CS 354 **Problem Set 4: Unique (and non-unique) Games**
Spring 2019 **Due:** Monday, May 13th

In this homework we will consider the following variant of the Unique Games Conjecture, sometimes called the 2-to-2 Conjecture. It will be more convenient to use the following variant of the LABEL COVER problem, where we may not assign colors to all the vertices:

Definition 1 (LABEL COVER)**.**
Input: A directed graph $G = (V, E)$, an alphabet Σ, and a constraint $\phi_e \subset \Sigma \times \Sigma$ for every $e \in E$.
Output: A set $S \subseteq V$ and an assignment $A : S \to \Sigma$ that satisfies the constraints on $(u, v) \in (S \times S)$. The goal is to (simultaneously) maximize the size of S and the fraction of constraints on $S \times S$ satisfied.

We say that a constraint is 2-to-2 if for every assignment to variable u there are (exactly) two assignments to its neighbor v that would satisfy the constraint.

Definition 2 (2-to-2 constraint)**.**
We say that a constraint $\phi \subset \Sigma \times \Sigma$ is 2-to-2 if $|\Sigma| = 2k$ is even and there exists permutations $\pi_{(u,v)}, \sigma_{(u,v)} : \Sigma \to \Sigma$ such that:

$$\phi_{(u,v)} = \left\{ (\pi(2i-1), \sigma(2i-1)), (\pi(2i-1), \sigma(2i)), (\pi(2i), \sigma(2i-1)), (\pi(2i), \sigma(2i)) \right\}_{i=1}^{k}.$$

Conjecture 1 (2-to-2 Conjecture)**.** *The following holds for every constant $\varepsilon > 0$. Given an instance of* LABEL COVER *with 2-to-2 constraints, it is* NP*-hard to distinguish between:*

Completeness *There exists an assignment to all variables that satisfies every constraint; and*

Soundness *any assignment to ε-fraction of the variables satisfies at most ε-fraction of the constraints between them.*

1. (**Warmup**) Suppose we replace the 2-to-2 constraints in Conjecture 1 with unique (i.e. 1-to-1) constraints. Explain[1] why this would be different from the Unique Games Conjecture and false assuming $\mathsf{P} \neq \mathsf{NP}$.

   Hint: This difference is crucial for proving hardness for 4-Coloring!

[1] Here and thereafter, *explain* means "write a couple of hand-wavy sentences".

2. (**Guided**) Prove that assuming Conjecture 1, the following holds for every constant integer $c > 4$. Given a graph $G' = (V', E')$, it is NP-hard to distinguish between:

   Completeness G' is 4-colorable[2]

   Soundness G' is not c-colorable.

   (a) **Defining** V': Similar to Unique-Games hardness Max-Cut, we will constuct a "cloud" cloud(v) of G'-vertices for every variable $v \in V$. But now the cloud will correspond to $\{0,1,2,3\}^\Sigma$ (instead of $\{0,1\}^\Sigma$ as in Max-Cut). For $x \in \{0,1,2,3\}^\Sigma$ and permutation $\pi : \Sigma \to \Sigma$, we abuse notation and let $\pi(x) \in \{0,1,2,3\}^\Sigma$ denote the vector whose i-th coordinate is $x_{\pi(i)}$.
   Do: Explain why we should use $\{0,1,2,3\}$ instead of binary $\{0,1\}$.

   (b) **Defining** E': For $u, v \in V$ and $x, y \in \{0,1,2,3\}^\Sigma$, we construct an edge $\Big((u,x),(v,y)\Big) \in E'$ iff for every $2i, 2i+1 \in \Sigma$,
   $$\{x_{\pi_{(u,v)}(2i)}, x_{\pi_{(u,v)}(2i+1)}\} \cap \{y_{\sigma_{(u,v)}(2i)}, y_{\sigma_{(u,v)}(2i+1)}\} = \emptyset.$$
   (In words: for every pair $(i,j) \in \Sigma \times \Sigma$ that satisfies constraint $\phi_{(u,v)}$, the colors in the respective coordinates must be different.)
   Do: Prove completeness, i.e. that if the original 2-to-2 instance G is satisfiable, then G' is 4-colorable.

   (c) **Assigning probabilities to the edges:** In the reduction to Max-Cut the probability of an edge corresponded to negating every coordinate of the corresponding Boolean vector, and adding noise. Here, we will modify this strategy in a few ways:
   - The actual graph G' is unweighted (for 4-coloring, weights are meaningless); we already defined the edges above, and will only use the weights in the analysis.
   - Because of the 2-to-2 constraints, we have to update the coordinates in pairs (i.e. they are no longer independent). Let $Q := \{0,1,2,3\}^2$. Below we define a Markov operator $T : Q \to Q$ that acts on each pair of coordinates and determines the weight on the edges.
   - We do not add noise.

   Do: Explain why *not* adding noise is a good idea for 4-coloring.

   (d) **Defining the Operator** T**:** We now want to define the operator $T : Q \to Q$. We have the following desiderata:
   - We want the operator to be *reversible*, i.e. $T^{-1} = T$. In other words, we need to define a weighted, *undirected* graph on Q.
   - To avoid trivialities like the identity operator, we want the operator to mix well. Formally, we require that the *spectral gap* of T is bounded away from 1. Since in our case Q is of constant size, it is enough to show that T (as a graph) is connected and non-bipartite.
   - Finally, we want T to assign positive probability only to the edges in E'. This means that for $(x^1, x^2), (y^1, y^2) \in Q$, we have an edge from (x^1, x^2) to (y^1, y^2) iff $x^1, x^2 \notin \{y^1, y^2\}$. In other words, let $T^{\otimes k} : Q^k \to Q^k$ be the operator that applies T independently to every pair of coordinates in its inputs. Then $\Big((u,x),(v,y)\Big) \in E'$ if and only if $\Pr[T^{\otimes k}(\pi_{(u,v)}x) \leftrightarrow \sigma_{(u,v)}(y)] > 0$.

   Do (optional): Construct such an operator T.

   Hint: Assign probabilities for edge $(x^1, x^2) \leftrightarrow (y^1, y^2)$ according to three different cases: (i) $x^1 = x^2$ and $y^1 = y^2$; (ii) $x^1 = x^2$ and $y^1 = y^2$; and (iii) $x^1 \neq x^2$ and $y^1 \neq y^2$.

[2] t-colorable means that we can color the vertices of the graph with t colors such that the endpoints of every edge have different colors.

(e) **Skipping the analysis of Boolean functions:** Like Unique-Games hardness results (and many other results in TCS), our analysis requires a theorem from the analysis of Boolean functions.

Theorem 1 (Informal)**.** *For every constant $\varepsilon > 0$ there exists a constant $\ell = \ell(\varepsilon) > 0$ such that the following holds. Suppose that a pair of functions $f, g : \underbrace{\{0,1,2,3\}^{\Sigma}}_{=Q^k} \to [0,1]$ satisfies:*

$$\mathbb{E}[f(x)], \mathbb{E}[g(x)] \geq \varepsilon \qquad AND \qquad \mathbb{E}[f(x) \cdot g(T(x))] = 0.$$

Then each of f, g has a small set $S_f, S_g \subset \Sigma$ of influential *coordinates ($|S_f|, |S_g| \leq \ell$), and there is a pair $(2i-1, 2i)$ such that both S_f and S_g contain at least one of the pair (but maybe not the same one).*

(The definition of "influential" is irrelevant for our purposes — we only need that there is *some* such set of variables.)

Do: nothing!

(f) **Soundness — do:** Prove that if the original 2-to-2 instance does not have an assignment for ε-fraction of the variables satisfying at least an $1/\ell(\varepsilon)^2$-fraction of the constraints, then G' is not $1/(2\varepsilon)$-colorable.

Hint: Assume by contradiction that G' has an independent set I of size $|I| = 2\varepsilon|V'|$. For each variable/cloud consider the indicator function of this independent set. Use the set of influential coordinates of the indicator function to assign a value to this variable.