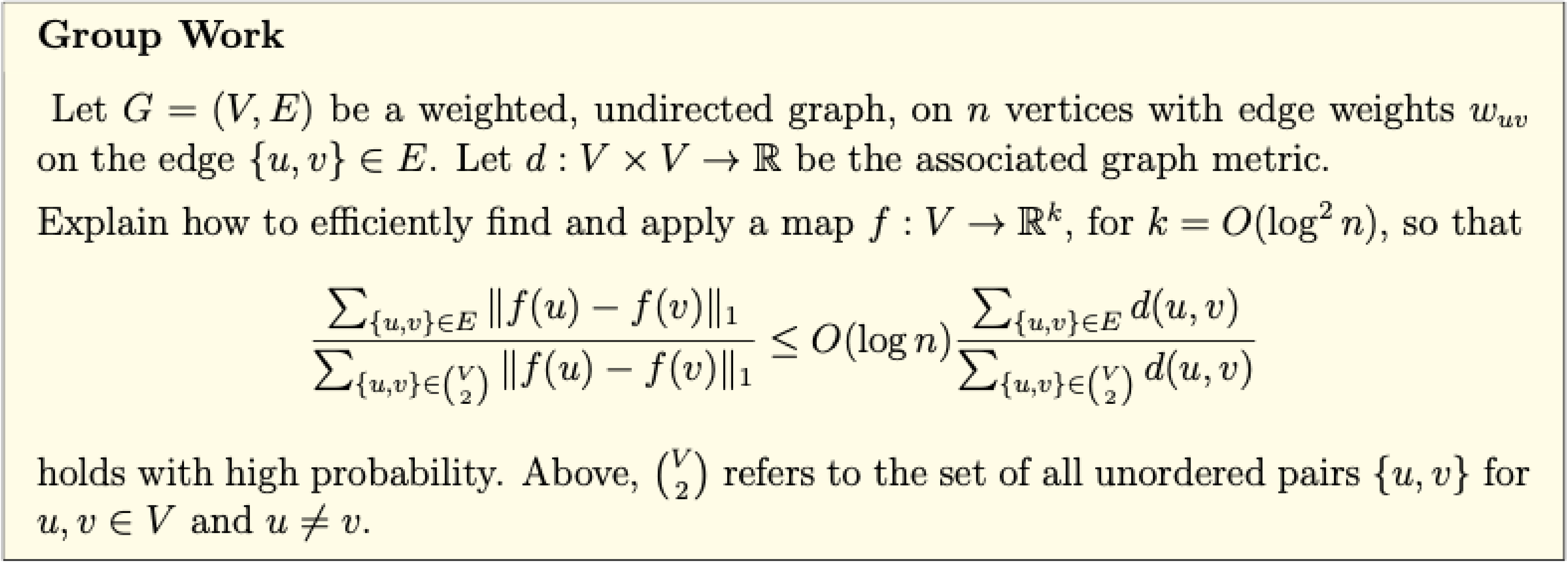
Class 7

Sparsest Cuts from Metric Embeddings

Warm-Up



Group Work

Let $G = (V, E)$ be a weighted, undirected graph, on n vertices with edge weights w_{uv} on the edge $\{u, v\} \in E$. Let $d : V \times V \to \mathbb{R}$ be the associated graph metric.

Explain how to efficiently find and apply a map $f : V \to \mathbb{R}^k$, for $k = O(\log^2 n)$, so that

$$\frac{\sum_{\{u,v\}\in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u) - f(v)\|_1} \leq O(\log n) \frac{\sum_{\{u,v\}\in E} d(u, v)}{\sum_{\{u,v\}\in\binom{V}{2}} d(u, v)}$$

holds with high probability. Above, $\binom{V}{2}$ refers to the set of all unordered pairs $\{u, v\}$ for $u, v \in V$ and $u \neq v$.

Announcements

- HW3 due Friday!
- HW4 out now!

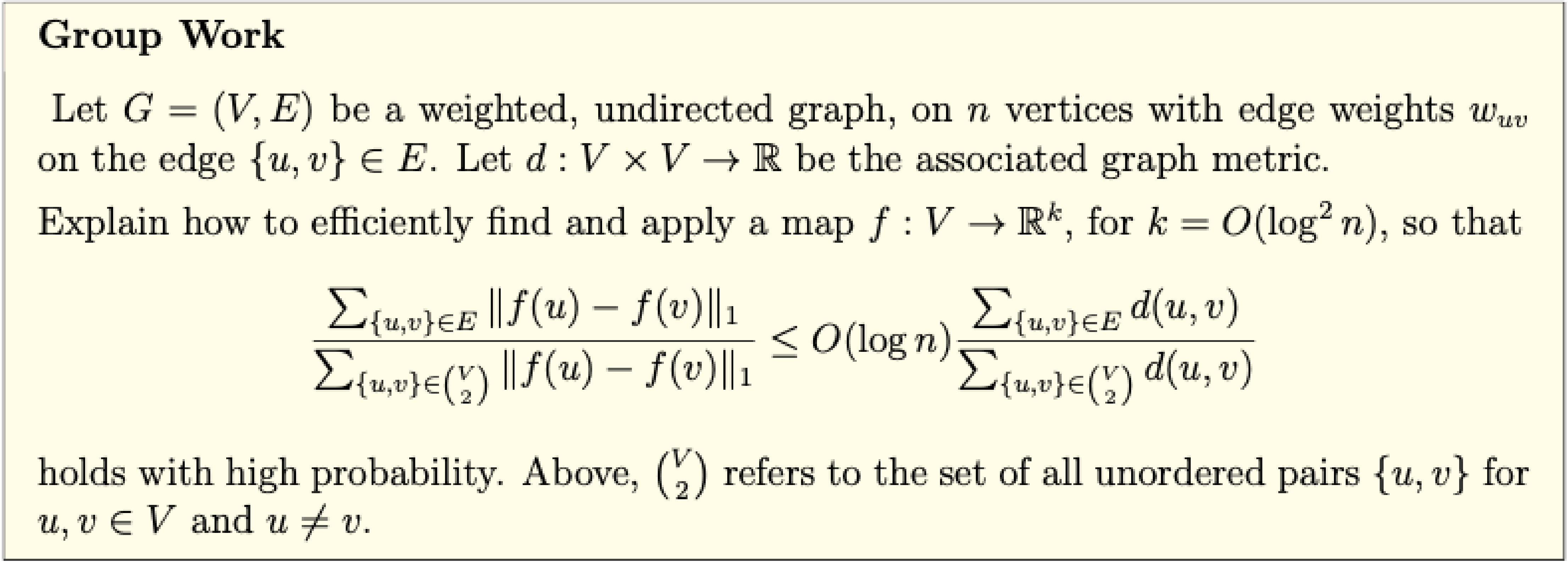
Recap

- Bourgain's embedding!
  - Randomized embedding from *any* X of size n into $(\mathbb{R}^k, \ell_1)$
  - Distortion $\mathrm{O}(\log n)$
  - $k = O(\log^2 n)$

Questions?

Minilectures, quiz, warmup?

- Warm-Up:



Group Work

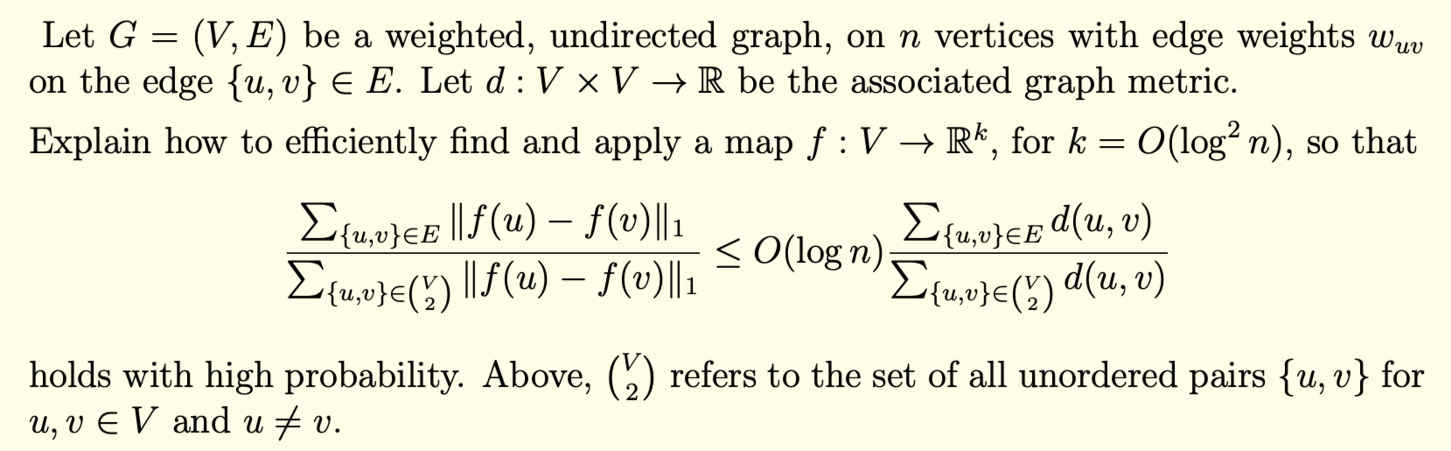
Let $G = (V, E)$ be a weighted, undirected graph, on n vertices with edge weights w_{uv} on the edge $\{u, v\} \in E$. Let $d : V \times V \to \mathbb{R}$ be the associated graph metric.

Explain how to efficiently find and apply a map $f : V \to \mathbb{R}^k$, for $k = O(\log^2 n)$, so that

$$\frac{\sum_{\{u,v\}\in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\}\in \binom{V}{2}} \|f(u) - f(v)\|_1} \leq O(\log n) \frac{\sum_{\{u,v\}\in E} d(u,v)}{\sum_{\{u,v\}\in \binom{V}{2}} d(u,v)}$$

holds with high probability. Above, $\binom{V}{2}$ refers to the set of all unordered pairs $\{u, v\}$ for $u, v \in V$ and $u \neq v$.

Warm-Up



Let $G = (V, E)$ be a weighted, undirected graph, on n vertices with edge weights w_{uv} on the edge $\{u, v\} \in E$. Let $d : V \times V \to \mathbb{R}$ be the associated graph metric.

Explain how to efficiently find and apply a map $f : V \to \mathbb{R}^k$, for $k = O(\log^2 n)$, so that

$$\frac{\sum_{\{u,v\}\in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u) - f(v)\|_1} \le O(\log n) \frac{\sum_{\{u,v\}\in E} d(u,v)}{\sum_{\{u,v\}\in\binom{V}{2}} d(u,v)}$$

holds with high probability. Above, $\binom{V}{2}$ refers to the set of all unordered pairs $\{u, v\}$ for $u, v \in V$ and $u \neq v$.

- Use Bourgain’s embedding!

$$\frac{k}{b \log n} d(u,v) \le ||f(u) - f(v)||_1 \le kd(u,v)$$

- Apply to top and bottom.

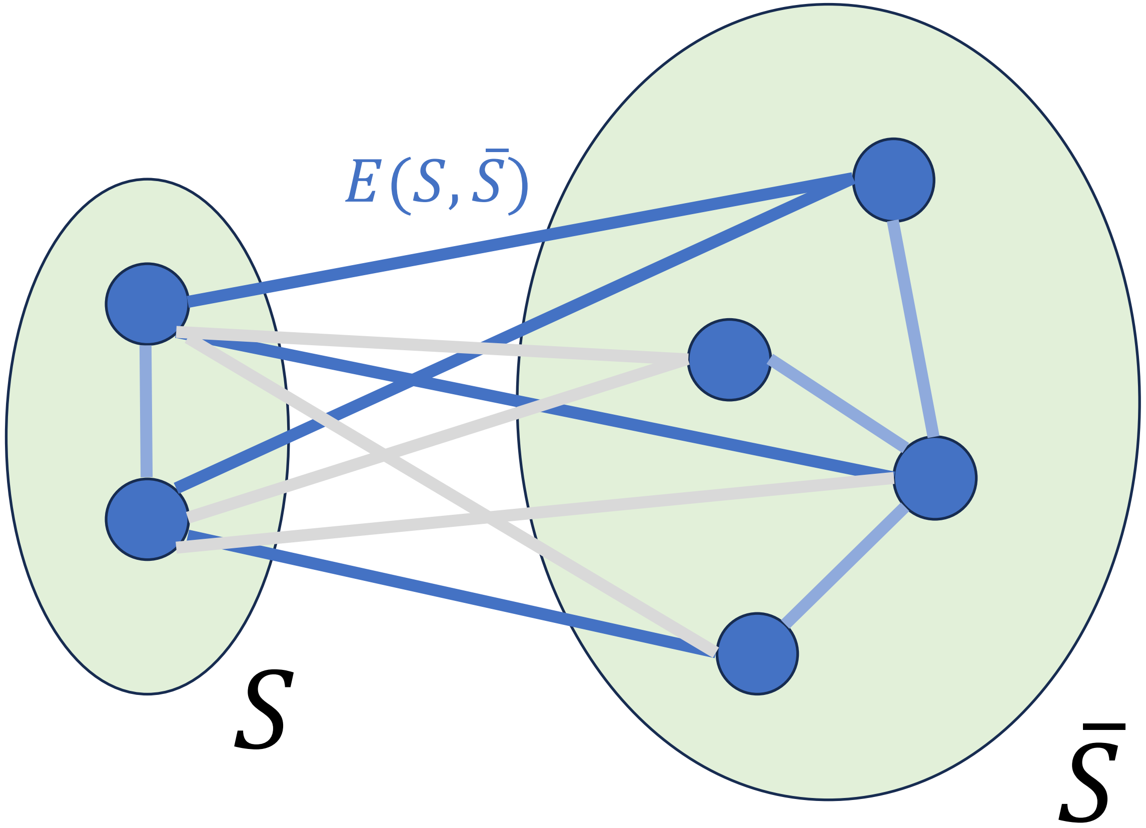
Plan for today

- Application of Bourgain’s embedding to sparsest cuts!

Sparsest Cuts

$$\phi(G,S) = \frac{|E(S,\bar{S})|}{|S||\bar{S}|}$$

$$\phi(G) = \min_{S} \phi(G,S)$$



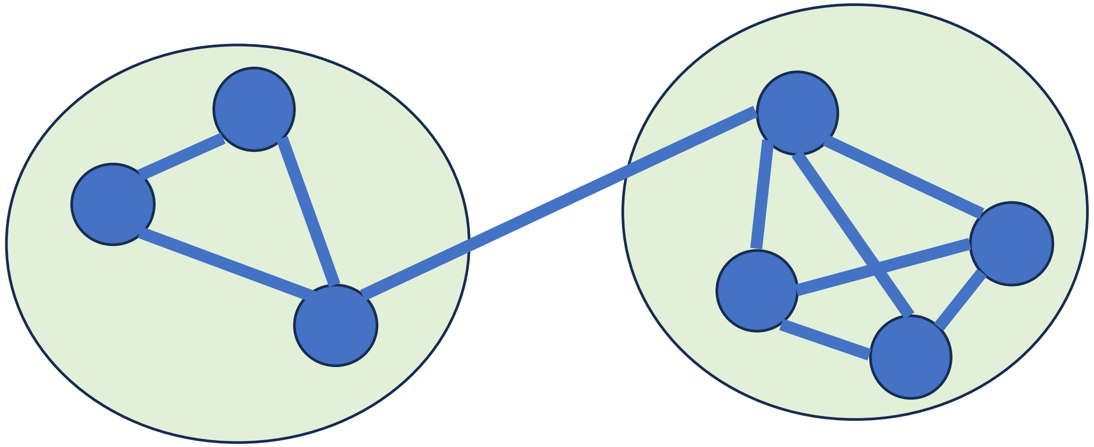
Number of possible edges between $S,\bar{S}$: $|S||\bar{S}|$

$$\phi(G,S) = \frac{|E(S,\bar{S})|}{|S||\bar{S}|}$$

Sparsest Cut = cut that realizes $\phi(G)$

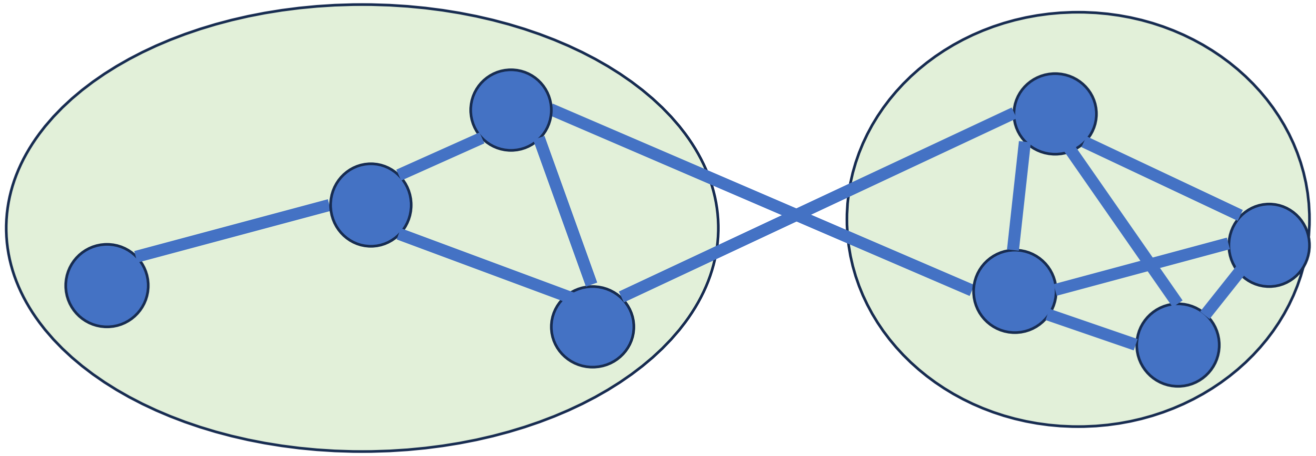
$$\phi(G) = \min_S \phi(G,S)$$

- What’s the sparsest cut of this graph?



$\phi(G) = \frac{1}{12}$

- This one?



$\phi(G) = \frac{2}{16} = \frac{1}{8}$

NOT the same as a **minimum** cut!

Efficient algorithms for sparsest cut?

- We saw an algorithm for **min-cut** in Week 1!
  - Karger's algorithm!
- …But it turns out, **sparsest-cut** is NP hard!

- Today, we'll see a randomized **approximation algorithm** for sparsest cut.
  - Returns S so that $\phi(G,S) \leq O(\log n) \cdot \phi(G)$
  - (Probably)

Assuming plausible complexity-theoretic assumptions, it's NP-hard even to approximate $\phi(G)$ to within a constant factor.

So the $O(\log n)$ is pretty good!

Outline

- First group(-ish) work: Show that

$$\phi(G) = \inf_{f:V\to\mathbb{R}^k} \frac{\sum_{\{u,v\}\in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u) - f(v)\|_1}$$

We'll walk through this one together!

- Second group work:
  - Use this to design an approximation algorithm.

Group Work!

1. $$\phi(G) = \min_{f:V\to\boxed{\{0,1\}}} \frac{\sum_{\{u,v\}\in E} |f(u)-f(v)|}{\sum_{\{u,v\}\in\binom{V}{2}} |f(u)-f(v)|},$$

This one is the conceptually important one

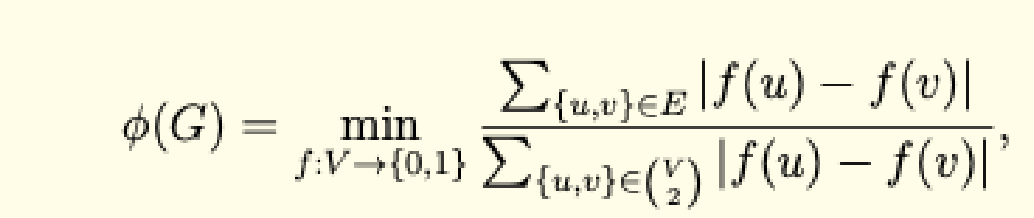
2. $$\phi(G) = \inf_{f:V\to\boxed{\mathbb{R}}} \frac{\sum_{\{u,v\}\in E} |f(u)-f(v)|}{\sum_{\{u,v\}\in\binom{V}{2}} |f(u)-f(v)|},$$

3. $$\phi(G) = \min_{f:V\to\boxed{\mathbb{R}^k}} \frac{\sum_{\{u,v\}\in E} \|f(u)-f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u)-f(v)\|_1},$$

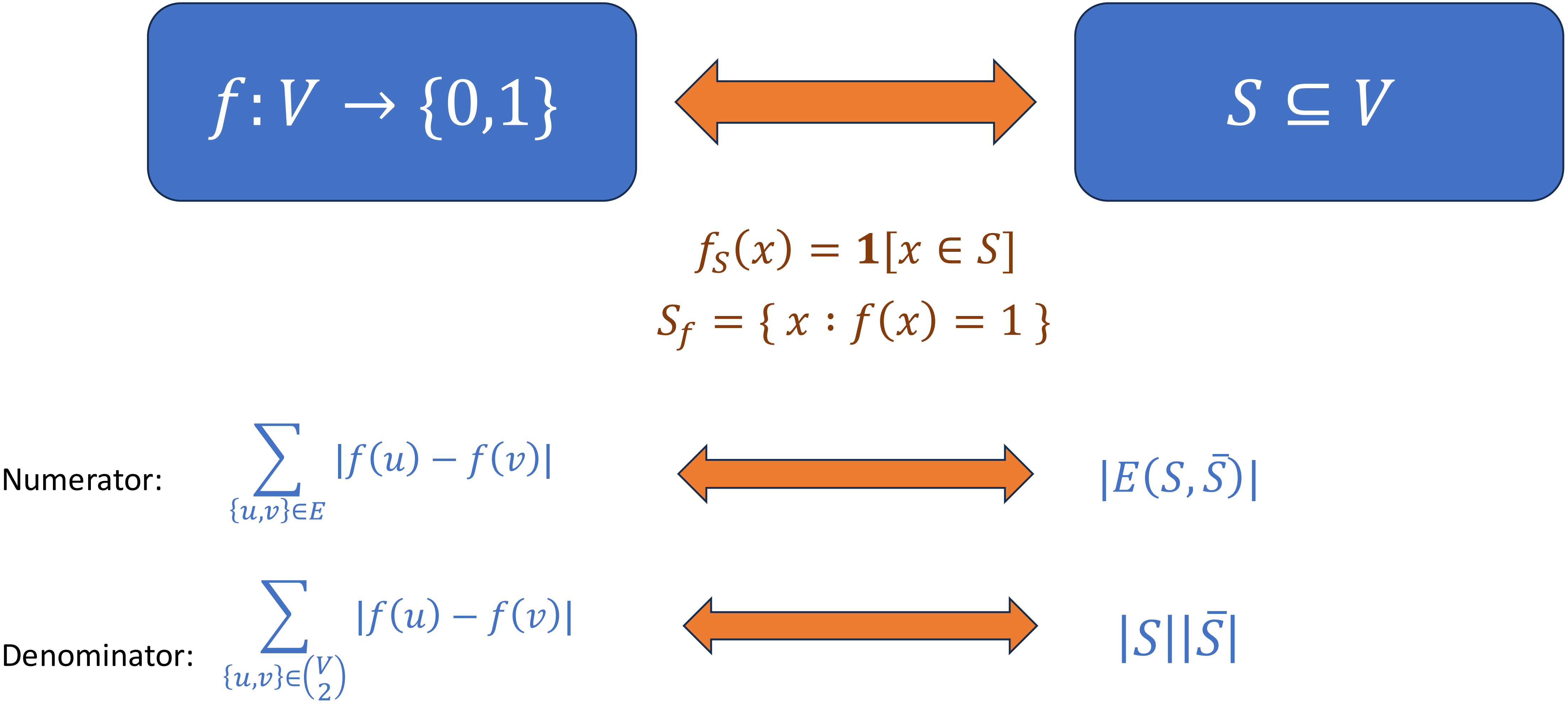
If time, try to get some intuition for these.

If not time, that’s fine, we’ll go over it together!

Solutions: Part 1, Problem 1



$$\phi(G) = \min_{f:V\to\{0,1\}} \frac{\sum_{\{u,v\}\in E} |f(u) - f(v)|}{\sum_{\{u,v\}\in\binom{V}{2}} |f(u) - f(v)|},$$



$f: V \to \{0,1\}$ $\longleftrightarrow$ $S \subseteq V$

$$f_S(x) = \mathbf{1}[x \in S]$$

$$S_f = \{\, x : f(x) = 1 \,\}$$

Numerator: $\sum_{\{u,v\}\in E} |f(u) - f(v)|$ $\longleftrightarrow$ $|E(S,\bar{S})|$

Denominator: $\sum_{\{u,v\}\in\binom{V}{2}} |f(u) - f(v)|$ $\longleftrightarrow$ $|S||\bar{S}|$

Part 2!

Some intuition

$$\phi(G) = \min_{f:V\to\mathbb{R}} \underbrace{\frac{\sum_{\{u,v\}\in E}\|f(u)-f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}}\|f(u)-f(v)\|_1}}_{\text{Call this ratio } R(f)}$$

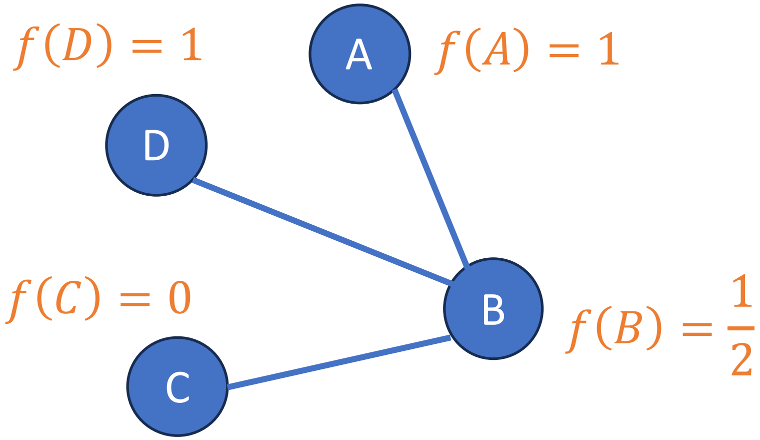
- Idea: Show that this minimum is actually attained by some $f: V \to \{0,1\}$
- So we can just use part 1 to say that it's equal to $\phi(G)$

Part 2!

Some intuition

$$\phi(G) = \min_{f:V\to\mathbb{R}} \underbrace{\frac{\sum_{\{u,v\}\in E} \|f(u)-f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u)-f(v)\|_1}}_{\text{Call this ratio } R(f)}$$

- Example: Say $f: V \to \{0, ½, 1\}$



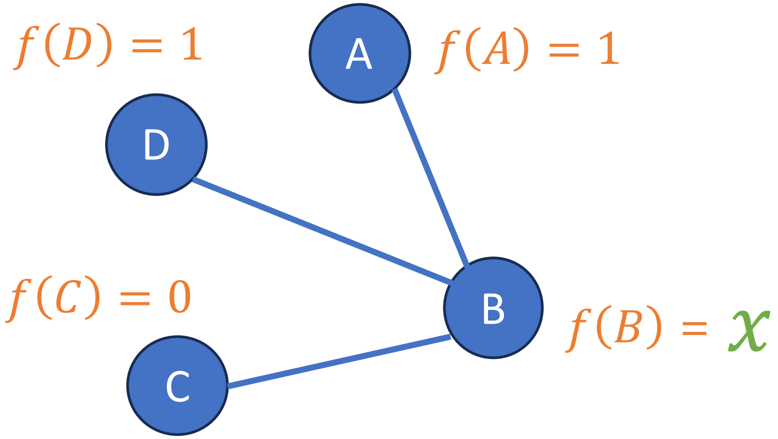
$$R(f) = \frac{\left|1-\frac{1}{2}\right| + \left|\frac{1}{2}-0\right| + \left|\frac{1}{2}-1\right|}{\left|1-\frac{1}{2}\right| + \left|\frac{1}{2}-0\right| + \left|\frac{1}{2}-1\right| + |1-0| + |1-1| + |1-0|}$$

Part 2!

Some intuition

$$\phi(G) = \min_{f:V\to\mathbb{R}} \underbrace{\frac{\sum_{\{u,v\}\in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u) - f(v)\|_1}}_{\text{Call this ratio } R(f)}$$

- Example: Say $f: V \to \{0, ½, 1\}$



$$R(f) = \frac{\left|1 - \frac{1}{2}\right| + \left|\frac{1}{2} - 0\right| + \left|\frac{1}{2} - 1\right|}{\left|1 - \frac{1}{2}\right| + \left|\frac{1}{2} - 0\right| + \left|\frac{1}{2} - 1\right| + |1 - 0| + |1 - 1| + |1 - 0|}$$

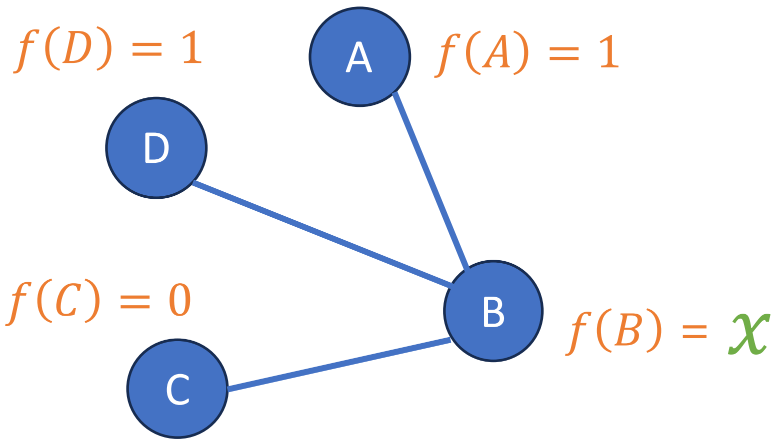
Now suppose we change $f(B)$
to some variable $x \in [0,1]$

Part 2!

Some intuition

$$\phi(G) = \min_{f:V\to\mathbb{R}} \underbrace{\frac{\sum_{\{u,v\}\in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u) - f(v)\|_1}}_{\text{Call this ratio } R(f)}$$

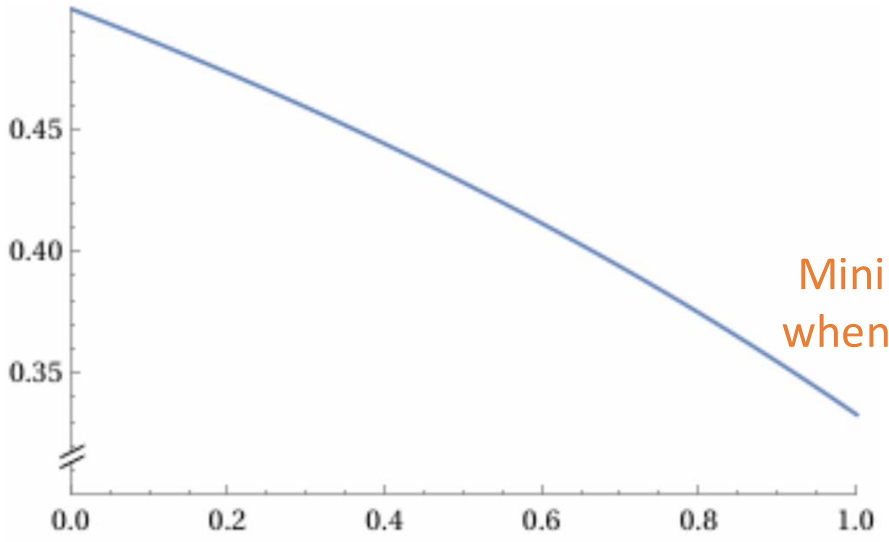
- Example: Say $f: V \to \{0, ½, 1\}$



Now suppose we change $f(B)$ to some variable $x \in [0,1]$

$$R(f) = \frac{|1 - x| + |x - 0| + |x - 1|}{|1 - x| + |x - 0| + |x - 1| + |1 - 0| + |1 - 1| + |1 - 0|}$$

$$= \frac{2 - x}{4 - x}$$



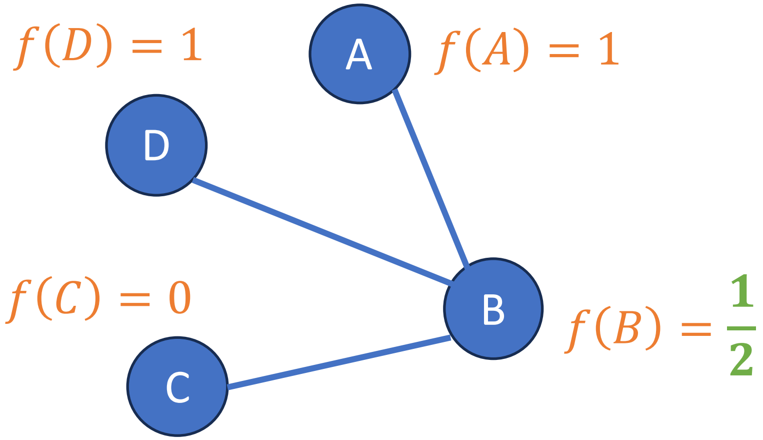
Minimized when $x = 1$

Part 2!

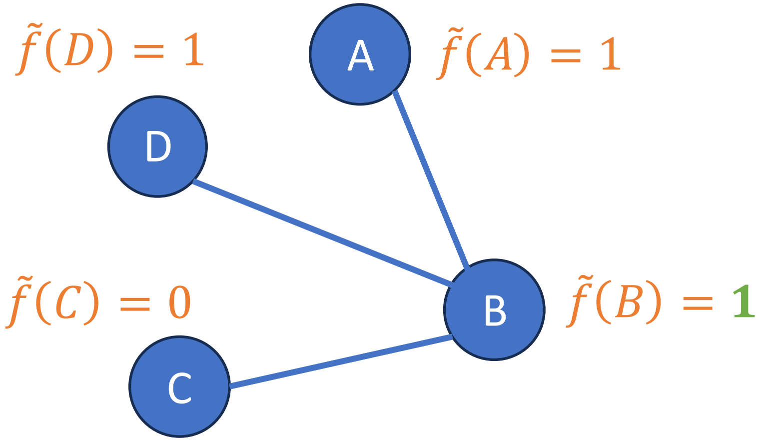
Some intuition

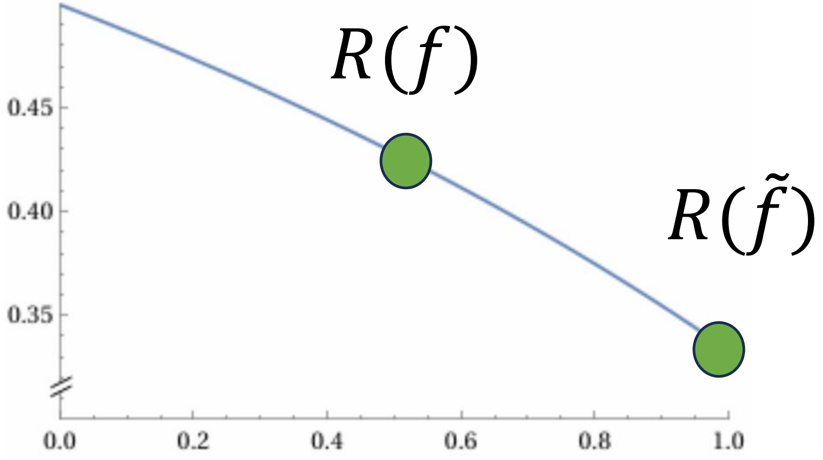
$$\phi(G) = \min_{f:V\to\mathbb{R}} \underbrace{\frac{\sum_{\{u,v\}\in E} \|f(u)-f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u)-f(v)\|_1}}_{\text{Call this ratio } R(f)}$$

- Example: Say $f: V \to \{0, ½, 1\}$



--- vs ---





Part 2!

Some intuition

$$\phi(G) = \min_{f:V\to\mathbb{R}} \underbrace{\frac{\sum_{\{u,v\}\in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u) - f(v)\|_1}}$$

Call this ratio $R(f)$

- Example: Say $f: V \to \{0, ½, 1\}$
- Then actually there is some $\tilde{f}: V \to \{0,1\}$ so that $R\left(\tilde{f}\right) \leq R(f)$

Part 2!

Some intuition

$$\phi(G) = \min_{f:V\to\mathbb{R}} \underbrace{\frac{\sum_{\{u,v\}\in E}\|f(u)-f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}}\|f(u)-f(v)\|_1}}_{\text{Call this ratio } R(f)}$$

- More generally, suppose $f: V \to \mathbb{R}$ takes on at least three distinct values $a < b < c$.

- What happens if we replace b with a variable $x \in [a, c]$?

- $R(f) = \frac{\text{Something linear in } x}{\text{Something else linear in } x}$

Monotonic!

- So if we replace b with either a or c, we get f with a smaller $R(f)$.

- Repeat until only two values left, and scale so they are 0 and 1.

Part 2!

Some intuition

$$\phi(G) = \min_{f:V\to\mathbb{R}} \underbrace{\frac{\sum_{\{u,v\}\in E} \|f(u)-f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u)-f(v)\|_1}}_{\text{Call this ratio } R(f)}$$

- Conclusion: This minimum is actually attained by some $f: V \to \{0,1\}$
- So we can just use part 1 to say that it’s equal to $\phi(G)$

Part 3!

Some intuition

$$\phi(G) = \min_{f:V\to\mathbb{R}^k} \frac{\sum_{\{u,v\}\in E}\|f(u)-f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}}\|f(u)-f(v)\|_1}$$

- If $f: V \to \mathbb{R}^k$, we can write $f(x) = (f_1(x), \ldots, f_k(x))$

$$\frac{\sum_{\{u,v\}\in E}\|f(u)-f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}}\|f(u)-f(v)\|_1} = \frac{\sum_{\{u,v\}\in E}\left(\sum_i |f_i(x)-f_i(v)|\right)}{\sum_{\{u,v\}\in\binom{V}{2}}\left(\sum_i |f_i(x)-f_i(v)|\right)}$$

$$\geq \min_i \frac{\sum_{\{u,v\}\in E}|f_i(x)-f_i(v)|}{\sum_{\{u,v\}\in\binom{V}{2}}|f_i(x)-f_i(v)|}$$

Fact:

$$\frac{\sum_i a_i}{\sum_i b_i} \geq \min_i \frac{a_i}{b_i}$$

This is the case where $f: V \to \mathbb{R}$, so use part 2.

Conclusion

$$\phi(G) = \min_{f:V\to\mathbb{R}^k} \frac{\sum_{\{u,v\}\in E}\|f(u)-f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}}\|f(u)-f(v)\|_1}$$

The sparsest cut of G: what we want to approximate!

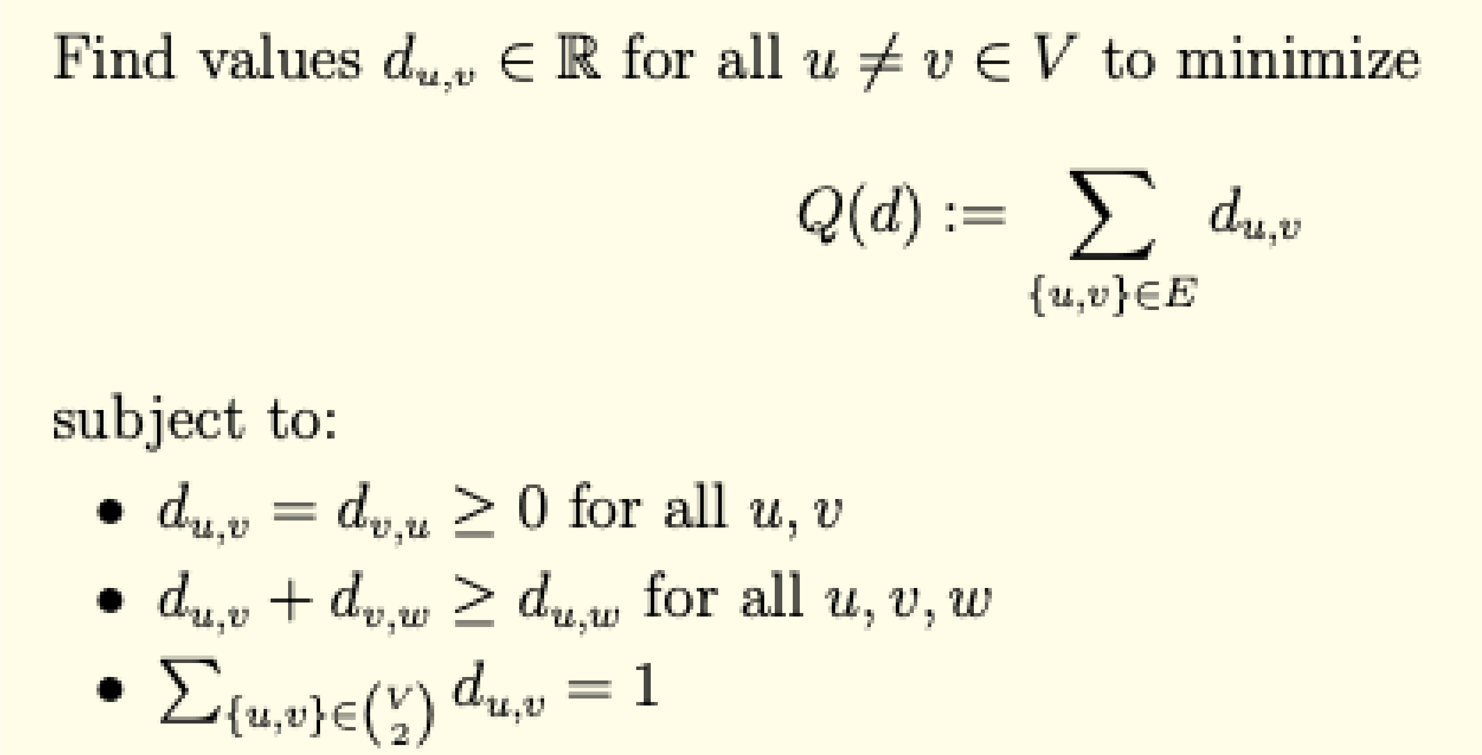
Something that we’ll try to approximate using metric embeddings…

Next up: Design an algorithm!

$$\phi(G) = \inf_{f:V\to\mathbb{R}^k} \frac{\sum_{\{u,v\}\in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u) - f(v)\|_1}$$

Let’s come up with an algorithm!

- Hope: find f to minimize $R(f) := \frac{\sum_{\{u,v\}\in E}\|f(u)-f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}}\|f(u)-f(v)\|_1}$
  - Unfortunately that’s not so easy…
- Instead,



Find values $d_{u,v} \in \mathbb{R}$ for all $u \neq v \in V$ to minimize

$$Q(d) := \sum_{\{u,v\}\in E} d_{u,v}$$

subject to:

- $d_{u,v} = d_{v,u} \geq 0$ for all u, v
- $d_{u,v} + d_{v,w} \geq d_{u,w}$ for all u, v, w
- $\sum_{\{u,v\}\in\binom{V}{2}} d_{u,v} = 1$

This is a **linear program.** Turns out we can solve it efficiently.

Group Work!

1. Suppose that d^* is the minimizer of the problem above.
   Explain why $Q(d^*) \leq \phi(G)$.
2. Find a randomized algorithm to approximate $\phi(G)$. More precisely, give a randomized algorithm that finds $f : V \to \mathbb{R}^k$ so that, with high probability,
   $$\frac{\sum_{\{u,v\}\in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u) - f(v)\|_1} \leq O(\log n)\phi(G).$$
3. Given f as in the previous part, explain how to efficiently find a set $S \subset V$ so that
   $$\phi(G, S) \leq O(\log n)\phi(G).$$

Find values $d_{u,v} \in \mathbb{R}$ for all $u \neq v \in V$ to minimize

$$Q(d) := \sum_{\{u,v\}\in E} d_{u,v}$$

subject to:

- $d_{u,v} = d_{v,u} \geq 0$ for all u, v
- $d_{u,v} + d_{v,w} \geq d_{u,w}$ for all u, v, w
- $\sum_{\{u,v\}\in\binom{V}{2}} d_{u,v} = 1$

$$\phi(G) = \min_{f:V\to\mathbb{R}^k} \frac{\sum_{\{u,v\}\in E}\|f(u)-f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}}\|f(u)-f(v)\|_1}$$

Solution: Problem 1

Say d^* minimizes this LP. Show that $Q(d^*) \leq \phi(G)$

- Consider d given by

$$d_{u,v} = \frac{\|f(u)-f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}}\|f(u)-f(v)\|_1}$$

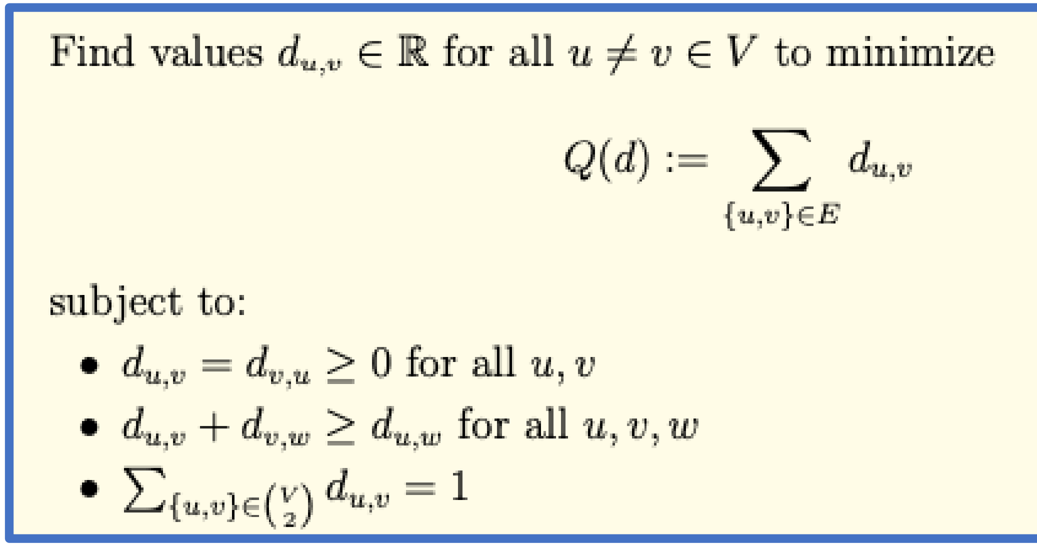
For the f that attains the min here.

- This satisfies all the constraints.

- $Q(d) = \frac{\sum_{\{u,v\}\in E}\|f(u)-f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}}\|f(u)-f(v)\|_1} = \phi(G)$

$$\rotatebox{90}{$\leq$}$$

$$Q(d^*)$$



Find values $d_{u,v} \in \mathbb{R}$ for all $u \neq v \in V$ to minimize

$$Q(d) := \sum_{\{u,v\}\in E} d_{u,v}$$

subject to:

- $d_{u,v} = d_{v,u} \geq 0$ for all u, v
- $d_{u,v} + d_{v,w} \geq d_{u,w}$ for all u, v, w
- $\sum_{\{u,v\}\in\binom{V}{2}} d_{u,v} = 1$

Solution: Problem 2

Find an algorithm! Find f s.t. $R(f) \le O(\log n)\phi(G)$

- Solve this LP to find some d^*
- Use Bourgain's embedding to to find some $f: V \to \mathbb{R}^k$ so that

$$\frac{k}{b \log n} d^*(u,v) \le \left\| f(u) - f(v) \right\|_1 \le k d^*(u,v)$$

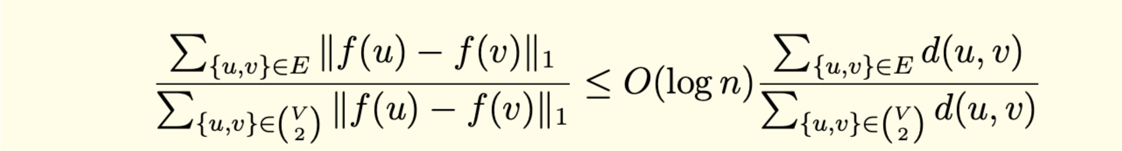
- By the Warm-Up,

$$\frac{\sum_{\{u,v\}\in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u) - f(v)\|_1} \le O(\log n)\, Q(d^*)$$

- By previous part:

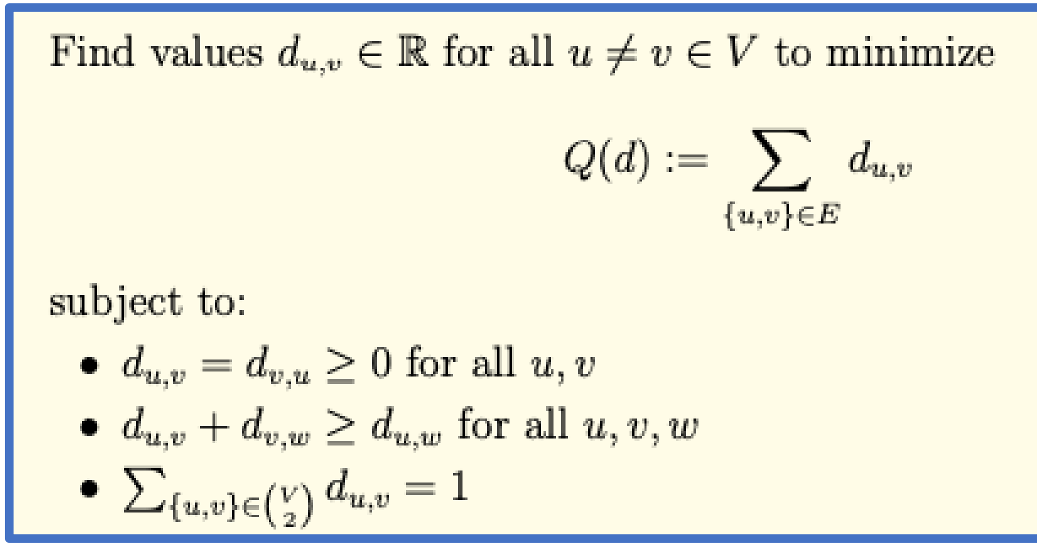
$$\le O(\log n)\phi(G)$$

Warm-Up:



$$\frac{\sum_{\{u,v\}\in E} \|f(u) - f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u) - f(v)\|_1} \le O(\log n) \frac{\sum_{\{u,v\}\in E} d(u,v)}{\sum_{\{u,v\}\in\binom{V}{2}} d(u,v)}$$

LP:



Find values $d_{u,v} \in \mathbb{R}$ for all $u \neq v \in V$ to minimize

$$Q(d) := \sum_{\{u,v\}\in E} d_{u,v}$$

subject to:

- $d_{u,v} = d_{v,u} \ge 0$ for all u, v
- $d_{u,v} + d_{v,w} \ge d_{u,w}$ for all u, v, w
- $\sum_{\{u,v\}\in\binom{V}{2}} d_{u,v} = 1$

Solution: Problem 3

Put it together!

- Find d^* by solving a linear program.
- Find $f: V \to \mathbb{R}^k$ via Bourgain's embedding.

We know that

$$\frac{\sum_{\{u,v\}\in E} \| f(u) - f(v) \|_1}{\sum_{\{u,v\}\in \binom{V}{2}} \| f(u) - f(v) \|_1} \leq O(\log n)\, \phi(G)$$

1. Write $f(x) = \big(f_1(x), f_2(x), \ldots, f_k(x)\big)$. Find $i^* = \mathrm{argmin_i} \frac{\sum_{\{u,v\}\in E} |f_i(u) - f_i(v)|}{\sum_{\{u,v\}\in \binom{V}{2}} |f_i(u) - f_i(v)|}$
2. While $f_{i^*}(x)$ takes on ≥ 3 distinct values $a_1 < a_2 < a_3 < \cdots$
   - Set a_2 to either a_1 or a_3, whichever makes $R(f_{i^*})$ smaller.
3. When f_{i^*} takes only two values, $a < b$, set $f_{i^*} \leftarrow \frac{f_{i^*} - a}{b - a}$
4. Let $S = \{x : f_{i^*}(x) = 1\}$ and celebrate!

Now $\phi(G, S) \leq O(\log n) \phi(G)$!

Recap

- We can find approximately-sparsest cuts efficiently!

- Step 1: $\phi(G) = \inf_{f:V\to\mathbb{R}^k} \frac{\sum_{\{u,v\}\in E} \|f(u)-f(v)\|_1}{\sum_{\{u,v\}\in\binom{V}{2}} \|f(u)-f(v)\|_1}$

- Step 2: Use an LP to to find **some** metric d^* (not necessarily an ℓ_1 metric) so that this quantity is small.
- Step 3: Use Bourgain's embedding to find some f so that $||f(u) - f(v)||_1 \approx d^*(u,v)$, so that this quantity is still pretty small.
- Step 4: Reverse-engineer Step 1 to find an actual cut $S, \bar{S}$.

Next time

- More embeddings! Into ℓ_2!