In this homework we will consider the following classes (see also `https://complexityzoo.uwaterloo.ca/Complexity_Zoo`).

PPAD Polynomial time Parity Argument, Directed:
The class of total search problems[1] that are poly-time reducible END-OF-A-LINE:
Given two circuits S, P, each with n input and output bits, such that $S(0^n) \neq 0^n = P(0^n)$, find an input $x \in \{0,1\}^n$ such that $P(S(x)) \neq x$ or $S(P(x)) \neq x \neq 0^n$.

PPP Polynomial time Pigeonhole Principle:
The class of total search problems that are poly-time reducible PIGEON:
Given a circuit H, with n input and output bits, either find an input $x \in \{0,1\}^n$ such that $H(x) = 0^n$, or find $x \neq y \in \{0,1\}^n$ such that $H(x) = H(y)$.

PWPP Polynomial time Weak Pigeonhole Principle:
The class of total search problems that are poly-time reducible WEAK-PIGEON:
Given a circuit H, with n input bits and $n-1$ output bits, find inputs $x \neq y \in \{0,1\}^n$ such that $H(x) = H(y)$. Wlog we assume that H has only XOR, OR, and **1** gates. We also consider the following variants of PWPP:

- $\mathsf{PWPP}_m(n)$ is same as PWPP, but H has only $m(n) < n$ output bits.
- $\mathsf{PWPP}^{\{\oplus,\vee\}}$ is same as PWPP, but H has only XOR and OR gates (no constant **1**).

PLS Polynomial time Local Search:
The class of total search problems that are poly-time reducible LOCAL-SEARCH:
Given circuits S and ϕ, with n input and output bits each, find an input x s.t. $\phi(S(x)) \leq \phi(x)$ (where we interperet the outout of ϕ as an n-bit integer).

[1] A search problem is *total* if it always has a solution.

1. **(Easier-than-medium)** This is the LOCAL MAX-CUT Game: Given an undirected graph G with integer edge weights (in $[0, 2^{|V|}]$), each vertex i is a player with two actions and utility function

$$U^i(\vec{a}) = \sum_{j : a_i \neq a_j} w(i,j).$$

(Here a_i denotes player i's action and $w(i,j)$ denotes the weight on edge (i,j).)

Now consider an asymmetric version of the LOCAL MAX-CUT Game, where vertex n is not a player but rather always deterministically stays on the left side of the cut. Prove that if finding a Nash equilibrium in this asymmetric LOCAL MAX-CUT Game is PPAD-complete, then PPAD $\subseteq$ PLS.

2. Pigeonhole Principle vs Parity Argument

   (a) **(Warmup)** Prove that PPAD $\subseteq$ PPP.

   (b) **(Open-ended)** Come up with an interesting analog of PPAD that is contained in PWPP.

3. **(Easier-than-medium)** Prove that if, for some constant $\varepsilon > 0$, finding an ε-Nash equilibrium (in two-player games) is PPAD-complete (assuming all payoffs are in $[-1,1]$), then END-OF-A-LINE can be solved in time $n^{O(\log(n))}$.

4. **(Guided)** Prove that PWPP$^{\{\oplus,\vee\}}$ = PWPP.

   (a) Prove that PWPP $\subseteq$ PWPP$_{n-2}$.

   (b) Prove that PWPP$_{n-2}$ $\subseteq$ PWPP$^{\{\oplus,\vee\}}$.

   Hint: Given instance $H' : \{0,1\}^n \to \{0,1\}^{n-2}$ of PWPP$_{n-2}$, we want to define an instance $H'' : \{0,1\}^n \to \{0,1\}^{n-1}$ of PWPP$^{\{\oplus,\vee\}}$. Recall that we want a **1** gate, and notice that $\bigvee_{i=1}^{n} x_i \neq 1$ only for one possible input of H''.