.]

Use the Azuma-Hoeffding inequality to show that

$$\Pr[|A - \mathbb{E}[A]| > c\sqrt{n}] \leq 2 \exp(-c^2/2).$$

(Notice that you may not know what $\mathbb{E}[A]$ is—that's okay!)

Hint: To use Azuma-Hoeffding, you need to bound $|Z_i - Z_{i-1}|$. How much can your expectation of the chromatic color change if I tell you additional information about a single vertex?

Hint: Bounding $|Z_i - Z_{i-1}|$ really formally is actually a bit tricky. Try to come up with an intuitive bound, and if you have time, try to work it out formally.

2. Repeat the same exercise with the *edge exposure* martingale:
 - Let X_i denote the the status of the i 'th edge, for $i \in \{1, \dots, \binom{n}{2}\}$.
 - $Z_i = \mathbb{E}[A|X_1, \dots, X_i]$

Do you get the same thing? Do you get something better? Worse?

3. (**Challenging**, but something to think about if you finish early.) What can you say about $\mathbb{E}[A]$?

Note: If you're interested, check out <https://arxiv.org/abs/0706.1725> for a surprisingly strong statement about the chromatic number of random graphs!!

Group Work: Solutions

See lecture notes (at the end of the notes).

Before we move on, note that in the above group work we essentially proved the *method of bounded differences*:

Theorem 1. Let X_1, \dots, X_n be independent, and suppose that $A = A(X_1, \dots, X_n)$ is a function of X_1, \dots, X_n . Suppose that for all $i \in [n]$, there is a $c_i > 0$ so that, for all values of x_1, \dots, x_n , and x'_i ,

$$|A(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) - A(x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n)| \leq c_i.$$

Then $\Pr[|A - \mathbb{E}A| > \lambda] \leq 2 \exp\left(\frac{-\lambda^2}{2 \sum_i c_i^2}\right)$.

As a fun bonus question, find a counter-example to show that the independence is necessary.

Group Work: Solutions

See Remark 7 in the lecture notes (and a footnote there) for such a counter-example.

3 More practice with Azuma-Hoeffding: Gambling

Group Work

Consider the following gambling game:

- At time t , you can choose to bet *any* amount you like in $[0, B]$, where B is a house limit.
- A fair coin is flipped. If it's heads, you win the amount that you bet; if tails, you lose the amount that you bet.

You're allowed to be in debt; you don't stop when you run out of money. Suppose you start with zero money.

1. Suppose that the amount you bet is a deterministic function of everything that's happened so far. Let Z_t be the amount of money you have at time t . Prove a bound of the form:

$$\Pr[|Z_n| \geq cB\sqrt{n}] \leq \text{-----}$$

Hint: Set up a martingale $\{Z_t\}$ (with respect to some sequence $\{X_t\}$ that you have to define)...

2. Now suppose that you can use *any* betting strategy you like, even a randomized one. Does your proof still work? If not, repeat the previous part with a possibly randomized betting strategy.

Group Work: Solutions

See lecture notes (at the end of the notes).