Problem Set 1: Query Complexity of Submodular Optimization
Due: Monday, April 15th

1. **(Warm-up)** Prove that the following functions are submodular:
   (a) Cover function: given a graph $G = (V, E)$, define $f : 2^V \to \mathbb{R}_+$ as:
   $$f(S) = |\{v \in V : v \text{ has a neighbor in } S\}|.$$
   (b) Cut function: given a graph $G = (V, E)$, define $f : 2^V \to \mathbb{R}_+$ as:
   $$f(S) = |E \cap S \times (V \setminus S)|.$$
2. **(Easier-than-medium)** Recall that query complexity is equivalent to decision tree complexity. For the next few questions, we consider submodular function maximization in a "symmetric decision tree complexity" model, where for each level of the decision tree, all the nodes have to query the function at the same set. (In other words, the queries cannot depend on the history of values the protocol already observed.)

   Prove that cut functions can be maximized exactly using $\mathsf{poly}(n)$ symmetric decision tree complexity.
3. For this question, consider the problem of (approximately) maximizing a monotone submodular function subject to a *cardinality constraint* (i.e. $|S| \le k$).
   (a) **(Medium)** Prove that any (possibly randomized) protocol with $\mathsf{poly}(n)$ symmetric decision tree complexity that is restricted to only querying feasible sets (i.e. sets S such that $|S| \le k$) cannot guarantee any constant factor approximation (in expectation).
   (b) **(Medium+)** Generalize your proof from the previous part to symmetric decision tree complexity of arbitrary protocols (i.e. remove the feasible set requirement).
4. **(Open)** What is the optimal approximation factor for unconstrained non-monotone submodular function maximization with $\mathsf{poly}(n)$ symmetric decision tree complexity? (It is known to be in the range $[1/3, 1/2]$. Making progress on either lower or upper bound would be nice!)