1. **(Medium)** The Different Problem.

   In the Different problem, Alice and Bob each receive an input array (A and B respectively) of n bits. Their goal is to determine whether their arrays are different or same. Prove the following:

   (a) The deterministic communication complexity of Different is $\Theta(n)$.

   (b) The non-deterministic communication complexity of Different is $\Theta(\log(n))$.

   (A non-deterministic protocol with communication complexity t takes as additional input a t-bit witness and: (i) if $A \neq B$, there exists a witness such that Alice returns True; and (ii) if $A = B$, no such witness exists.)

   (c) The co-non-deterministic communication complexity of Different is $\Theta(n)$.

   (A co-non-deterministic protocol with communication complexity t takes as additional input a t-bit witness and: (i) if $A = B$, there exists a witness such that Alice returns False; and (ii) if $A \neq B$, no such witness exists.)

   (d) The randomized communication complexity of Different is $\Theta(\log(n))$.

   (A one-sided error randomized protocol with communication complexity t, always terminates after communicating t bits, Alice always returns False if $A = B$, and returns True with probability at least half $A \neq B$. *Private coins* restriction: Alice and Bob may toss random coins, but sending the outcome of those coin tosses still costs communication.)

2. **Communication complexity of correlated equilibrium in two-player games**

   In a *correlated equilibrium* we consider, in addition to our players Alice and Bob, a neutral *correlating device* that sends Alice and Bob recommended actions (α, β) drawn from some joint (correlated) distribution. Each player only sees the recommended action for them. We say that they are in a correlated equilibrium if they are don't have a strategy that is better than following the recommendation. See `https://en.wikipedia.org/wiki/Correlated_equilibrium` for an example.

   Note that correlated equilibria are a generalization of Nash equilibria and are therefore easier to find. They are often strictly easier, i.e. tractable even in settings where Nash equilibria are intractable.

   In this question we consider the communication complexity of approximate correlated equilibrium. There are a couple of closely related ways of formalizing approximate (Nash or correlated) equilibrium. It will be easier to work with the following notion of "well-supported" correlated equilibrium, which means that *every* action in the support is approximately good.

   Definition 1 (ϵ-well supported correlated equilibrium)**.** We say that a (possibly correlated) distribution D over actions (α, β) for Alice and Bob is an *ϵ-well supported correlated equilibrium* if:

   - For every α, α' in Alice's action-set, we have that

     $$\mathsf{E}_{\beta|\alpha}\left[U^A(\alpha,\beta)\right] \geq \mathsf{E}_{\beta|\alpha}\left[U^A(\alpha',\beta)\right] - \epsilon.$$

     (Where the expectation is taken over strategy profiles $(a,b) \sim D$, conditioned on $a = \alpha$.)

   - And the analogous condition for Bob also holds:

     $$\mathsf{E}_{\alpha|\beta}\left[U^B(\beta,\alpha)\right] \geq \mathsf{E}_{\alpha|\beta}\left[U^B(\beta',\alpha)\right] - \epsilon.$$

   (a) **(Guided)** Prove that finding an $1/N^2$-well-supported correlated equilibrium in two-player $N \times N$ games requires $\Omega(N)$ communication.

      i. First, consider the following *zero-sum game*:

Zero-sum Game G_0

- Alice,Bob respectively pick $a, b \in [N]$;
- Bob pays Alice 1 if $a = b$;
- Alice pays Bob 1 if $a \neq b$.

Prove that in every $1/N^2$-well-supported correlated equilibrium, Alice's strategy has full support (i.e. she uses every action with nonzero probability).

ii. Zero-sum games are easy to analyze but they are not good candidates for hard instances since both players know each other's payoff. We modify Game G_0 as follows. Let $S \subset [N]$ of size $|S| = N/2$. We consider the following modified game:

S-Modified Game $G(S)$

- Alice,Bob respectively pick $a, b \in [N]$;
- Alice's payoffs are as in G_0 (earns 1 if $a = b$, loses 1 if $a \neq b$);
- For $b \in S$, Bob's payoffs are also as in G_0 (negative of Alice's payoffs);
- For $b \notin S$, Bob loses 2.

Prove that in every $1/N^2$-well-supported correlated equilibrium, Alice's support is exactly S. Conclude that Bob must send Alice the entire set S.

(b) **(Open)**

What is the communication complexity of ϵ-well-supported correlated equilibrium for arbitrarily small constant $\epsilon > 0$?

(Is it $\text{poly}(N)$? $\text{polylog}(N)$?)