CS265, Winter 2026

Class 16: Agenda/Questions

1 Questions?

Any questions from the minilectures and/or the quiz? (Martingales, the Doob martingale, Azuma-Hoeffding)

2 Practice with Azuma-Hoeffding: Chromatic numbers

Group Work

Let $G \sim G_{n,p}$ be a Erdos-Renyi random graph (so there are n vertices, and each edge is present independently with probability p). Let $A = \chi(G)$ be the chromatic number of G. That is, A is the minimum number of colors necessary to properly color G (ie color the nodes of the graph such that no pair of neighboring nodes are assigned the same color).

1. Consider the Doob *vertex exposure* martingale. That is:
   - For $i \in \{1, \ldots, n\}$, let X_i denote the the status of the edges between vertex i and vertices $\{1, \ldots, i-1\}$.
   - $Z_i = \mathbb{E}[A | X_1, \ldots, X_i]$

   [Note: this is a slightly different definition of the vertex exposure martingale than was in the lecture notes. Both work fine for this example.]

   Use the Azuma-Hoeffding inequality to show that

   $$\Pr[|A - \mathbb{E}[A]| > c\sqrt{n}] \leq 2\exp(-c^2/2).$$

   (Notice that you may not know what $\mathbb{E}[A]$ is—that's okay!)

   Hint: *To use Azuma-Hoeffding, you need to bound* $|Z_i - Z_{i-1}|$. *How much can your expectation of the chromatic color change if I tell you additional information about a single vertex?*

   Hint: *Bounding* $|Z_i - Z_{i-1}|$ *really formally is actually a bit tricky. Try to come up with an intuitive bound, and if you have time, try to work it out formally.*

2. Repeat the same exercise with the *edge exposure* martingale:
   - Let X_i denote the the status of the i'th edge, for $i \in \{1, \ldots, \binom{n}{2}\}$.
   - $Z_i = \mathbb{E}[A | X_1, \ldots, X_i]$

   Do you get the same thing? Do you get something better? Worse?

3. (**Challenging,** but something to think about if you finish early.) What can you say about $\mathbb{E}[A]$?

   Note: *If you're interested, check out* https://arxiv.org/abs/0706.1725 *for a surprisingly strong statement about the chromatic number of random graphs!!.*

Before we move on, note that in the above group work we essentially proved the *method of bounded differences:*

Theorem 1. *Let $X_1, \ldots, X_n$ be independent, and suppose that $A = A(X_1, \ldots, X_n)$ is a function of $X_1, \ldots, X_n$. Suppose that for all $i \in [n]$, there is a $c_i > 0$ so that, for all values of $x_1, \ldots, x_n$, and x'_i,*

$$|A(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n - A(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n)| \le c_i.$$

Then $\Pr[|A - \mathbb{E}A| > \lambda] \le 2\exp\left(\frac{-\lambda^2}{2\sum_i c_i^2}\right)$.

As a fun bonus question, find a counter-example to show that the independence is necessary.

3 More practice with Azuma-Hoeffding: Gambling

Group Work

Consider the following gambling game:

- At time t, you can choose to bet *any* amount you like in $[0, B]$, where B is a house limit.
- A fair coin is flipped. If it's heads, you win the amount that you bet; if tails, you lose the amount that you bet.

You're allowed to be in debt; you don't stop when you run out of money. Suppose you start with zero money.

1. Suppose that the amount you bet is a deterministic function of everything that's happened so far. Let Z_t be the amount of money you have at time t. Prove a bound of the form:

$$\Pr[|Z_n| \ge cB\sqrt{n}] \le \_\_\_\_\_$$

   Hint*: Set up a martingale $\{Z_t\}$ (with respect to some sequence $\{X_t\}$ that you have to define)...*

2. Now suppose that you can use *any* betting strategy you like, even a randomized one. Does your proof still work? If not, repeat the previous part with a possibly randomized betting strategy.