In this problem we will consider the following problems (the second problem is essentially the one you also saw in Problem Set 5).

Definition 1 (ε-Nash problem)**.**

Input: Payoff matrices $A, B \in [0,1]^{n\times n}$.

Output: Any ε-Nash equilibrium.

Definition 2 (ε-best ε-Nash problem)**.**

Input: Payoff matrices $A, B \in [0,1]^{n\times n}$ and parameter $\alpha \in [0,2]$.

Output- Distinguish between: Completeness The game has an exact Nash equilibrium with total payoff α; and

Soundness the maximum total payoff in any ε-Nash equilibrium is $\alpha - \varepsilon$.

(Note that unlike ε-Nash, this is a decision problem.)

1. (**Warm-up**) ETH vs NP-hardness:

   (a) For any constant $\varepsilon > 0$, ε-best ε-Nash can be solved in time $n^{O(\log(n)/\varepsilon^2)}$ [LMM03]. Prove that if ε-best ε-Nash is NP-hard then the Exponential Time Hypothesis (ETH) is false.

   (b) For any constant $\delta > 0$, there is a constant $\varepsilon > 0$ such that the ε vs $1-\varepsilon$ Unique Games[1] problem can be solved in time $2^{O(n^\delta)}$ [ABS15]. Explain why the Unique Games Conjecture (UGC) does not contradict ETH.

[1] I.e. given an instance of Unqiue Label Cover, decide if $\geq (1-\varepsilon)$ or $\leq \varepsilon$ fraction of the constraints can be satisfied.

2. (**Guided**) Let $\varepsilon > 0$ be a sufficiently small. Prove that assuming ETH, the ε-BEST ε-NASH problem is requires $n^{\tilde{\Omega}(\log(n))}$ time.

   You may use the following implication of the PCP Theorem:

   Theorem 1 ([Din07]). *Assuming ETH, given a (non-unique) instance of* LABEL COVER *with a bipartite and regular constraint graph $G_{LC} = (A_{LC}, B_{LC}, E_{LC})$ and constant size alphabet Σ, distinguishing between the following requires $2^{\tilde{\Omega}(n)}$ time:*

   Completeness *The instance is completely satisfied.*

   Soundness *No assignment can satisfy more than 0.9-fraction of the constraints.*

   (a) **Birthday Repetition**

   Consider random partitions of A_{LC} (respectively A_{LC}) into $\ell = \sqrt{n}/100\log(n)$ subsets $A_{LC} = A_1 \cup \cdots \cup A_\ell$ of size $100\sqrt{n}\log(n)$ each. Prove that:

   i. W.h.p. for every $i, j \in [\ell]$, the number of constraints between A_i and B_j is roughly the expected number of constraints, $|E_{LC}|/\ell^2$:

   $$|E_{LC}|/\ell^2/2 < |(A_i \times B_j) \cap E_{LC}| < 2|E_{LC}|/\ell^2.$$

   ii. Prove that in the soundness case, for any assignment to the LABEL COVER variables, for at least 0.05-fraction of the pairs i, j some constraints between A_i and B_j are unsatisfied.

   (b) **The FGLSS graph/game**

   Construct an $(\ell \times \Sigma^{n/\ell}) \times (\ell \times \Sigma^{n/\ell})$ identical interests[2] game with payoffs in $\{0, 1\}$, where Alice's actions correspond to a choice of $i \in [\ell]$, and an assignment to A_i (resp. for Bob with B_j).

   We say that a mixed strategy x for Alice is *ε-well-spread* if it's marginal distribution over choice of $i \in [\ell]$ is ε-close to uniform in total variation distance. In other words, if we look only at the $[\ell]$-component of x (ignoring $\Sigma^{n/\ell}$-component), then it is close to uniform. Define for Bob's strategies analogously.

   Design the payoffs for the game such that the following hold:

   Completeness If the LABEL COVER instance is satisfiable, then there exist 0-well-spread strategies x, y such that $x^\top U y = 1$. (Where U denotes the payoff matrix).

   Soundness If only ≤ 0.9-fraction of the LABEL COVER constraints can be satisfied, then for every ε-well-spread strategies x, y, we have that $x^\top U y \leq 0.95 + O(\varepsilon)$.

   (c) **Forcing a large support**

   Consider the $\ell \times \binom{\ell}{\ell/2}$ zero-sum game where Alice chooses an index $i \in [\ell]$, and Bob chooses a subset $T \subset [\ell]$ of size $|T| = \ell/2$. Bob receives payoff 2 if his subset contains Alice's index, and 0 otherwise. Prove:

   Completeness If Alice's strategy is uniform over all her actions, then her expected payoff (regardless of Bob's strategy) is at least -1.

   Soundness If Alice's strategy is ε-far from uniform, then Bob can guarantee an expected payoff of $1 + \Omega(\varepsilon)$.

   (d) **Putting it all together**

   Complete the proof that assuming ETH, ε-BEST ε-NASH requires $n^{\tilde{\Omega}(\log(n))}$ time.

[2] A game is called *identical interests* if for any choice of actions all players receive the same utility.

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