Recall, Constant (Zero)-Sum Games

- Definition:
  - A constant-sum game is a 2-player normal-form game in which:
    - there exists a constant c such that for each strategy profile $a = (a_1, a_2) \in A$, we have $u_1(a) + u_2(a) = c$.
    - if $c = 0$, this is called a zero-sum game.
  - Generally, competitive or adversarial games
  - For example (Matching Pennies):
    - If penny sides match, row player wins both, otherwise loses both

	Heads	Tails
Heads	(1, -1)	(-1, 1)
Tails	(-1, 1)	(1, -1)

Coordinated-Competitive Games

- Some games have elements of both coordination and competition:
  - Classic example: “Battle of the Sexes”
    - Yes, that’s the standard name, but let’s look beyond gender
  - A couple wants to go to the movies
    - Would prefer going together, but have different preferences
    - Partner 1 prefers Star Wars (SW)
    - Partner 2 prefers Harry Potter (HP)

	SW	HP
SW	(2, 1)	(0, 0)
HP	(0, 0)	(1, 2)

Strategies

- Different strategy forms in games
  - <u>Pure strategy</u>: select single action and play it
  - <u>Mixed strategy</u>: For a set X, let $\Pi(X)$ be set of all probability distributions over X. Set of mixed strategies for player i is $S_i = \Pi(A_i)$
  - <u>Mixed strategy profile</u>: cartesian product of individual mixed strategy sets, $S_1 \times S_2 \times \ldots \times S_n$
  - <u>Expect utility of mixed strategy</u>:

Expected utility for player i under strategy s

$$\boxed{u_i(s)} = \sum_{a \in A} \boxed{u_i(a)} \boxed{\prod_{j=1}^{n} s_j(a_j)}$$

Probability of all players taking action profile *a*

Utility for player i under action profile *a*

Pareto Optimality

- Strategy profile s Pareto dominates strategy profile s' if for all $i \in N$, $u_i(s) \geq u_i(s')$ and there exists some $k \in N$, $u_k(s) > u_k(s')$
  - That is, in Pareto dominated strategy, some player (k) can be made better off while no one else is worse off
- Strategy profile s is Pareto optimal (efficient) if there does not exist another strategy profile $s' \in S$ that dominates s
  - Every game has at least one Pareto optimal strategy which is pure for all players
  - Possible to have multiple Pareto optimal strategies (e.g., in zero-sum game, all strategies Pareto optimal)

Nash Equilibrium

- Player i's <u>best response</u> to strategy profile $s_{-i,}$
  - is a mixed strategy $s_i^* \in S_i$
  - such that $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ for all strategies $s_i \in S_i$
  - Not necessarily unique
- Strategy profile $s = (s_1, \ldots, s_n)$ is <u>Nash equilibrium</u> if, for all players i, s_i is a best response to s_{-i}
  - Nash equilibrium is a stable strategy in that no player would want to change strategy if they knew the strategies the other players were following
  - Theorem (Nash, 1951): Every game with finite players and action profiles has at least one Nash equilibrium

Finding Nash Equilibria

- Generally involved, but we consider special case
- Consider the Battle of the Sexes game:

	SW	HP
SW	(2, 1)	(0, 0)
HP	(0, 0)	(1, 2)

  - Two pure strategy Nash equilibria: (SW, SW), (HP, HP)
  - Mixed strategy Nash equilbrium
    - Players need to have equal utility for either action
    - Let p = probability player 2 picks SW and $U_1(x)$ = player 1 utility
      $U_1(SW) = U_1(HP) \Rightarrow 2p + 0(1 – p) = 0p + 1(1 – p) \Rightarrow p = 1/3$
    - Nash equilibrium: player 2: (SW 1/3 of time & HP 2/3 of time); (similar calculation) player 1: (SW 2/3 of time & HP 1/3 of time)
    - Expected payoff here = (1/3)(2/3)(2) + (2/3)(1/3)(1) = 2/3

Finding Nash Equilibria

- Another example, recall Matching Pennies game:

	Heads (H)	Tails (T)
Heads (H)	(1, -1)	(-1, 1)
Tails (T)	(-1, 1)	(1, -1)

  - No pure strategy Nash equilibrium
  - Mixed strategy Nash equilbrium
    - As before, players need to have equal utility for either action
    - Let p = probability player 2 picks H and $U_1(x)$ = player 1 utility
      $U_1(H) = U_1(T) \Rightarrow \quad 1p + -1(1 - p) = -1p + 1(1 - p)$
      $2p - 1 = -2p + 1 \Rightarrow p = 1/2$
    - Nash equilibrium is 1/2 of time pick H, 1/2 of time pick T
    - Expected payoff here = 0
    - Recall, it is a zero-sum game

Maxmin and Minmax Strategies

- Maxmin strategy for player i is:

$$\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

  - Intuitively, best choice for player i to play if:
    - player i plays first
    - all other players see strategy (but not action)
    - then all other players choose strategies to minimize i's payoff

- Minmax strategy (2-player game) for player i is:

$$\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$$

  - Intuitively, player i is trying to minimize the maximum payoff of other player
    - Theorem (von Neumann, 1928): In a finite, 2-player, zero-sum game, in Nash equilibrium, both players receive payoff equal to both maxmin and minmax value

Dominated Strategies

- Let s_i and s_i' be two strategies of player i and S_{-i} be set of strategy profiles of all remaining players
  - s_i <u>strictly dominates</u> s_i' if for all $s_{-i} \in S_{-i}$, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$
  - s_i <u>weakly dominates</u> s_i' if for all $s_{-i} \in S_{-i}$, $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ and $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for at least one $s_{-i} \in S_{-i}$
  - s_i <u>very weakly dominates</u> s_i' if for all $s_{-i} \in S_{-i}$, $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$
- Consider Prisoner's Dilemma
  - Nash equilibrium (D, D) results from strictly dominant strategy, but is only outcome that is not Pareto optimal

	C	D
C	(-1, -1)	(-4, 0)
D	(0, -4)	(-3, -3)

Dominated Strategy Removal

- Consider the following game:

	E	F	G
A	(3, 1)	(0, 1)	(0, 0)
B	(1, 1)	(1, 1)	(5, 0)
C	(0, 1)	(4, 1)	(0, 0)

  - Note: G is dominated (for player 2) by E or F, so we remove it:

	E	F
A	(3, 1)	(0, 1)
B	(1, 1)	(1, 1)
C	(0, 1)	(4, 1)

  - Now, B is dominated (for player 1) by mixed strategy that chooses A or C with equal probability, so we remove it:

	E	F
A	(3, 1)	(0, 1)
C	(0, 1)	(4, 1)

Signaling

- Signaling is a way for players to communicate
  - Allows players to potentially coordinate strategies
  - Could be open communication (state declaration, corporate policy, college degree, etc.)
  - Could be private channel communication
    - E.g., spectrum auction in 1997
  - Hawk/Dove (aka, Chicken) game (in foreign policy)
    - Some scarce good (utility 6). Hawk takes it from Dove. Two Doves will split it without fight (3 each). Two Hawks split it (after fight that costs each 5) for total payoff of -2 each.

	Hawk	Dove
Hawk	(-2, -2)	(6, 0)
Dove	(0, 6)	(3, 3)