Class 17

Martingale Stopping Theorem

Warm-Up

- Let $\{Z_t\}$ be a random walk on a line:
  - $Z_t = Z_{t-1} - 1$ with probability ½
  - $Z_t = Z_{t-1} + 1$ with probability ½

(all independent)

- Let T be the first time t that $Z_t = 10$.

- Prove that $E[T] = \infty$, but that $\Pr[\, \exists t, Z_t = 10 \,] = 1$.
  - We asserted this in the videos but didn't formally prove it!

- **Extra:** Now say $\Pr[Z_t = Z_{t-1} + 1] = p$ for some $p \neq \frac{1}{2}$ …

  What is $\Pr[\, \exists t, Z_t = 10]$?

Announcements

- HW7 due Friday!
- FINAL EXAM!
  - Friday March 21, 8:30am – 11:30am, Hewlett 201.
  - Practice exam (with our cheat sheet) out now.
- More info (same as has been on the website, just a reminder):
  - The exam is not collaborative
  - It is closed book and closed notes EXCEPT for a “cheat sheet” you can prepare
    - 3 pages, front and back, put whatever you want on it.
  - We’ll also include a “cheat sheet” at the end of the exam with generally useful formulas and equations.

Recap!

- Stopping times
- Martingale Stopping theorem
- Applications

Stopping times

- T is a **stopping time** for $\{X_t\}$ if the event that $T = i$ is mutually independent of all the random variables $X_j \mid X_0, \ldots, X_i$, for all $j > i$.
  - Informally, you should be able to tell that T has occurred at time T, without looking into the future.
  - Example: The first time X_t hits 100.
  - Non-example: The last time X_t hits 100.

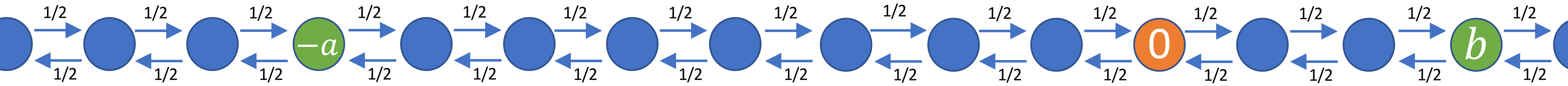
Martingale Stopping Theorem

- Let $\{Z_t\}$ be a martingale w.r.t. $\{X_t\}$.
- Let T be a stopping time for $\{X_t\}$.
- Then $\mathbf{E}[Z_T] = \mathbf{E}[Z_0]$ if at least one of the following hold:
  - There is a c so that $|Z_i| < c$ for all i
  - There is a c so that $T < c$ with probability 1
  - There is a c so that $\mathbf{E}[|Z_{i+1} - Z_i| \mid X_0, \ldots, X_i] < c$ for all i, and $\mathbf{E}[T] < \infty$

Why do we care?

- Hitting time of random walks!

- Set up a martingale Z_t so that $\mathbf{E}[Z_T]$ has something to do with something you care about.
  - E.g., $\mathbf{E}[Z_T] = \Pr[\,Z_T = b] \cdot b - \Pr[Z_T = -a] \cdot a$
- You know what $\mathbf{E}[Z_0]$ is.
  - E.g., $\mathbf{E}[Z_0] = 0.$
- Solve FTW.
  - E.g., $\Pr[Z_T = b] = \frac{a}{a+b}$



Questions?

Warm-up, stopping times, martingale stopping thm, quiz?

Warm-Up

- Let $\{Z_t\}$ be a random walk on a line:
  - $Z_t = Z_{t-1} - 1$ with probability ½
  - $Z_t = Z_{t-1} + 1$ with probability ½

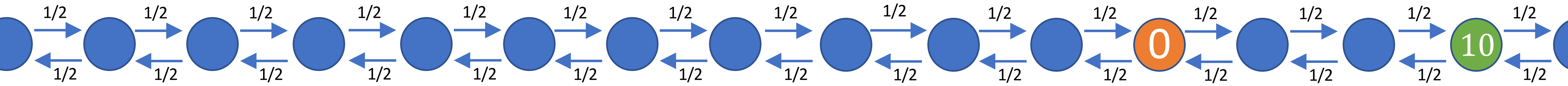
(all independent)

- Let T be the first time t that $Z_t = 10$.

- Prove that $E[T] = \infty$, but that $\Pr[\exists t, Z_t = 10] = 1$.

- Now say $\Pr[Z_t = Z_{t-1} + 1] = p$ for some $p \neq \frac{1}{2}$ …

  What is $\Pr[\exists t, Z_t = 10]$?

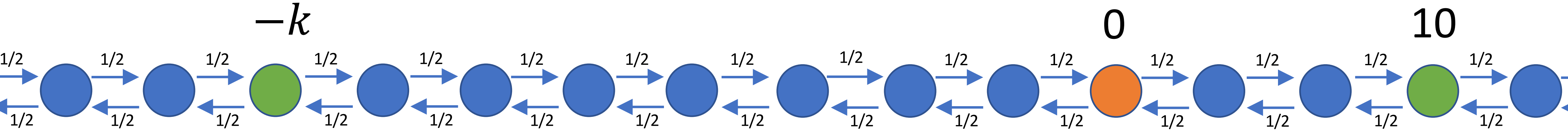


1st Solution

- We actually saw a sketch of something like this (in a footnote in the lecture notes…) when we analyzed the Gambler's ruin example in Lecture 14.
  - Check out solutions to group work or the Lecture 14 notes (footnote 1) for the basic idea

2^{nd} Solution

- Use Theorem 2! Let $T_k = \min\{\, t : Z_t = 10 \text{ or } Z_t = -k \,\}$
- $E[T] \geq E[T_k] = 10k \to \infty$ as $k \to \infty$.
  - So $E[T] = \infty$.
- $\Pr[Z_t \text{ ever reaches } 10] \geq \Pr\left[\, Z_{T_k} = 10 \,\right] = \frac{k}{k+10} \to 1$ as $k \to \infty$
  - So $\Pr[\, Z_t \text{ ever reaches } 10 \,] = 1$.



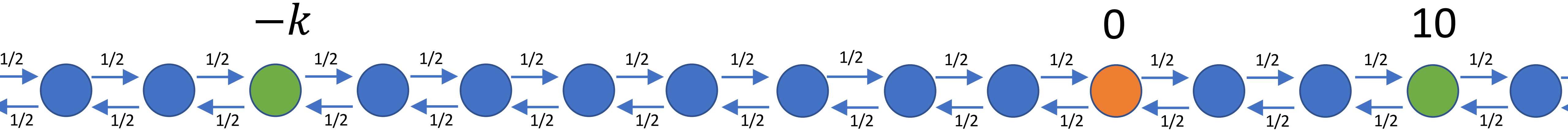
What if the prob of moving up is p?

- Use Theorem 3! Let $T_k = \min\{\, t : Z_t = 10 \text{ or } Z_t = -k \,\}$

- If $p > \frac{1}{2}$, $\Pr[Z_t \text{ ever reaches } 10] = 1$ (no smaller than for $p = 1/2$, and that was 1).
- If $p < \frac{1}{2}$, our intuition is that

$$\Pr[\, Z_t \text{ ever reaches } 10 \,] = \lim_{k\to\infty} \Pr\left[\, Z_{T_k} = 10 \,\right] = \left(\frac{p}{p-1}\right)^{10}$$

- (This is true, though one needs to be a bit careful to show this formally – see solns after class)



Plan for today

- Practice with Martingale Stopping Theorem!
- Wald's Equation
- (If time) Ballot counting thm

Wald's Equation

Theorem.

- Suppose that $X_1, X_2, \ldots$ are non-negative i.i.d. random variables, $X_i \sim X$.
- Let T be a stopping time for $\{X_i\}$.
- Suppose that $\mathbf{E}[T], \mathbf{E}[X] < \infty$.
- Then

$$\mathbf{E}\left[\sum_{i=1}^{T} X_i\right] = \mathbf{E}[T] \cdot \mathbf{E}[X]$$

Theorem.

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- Then $\mathbf{E}\left[\sum_{i=1}^{T} X_i\right] = \mathbf{E}[T] \cdot \mathbf{E}[X]$

Group Work

1. Find an example where Thm 1 fails if the hypotheses aren't met.
   - Try to violate as few of the hypothesis as you can!
2. Let $Z_i = \sum_{j=1}^{i} \left(X_j - E[X]\right)$. Prove that $\{Z_t\}$ is a martingale wrt $\{X_t\}$
3. Show that the martingale stopping thm applies to $\{Z_t\}$ and T.
4. Use the martingale stopping thm to prove Wald's equation.
5. Consider rolling a fair six-sided die repeatedly. Let X be the the sum of all of the rolls up until the first "6" is rolled. (Not including that first "6"). What is E[X]?

1. Example where the Theorem fails

Here's one where **E[T]=∞**, and **the $\boldsymbol{X_i}$ might be negative**

- Let $X_1, X_2, \ldots,$ be i.i.d. $\{\pm 1\}$ random variables, $\Pr[X_i = 1] = p < \frac{1}{2}$
- Let T be the first time $\sum_i X_i = 10$.
  - From our warm-up, we know that $\mathbf{E}[T] = \infty$.

- Then $\mathbf{E}\left[\sum_{i=1}^{T} X_i\right] = 10$.
- $\mathbf{E}[X] = 2p - 1 \neq 0$
- But $\mathbf{E}[T] \cdot \mathbf{E}[X] = \infty \cdot (2p - 1) = \infty \neq 10$

Theorem.

- Suppose that $X_1, X_2, \ldots$ are non-negative i.i.d., $X_i \sim X$.
- Let T be a stopping time for $\{X_i\}$.
- Suppose that $\mathbf{E}[T], \mathbf{E}[X] < \infty$.
- Then $\mathbf{E}\left[\sum_{i=1}^{T} X_i\right] = \mathbf{E}[T] \cdot \mathbf{E}[X]$

1. Example where the Theorem fails

Here's one where T isn't a stopping time

- Let $X_1, X_2, \ldots$, be i.i.d. $\{0,1\}$ random variables, mean $1/2$.
  - Actually we only need X_1
- Let T be $1 - X_1$. (Note that T is not a stopping time!)
- Then $\mathbf{E}\left[\sum_{i=1}^{T} X_i\right] = 0$.
  - If $X_1 = 1$ then we sum zero things
  - If $X_1 = 0$ then we sum one thing which is equal to 0
- But $\mathbf{E}[T] \cdot \mathbf{E}[X] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

Theorem.

- Suppose that $X_1, X_2, \ldots$ are non-negative i.i.d., $X_i \sim X$.
- Let T be a stopping time for $\{X_i\}$.
- Suppose that $\mathbf{E}[T], \mathbf{E}[X] < \infty$.
- Then $\mathbf{E}\left[\sum_{i=1}^{T} X_i\right] = \mathbf{E}[T] \cdot \mathbf{E}[X]$

2. Let $Z_i = \sum_{j=1}^{i} \left(X_j - E[X] \right)$.

Prove that $\{Z_t\}$ is a martingale wrt $\{X_t\}$

- $\mathbf{E}[Z_i | X_1, \dots, X_{i-1}] = Z_{i-1}$

$$\mathbf{E}\left[\sum_{j=1}^{i-1} \left(X_j - \mathbf{E}[X] \right) + \left(X_i - \mathbf{E}[X] \right) \mid X_1, .., X_{i=1} \right] = Z_{i-1} + \overbrace{\mathbf{E}[\, X_i - \mathbf{E}[X] \mid X_{\cdot}, \dots, X_{i-1}]}^{0}$$

- $\forall i < \infty, \mathbf{E}[|Z_i|] < \infty$

$$\mathbf{E}[|Z_i|] = \mathbf{E}\left| \sum_{j=1}^{i} \left(X_j - \mathbf{E}[X] \right) \right| \leq \sum_{j=1}^{i} \mathbf{E}\left| X_j - \mathbf{E}[X] \right| \leq \sum_{j=1}^{i} 2 \cdot \mathbf{E}[X] = 2 \cdot i \cdot \mathbf{E}[X] < \infty$$

3. The Martingale Stopping Thm applies

Theorem.

- Suppose that $X_1, X_2, \ldots$ are non-negative i.i.d., $X_i \sim X$.
- Let T be a stopping time for $\{X_i\}$.
- Suppose that $\mathbf{E}[T], \mathbf{E}[X] < \infty$.
- Then $\mathbf{E}\left[\sum_{i=1}^{T} X_i\right] = \mathbf{E}[T] \cdot \mathbf{E}[X]$

- $\mathbf{E}[T] < \infty$ by assumption.
- $\mathbf{E}\left[|Z_{i+1} - Z_i| \middle| X_0, \ldots, X_i\right]$

$$= \mathbf{E}\left[\left|\sum_{j=1}^{i+1}(X_j - \mathbf{E}X) - \sum_{j=1}^{i}(X_j - \mathbf{E}X)\right| \middle| X_0, \ldots, X_i\right]$$

$$= \mathbf{E}[|X_{i+1} - \mathbf{E}X| \mid ~~X_0, \ldots, X_i~~]$$

$$\leq \mathbf{E}|X_{i+1}| + |\mathbf{E}X|$$

$$= 2\mathbf{E}[X]$$ 👍

Theorem 1 (Martingale Stopping Theorem). *Letting $\{Z_t\}$ denote a martingale with respect to $\{X_t\}$, and T a stopping time for $\{X_t\}$, then $\mathbf{E}[Z_T] = \mathbf{E}[Z_0]$ if at least one of the following conditions hold:*

1. *If there exists a constant c such that for all i, $|Z_i| < c$.*
2. *If there exists a constant c such that with probability 1, $T < c$.*
3. *If $\mathbf{E}[T] < \infty$, and there exists a constant c such that for all i, $\mathbf{E}[|Z_{i+1} - Z_i| | X_0, \ldots, X_i] < c$.*

4. Apply the martingale stopping theorem

Theorem.

- Suppose that $X_1, X_2, \ldots$ are non-negative i.i.d., $X_i \sim X$.
- Let T be a stopping time for $\{X_i\}$.
- Suppose that $\mathbf{E}[T], \mathbf{E}[X] < \infty$.
- Then $\mathbf{E}\left[\sum_{i=1}^{T} X_i\right] = \mathbf{E}[T] \cdot \mathbf{E}[X]$

$$\mathbf{E}[Z_T] = \mathbf{E}[Z_0] = \underline{\quad 0 \quad}$$

$$0 = \mathbf{E}[Z_T] = \mathbf{E}\left[\sum_{j=1}^{T} (X_j - \mathbf{E}X)\right] = \mathbf{E}\left[\sum_{j=1}^{T} X_j\right] - \mathbf{E}\left[\sum_{j=1}^{T} \mathbf{E}X\right]$$

$$= \mathbf{E}\left[\sum_{j=1}^{T} X_j\right] - \mathbf{E}[T] \cdot \mathbf{E}[X]$$

👍 Rearrange to prove the theorem.

5. Application of Wald's equation

- X = sum of die-rolls up until you get a six. (Not including that six).
- $\mathbf{E}[X] =$ ____15____

$X_i = i^{th}$ roll

$T = 1^{\text{st}}$ time you roll a 6

$$X = \sum_{i=1}^{T-1} X_i$$

$$\mathbf{E}\left[\sum_{i=1}^{T} X_i\right] = \mathbf{E}[T] \cdot \mathbf{E}[X_1]$$

$$= 6 \cdot \frac{7}{2} = 21$$

(And then subtract off the last 6)

Ballot counting (if time)

- Election with two candidates, A and B, and n voters.
- A will win, receiving $N_A > N_B$ votes. (so $N_A + N_B = n$).
- Ballots are counted in a random order.
- What is the probability that A remains ahead the whole time?

Ballot counting

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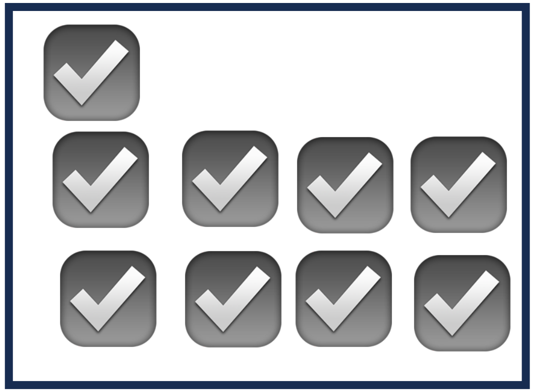
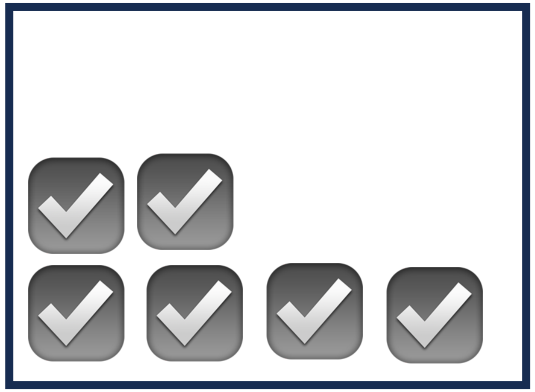
- A will win, receiving $N_A > N_B$ votes. (so $N_A + N_B = n$).
- Ballots are counted in a random order.
- What is the probability that A remains ahead the whole time?

- Let A_t be number of votes for A at time t
- Let B_t be number of votes for B at time t
- Let $Z_t = \frac{A_{n-t} - B_{n-t}}{n-t}$



Votes for A



Votes for B

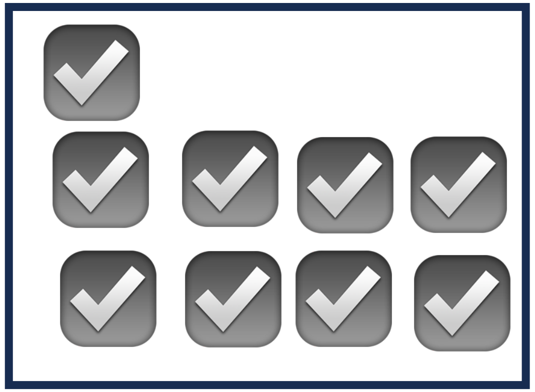
Ballot counting

- Election with two candidates, A and B, and n voters.
- A will win, receiving $N_A > N_B$ votes. (so $N_A + N_B = n$).
- Ballots are counted in a random order.
- What is the probability that A remains ahead the whole time?

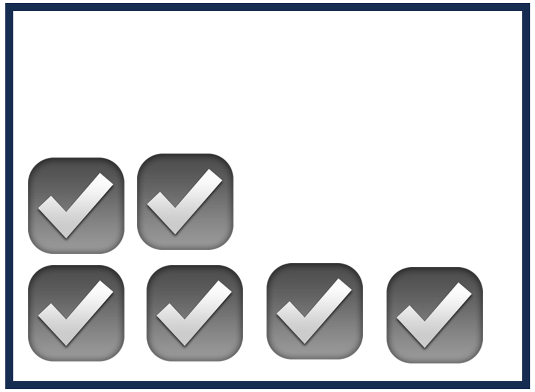
- Let A_t be number of votes for A at time t
- Let B_t be number of votes for B at time t
- Let $Z_t = \frac{A_{n-t} - B_{n-t}}{n-t}$

$t = 5$

$Z_t = \frac{6-4}{10}$



Votes for A



Votes for B

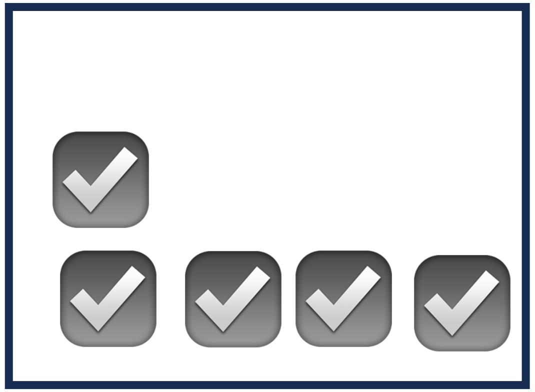
Ballot counting

- Election with two candidates, A and B, and n voters.
- A will win, receiving $N_A > N_B$ votes. (so $N_A + N_B = n$).
- Ballots are counted in a random order.
- What is the probability that A remains ahead the whole time?

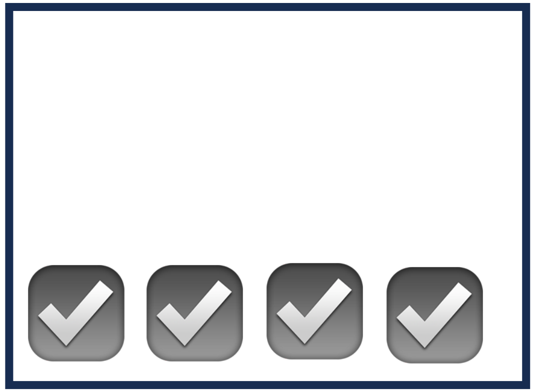
- Let A_t be number of votes for A at time t
- Let B_t be number of votes for B at time t
- Let $Z_t = \frac{A_{n-t} - B_{n-t}}{n-t}$

$t = 6$

$Z_t = \frac{5-4}{9}$



Votes for A



Votes for B

$$Z_t = \frac{A_{n-t} - B_{n-t}}{n-t}$$

Group Work

1. Let T be smallest t so that $Z_t = 0$; if this never occurs, $T = n - 1$. Explain why MST applies to $\{Z_t\}$ and T.
   - Assume for now that $\{Z_t\}$ is a martingale.

2. Apply MST to $\{Z_t\}$ and T, and use it to compute the probability that A was ahead the whole time.

3. Show that $\{Z_t\}$ is indeed a martingale.

1. Let T be smallest t so that $Z_t = 0$; if this never occurs, $T = n - 1$. Explain why MST applies to $\{Z_t\}$ and T.

- T is a stopping time 👍
- $T \leq n$, so set $c = n + 1$. 👍

Theorem 1 (Martingale Stopping Theorem). *Letting $\{Z_t\}$ denote a martingale with respect to $\{X_t\}$, and T a stopping time for $\{X_t\}$, then $\mathbf{E}[Z_T] = \mathbf{E}[Z_0]$ if at least one of the following conditions hold:*

1. *If there exists a constant c such that for all i, $|Z_i| < c$.*
2. *If there exists a constant c such that with probability 1, $T < c$.* ⬅
3. *If $\mathbf{E}[T] < \infty$, and there exists a constant c such that for all i, $\mathbf{E}[|Z_{i+1} - Z_i| | X_0, \ldots, X_i] < c$.*

$$Z_t = \frac{A_{n-t} - B_{n-t}}{n-t}$$

2. Apply MST to compute the prob. A was ahead the whole time.

- $E[Z_T] = E[Z_0] = \underline{\frac{A_n - B_n}{n} = \frac{N_A - N_B}{n}}$

- OTOH, $E[Z_T] = \quad (1-p) \cdot 0 + p \cdot 1 = p$

  - If $T < n-1$, then $Z_T = 0$ and there was some point when B was ahead.
  - If $T = n-1$, then $Z_T = 1$ and A was ahead the whole time.

  - Let $p = \Pr[\, A \text{ ahead the whole time} \,]$

- Solve for p: $\quad p = \frac{N_A - N_B}{n}$

$$Z_t = \frac{A_{n-t} - B_{n-t}}{n-t}$$

3. Show Z_t is indeed a martingale

- Start two piles of ballots. Take one from a random pile at each step.

$$E[Z_{t+1} \mid Z_1, \ldots, Z_t] = \frac{E[A_{n-t-1}]}{n-t-1} - \frac{E[B_{n-t-1}]}{n-t-1}$$

Case 1: we remove a ballot from A

Case 2: we remove a ballot from A

$$= \frac{1}{n-t-1}\left(\frac{A_{n-t}}{n-t}(A_{n-t}-1) + \frac{B_{n-t}}{n-t}A_{n-t}\right)$$

$$-\frac{1}{n-t-1}\left(\frac{B_{n-t}}{n-t}(B_{n-t}-1) + \frac{A_{n-t}}{n-t}B_{n-t}\right)$$

… simplify using $A_{n-t} + B_{n-t} = n - t$ …

$$= \frac{A_{n-t}}{n-t} - \frac{B_{n-t}}{n-t} = Z_t$$

Recap

- The Martingale Stopping Theorem is useful!

Next time

- Pseudorandomness!