CS 357: Advanced Topics in Formal Methods

Fall 2019

Lecture 7

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Logical Definitions

Suppose Σ is a signature. A Σ-*formula* is a well-formed formula whose non-logical symbols are contained in Σ.

Let Γ be a set of Σ-formulas. We write $\models_M \Gamma[s]$ to signify that $\models_M \phi[s]$ for every $\phi \in \Gamma$.

If Γ is a set of Σ-formulas and ϕ is a Σ-formula, then Γ *logically implies* ϕ, written $\Gamma \models \phi$, iff for every model M of Σ and every variable assignment s, if $\models_M \Gamma[s]$ then $\models_M \phi[s]$.

We write $\psi \models \phi$ as an abbreviation for $\{\psi\} \models \phi$.

ψ and ϕ are *logically equivalent* (written $\psi \models\!\!=\!\phi$) iff $\psi \models \phi$ and $\phi \models \psi$.

A Σ-formula ϕ is *valid*, written $\models \phi$ iff $\emptyset \models \phi$ (i.e. $\models_M \phi[s]$ for every M and s).

Examples

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2. $Pv_1 \models \forall v_1 \, Pv_1$

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3.	$\forall v_1\, Pv_1 \models \exists v_2\, Pv_2$	*True*
4.	$\exists x\, \forall y\, Qxy \models \forall y\, \exists x\, Qxy$	

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3. $\forall x \, \exists y \, Qxy$ Models (A, Q) where $dom(Q) = A$.

Invariance of Truth Values

Theorem

Suppose s_1 and s_2 are variable assignments over a model M which agree at all variables (if any) which occur free in the *wff* ϕ. Then $\models_M \phi[s_1]$ iff $\models_M \phi[s_2]$.

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Proof

The proof is by induction on well-formed formulas ϕ.

1. If ϕ is an atomic formula, then all variables in ϕ occur free. Thus s_1 and s_2 agree on all variables in ϕ. It follows that $\bar{s}_1(t) = \bar{s}_2(t)$ for each term t in ϕ (technically we should prove this by induction too). The result follows.

2. If ϕ is $(\neg\alpha)$ or $(\alpha \to \beta)$, the result is immediate from the inductive hypothesis.

3. Suppose $\phi = \forall x\, \psi$. The variables free in ϕ are the same as those free in ψ except for x. Thus, for any d in $dom(M)$, $s_1(x|d)$ and $s_2(x|d)$ agree at all variables free in ψ. The result follows from the inductive hypothesis.

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As a corollary of this theorem, we have that for sentences, satisfaction is independent of the variable assignment.

Definability Within a Model

Consider a fixed model M.

If ϕ is a formula whose free variables are among $v_1, \ldots, v_k$, and $a_1, \ldots, a_k$ are elements of M, then we write

$$\models_M \phi[[a_1, \ldots, a_k]]$$

to mean that M satisfies ϕ with some (and hence every) variable assignment s such that $s(v_i) = a_i$.

We can then associate with every such formula ϕ the k-ary relation:

$$\{\langle a_1, \ldots, a_k \rangle \mid \models_M \phi[[a_1, \ldots, a_k]]\}.$$

We say that this is the relation *defined by* ϕ in M.

In general, a k-ary relation on $dom(M)$ is said to be *definable* in M iff there is a formula which defines it there.

Example

Consider the model $(\mathcal{N}, 0, S, +, \times)$.

▶ The ordering relation $\{\langle m, n \rangle \mid m < n\}$ is defined by $\exists\, v_3\, (v_1 + Sv_3 = v_2)$.

▶ For any natural number n, $\{n\}$ is definable. For example, $\{2\}$ is defined by $v_1 = SS0$.

▶ The set of primes is definable in $(\mathcal{N}, 0, S, +, \times)$:

   $1 < v_1 \wedge \forall\, v_2\, \forall\, v_3\, (v_1 = v_2 \times v_3 \rightarrow v_2 = 1 \vee v_3 = 1)$.

   where $<$ can be defined as above, and 1 is an abbreviation for $S0$.

Notice that because there are uncountably many relations on $\mathcal{N}$ and only countably many possible defining formulas, some relations on $\mathcal{N}$ are not definable.

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In first-order logic, it means that every model and variable assignment which makes every formula in Γ true also makes ϕ true.

How do we demonstrate or convince a skeptic of these facts?

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How do we demonstrate or convince a skeptic of these facts?

In propositional logic, a skeptic could use a truth table to check whether $\Gamma \models \phi$.

In first-order logic, no such method exists: we cannot, in general, enumerate all possible models and variable assignments.

Instead, we rely on the notion of a formal mathematical proof.

What is a Proof?

A *proof* is a convincing argument made up of a finite sequence of fixed indisputable steps.

Mathematicians, including the author of the textbook, rely on proofs to convince their audience of the mathematical truth of a proposition.

For the most part, proofs are arguments *about* mathematical objects, but are not mathematical objects themselves. Let's call these "informal" proofs.

In logic, because part of our purpose is to study proofs themselves, we will define proofs that are themselves mathematical objects. We refer to such proofs as *formal proofs* or *deductions*.

Deductions are built from *axioms*, facts which we accept without proof, *assumptions*, facts assumed to be true for the purpose of the deduction, and *rules of inference*, an agreed-upon set of rules for creating new facts from old. Any fact that can be derived in this manner is a (formal) *theorem*.

A particular choice of axioms and rules of inference is often referred to as a *calculus*.

Proofs

Note that because a proof is finite and each step conforms to a pre-determined set of rules, the question of whether a given sequence of steps is a valid proof is decidable.

Thus, a proof is an effective mechanism for convincing a skeptic.

The interplay of formal and informal proofs is a potential source of confusion, especially since we often use informal proofs to prove things about formal proofs! To avoid confusion, remember:

- An informal proof is a convincing argument *about* mathematical objects.
- A formal proof or deduction is a mathematical object *itself*: a sequence of theorems obtained using a specific set of axioms, assumptions, and rules of inference.

Remember that an "informal" proof must still be convincing!

A Deductive Calculus for First-Order Logic

A calculus whose axioms are valid first-order formulas and whose rules of inference preserve first-order validity can be used to derive theorems which are valid first-order formulas.

There are many possible choices for axioms and rules of inference.

We present a calculus for first-order logic with an infinite number of axioms, which we will denote as Λ, and which uses only a single rule of inference, known as *modus ponens*.

This rule states that given formulas α and $\alpha \to \beta$ we may infer β.

Rules of inference are often written in the following format with the given formulas above and the deduced formula below:

$$\frac{\alpha, \alpha \to \beta}{\beta}.$$

A Deductive Calculus

A *deduction of ϕ from* Γ is a sequence $\langle \alpha_0, \ldots, \alpha_n \rangle$ of formulas such that $\alpha_n = \phi$ and for each $i \leq n$ either

- α_i is in $\Gamma \cup \Lambda$, or
- for some j and k less than i, α_i is obtained by modus ponens from α_j and α_k (i.e. $\alpha_k = \alpha_j \rightarrow \alpha_i$).

If such a deduction exists, we say that ϕ is *deducible* from Γ or that ϕ is a *theorem* of Γ, and we write $\Gamma \vdash \phi$.

The set of theorems of Γ is the set generated from $\Gamma \cup \Lambda$ by modus ponens.

A Deductive Calculus

A set of formulas Δ is *closed under modus ponens* iff whenever α and $\alpha \rightarrow \beta$ are in Δ, so is β.

Our definition of deduction gives rise to the following induction principle.

Induction Principle

Suppose that S is a set of wffs that includes $\Gamma \cup \Lambda$ and is closed under modus ponens. Then S contains every theorem of Γ.

Axioms

A *wff* ϕ is a *generalization* of ψ iff for some variables $x_1, \ldots, x_n$, where $n \geq 0$, we have $\phi = \forall x_1 \cdots \forall x_n \, \psi$.

The axioms Λ are made up of all generalizations of *wffs* of the following forms, where x and y are variables and α and β are *wffs*.

1. Tautologies
2. $\forall x \, \alpha \rightarrow \alpha_t^x$, where t is substitutable for x in α;
3. $\forall x \, (\alpha \rightarrow \beta) \rightarrow (\forall x \, \alpha \rightarrow \forall x \, \beta)$;
4. $\alpha \rightarrow \forall x \, \alpha$, where x does not occur free in α;
5. $x = x$;
6. $x = y \rightarrow (\alpha \rightarrow \alpha')$, where α is atomic and α' is obtained from α by replacing x in zero or more places by y.

Note that the axioms depend on the definition of a well-formed formula which requires that a language be specified. The last two items are only included if the language includes equality.

Axioms

Tautologies

Axiom group 1 consists of *tautologies*. These are the *wffs* obtainable from tautologies of propositional logic by replacing each propositional symbol by a *wff* of the first-order language.

For example, consider the propositional tautology

$$(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A).$$

A corresponding axiom is the formula

$$\forall x \, [(\forall y \, \neg Py \rightarrow \neg Px) \rightarrow (Px \rightarrow \neg \forall y \, \neg Py)].$$

There is a more direct way to view the relationship of first-order and propositional logic.

A first-order formula is *prime* if it is atomic or of the form $\forall x \, \alpha$.

First-order formulas correspond exactly to propositional logic formulas in which the set of propositional symbols is taken to be all prime first-order formulas.

Tautologies

By viewing first-order formulas as instances of propositional logic formulas, all propositional notions are also defined for first-order formulas.

Thus, the notions of *tautology*, *tautological consequence*, and *tautological implication* are thus directly applicable to first-order formulas.

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Note that the converse fails. **Theorem**

$\Gamma \vdash \phi$ iff $\Gamma \cup \Lambda$ tautologically implies ϕ.

Proof

$\Rightarrow$: Follows from the fact that modus ponens is propositionally valid.

$\Leftarrow$: By the compactness theorem for propositional logic, there is a finite subset $\Delta = \{\delta_1, \ldots, \delta_m\}$ of $\Gamma \cup \Lambda$ which tautologically implies ϕ. Thus, $\delta_1 \to \cdots \to \delta_m \to \phi$ is a tautology and hence is in Λ. By applying modus ponens m times, we obtain ϕ. $\qquad\square$

Substitution

The second axiom group contains formulas of the form $\forall x\, \alpha \to \alpha_t^x$.

The notation α_t^x denotes the expression obtained from α by replacing x, wherever it occurs free in α, by the term t.

We must also impose the restriction that t be *substitutable* for x in α. Informally, t is substitutable for x in α if no variables from t become bound when the substitution is made. Formally, this is defined as follows.

- ▶ For atomic α, t is substitutable for x in α.
- ▶ t is substitutable for x in $(\neg\alpha)$ iff it is substitutable for x in α, and t is substitutable for x in $(\alpha \to \beta)$ iff it is substitutable for x in both α and β.
- ▶ t is substitutable for x in $\forall y\, \alpha$ iff either
  - ▶ x does not occur free in $\forall y\, \alpha$, or
  - ▶ y does not occur in t and t is substitutable for x in α.

Example

For convenience, we repeat the first three axiom groups here:

1. Tautologies
2. $\forall x\, \alpha \to \alpha_t^x$, where t is substitutable for x in α;
3. $\forall x\,(\alpha \to \beta) \to (\forall x\, \alpha \to \forall x\, \beta)$;

We will give a deduction of: $\vdash \forall x\,(Px \to \exists y\, Py)$.

1.	$\forall x\,[(\forall y\, \neg Py \to \neg Px) \to (Px \to \neg \forall y\, \neg Py)]$	Tautology
2.	$(1) \to [\forall x\,(\forall y\, \neg Py \to \neg Px) \to \forall x\,(Px \to \neg \forall y\, \neg Py)]$	Axiom group 3
3.	$\forall x\,(\forall y\, \neg Py \to \neg Px) \to \forall x\,(Px \to \neg \forall y\, \neg Py)$	MP(1, 2)
4.	$\forall x\,(\forall y\, \neg Py \to \neg Px)$	Axiom group 2
5.	$\forall x\,(Px \to \neg \forall y\, \neg Py)$	MP(4, 3)

Reasoning About Deductions

Recall that our goal is to have a method to convince skeptics that $\Gamma \vdash \phi$.

If we can get the skeptic to believe in our deductive system (we'll tackle that issue shortly when we discuss *soundness*), then all we have to do is give a deduction of ϕ from Γ.

However, as you might imagine from the previous example, a deduction can be tedious and lengthy. For this reason, we introduce a number of shortcuts which can be used to show that $\Gamma \vdash \phi$ without giving an explicit deduction.

In each case we will have to justify that the given shortcut or rule is just as good as giving a deduction.

Generalization Theorem

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Theorem (rule T)

If $\Gamma \vdash \alpha_1, \ldots, \Gamma \vdash \alpha_n$ and $\{\alpha_1, \ldots, \alpha_n\}$ tautologically implies β, then $\Gamma \vdash \beta$.

Deduction Theorem

Theorem

If $\Gamma \cup \{\gamma\} \vdash \phi$ then $\Gamma \vdash (\gamma \to \phi)$.

Proof

$\Gamma \cup \{\gamma\} \vdash \phi$ iff $\Gamma \cup \{\gamma\} \cup \Lambda$ tautologically implies ϕ

                   iff $\Gamma \cup \Lambda$ tautologically implies $\gamma \to \phi$

                   iff $\Gamma \vdash (\gamma \to \phi)$.

$\square$

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If $\Gamma \cup \{\gamma\} \vdash \phi$ then $\Gamma \vdash (\gamma \rightarrow \phi)$.

Proof

$$\begin{array}{lll}
\Gamma \cup \{\gamma\} \vdash \phi & \text{iff} & \Gamma \cup \{\gamma\} \cup \Lambda \text{ tautologically implies } \phi \\
 & \text{iff} & \Gamma \cup \Lambda \text{ tautologically implies } \gamma \rightarrow \phi \\
 & \text{iff} & \Gamma \vdash (\gamma \rightarrow \phi).
\end{array}$$

$\square$

Corollary (contraposition)

$\Gamma \cup \{\phi\} \vdash \neg\psi$ iff $\Gamma \cup \{\psi\} \vdash \neg\phi$.

A set of formulas is *inconsistent* iff for some *wff* β, both β and $\neg\beta$ are theorems of the set.

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Corollary (reductio ad absurdum)

If $\Gamma \cup \{\phi\}$ is inconsistent, then $\Gamma \vdash \neg\phi$.

Example

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which is equivalent to

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And again, by generalization, it suffices to show that

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Example

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inconsistent. But this is easy to see, since

$$
\begin{aligned}
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\forall y \, \phi & \quad \vdash \quad \phi.
\end{aligned}
$$

Generalization on Constants

Theorem

Suppose $\Gamma \vdash \phi$ and c is a constant symbol which does not occur in Γ. Then there is a variable y which does not occur in Γ such that $\Gamma \vdash \forall y\, \phi_y^c$. Furthermore, there is a deduction of $\forall y\, \phi_y^c$ in which c does not occur.

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Proof

Let $\langle \alpha_0, \ldots, \alpha_n \rangle$ be a deduction of ϕ from Γ. Let y be a variable which does not occur in any α_i. We claim that $\langle (\alpha_0)_y^c, \ldots, (\alpha_n)_y^c \rangle$ is a deduction from Γ of ϕ_y^c.

- ▶ Case 1: $\alpha_k \in \Gamma$. Then c does not occur in α_k, so $(\alpha_k)_y^c = \alpha_k$, which is in Γ.
- ▶ Case 2: $\alpha_k \in \Lambda$. A careful examination of the axioms reveals that if $\alpha_k \in \Lambda$, then $(\alpha_k)_y^c$ must also be in Λ.
- ▶ Case 3: α_k is obtained by modus ponens from α_i and α_j. It follows that $(\alpha_k)_y^c$ is obtained by modus ponens from $(\alpha_i)_y^c$ and $(\alpha_j)_y^c$.

It follows from the generalization theorem that there is a deduction of $\forall y\, \phi_y^c$, and it is not hard to see that c does not occur in the deduction. $\qquad \square$

Corollaries

Corollary

If $\Gamma \vdash \phi_c^x$, where c does not occur in Γ or in ϕ, then $\Gamma \vdash \forall x\, \phi$, and there is a deduction in which c does not occur.

Corollary (rule EI)

If $\Gamma \cup \{\phi_c^x\} \vdash \psi$ and c does not occur in any of Γ, ϕ, or ψ, then $\Gamma \cup \{\exists x\, \phi\} \vdash \psi$, and there is a deduction in which c does not occur.

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By contraposition, we have $\Gamma \cup \{\neg\psi\} \vdash \neg\phi_c^x$.

By the above corollary, it follows that $\Gamma \cup \{\neg\psi\} \vdash \forall x\, \neg\phi$.

Applying contraposition again, yields $\Gamma \cup \{\exists x\, \phi\} \vdash \psi$.

$\square$

Alphabetic Variants

An *alphabetic variant* of a formula ϕ is a formula ϕ' obtained by renaming some of the bound variables of ϕ. This is useful when we want to substitute t into ϕ, but t is not substitutable.

Theorem (Existence of Alphabetic Variants)

Let ϕ be a formula, t a term, and x a variable. Then there exists ϕ' such that

1. $\phi \vdash \phi'$ and $\phi' \vdash \phi$; and
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Proof

We construct ϕ' by recursion on ϕ.

- If ϕ is atomic, $\phi' = \phi$
- $(\neg\phi)' = (\neg\phi')$ and $(\phi \to \psi)' = (\phi' \to \psi')$
- $(\forall y\, \phi)' = \forall z\, (\phi')^y_z$, where z does not appear in ϕ' or t or x.

It is not hard to show that the two conditions are satisfied by this definition. $\square$

Equality

Assuming the language includes equality, the following are the standard set of common facts about equality. Their proofs are straightforward.

1. Reflexivity: $\vdash \forall x\, x = x$.
2. Symmetry: $\vdash \forall x \, \forall y \, (x = y \rightarrow y = x)$.
3. Transitivity: $\vdash \forall x \, \forall y \, \forall z \, (x = y \rightarrow y = z \rightarrow x = z)$.
4. Substitutivity in predicates: if P is an n-place predicate symbol, then $\vdash \forall x_1 \ldots \forall x_n \, \forall y_1 \ldots \forall y_n \, (x_1 = y_1 \rightarrow \cdots \rightarrow x_n = y_n \rightarrow Px_1 \ldots x_n \rightarrow Py_1 \ldots y_n)$.
5. Substitutivity in functions: if f is an n-place function symbol, then $\vdash \forall x_1 \ldots \forall x_n \, \forall y_1 \ldots \forall y_n \, (x_1 = y_1 \rightarrow \cdots \rightarrow x_n = y_n \rightarrow fx_1 \ldots x_n = fy_1 \ldots y_n)$.

Syntactic strategies

Often, a strategy for showing the existence of a deduction can be chosen by looking at the syntax of the formula to be deduced.

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► Suppose that ϕ is $\forall x \, \psi$. If x does not occur free in Γ, then it will suffice to show that $\Gamma \vdash \psi$. If x does occur free in Γ, then an alphabetic variant can be constructed where x is renamed to a variable that doesn't occur free in Γ.

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- Suppose ϕ is the negation of another formula.
  - If $\phi = \neg(\psi \rightarrow \theta)$, then it suffices (by rule T) to show that $\Gamma \vdash \psi$ and $\Gamma \vdash \neg\theta$.
  - If $\phi = \neg\neg\psi$, then it suffices to show $\Gamma \vdash \psi$.
  - If $\phi = \neg\forall x\, \psi$, then it suffices to show that $\Gamma \vdash \neg\psi^x_t$, where t is substitutable for x in ψ. Unfortunately, this is not always possible.

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  - If $\phi = \neg\forall x \, \psi$, then it suffices to show that $\Gamma \vdash \neg\psi_t^x$, where t is substitutable for x in ψ. Unfortunately, this is not always possible.

As an example of when the last strategy fails, consider $\neg\forall x \, \neg(Px \to \forall y \, Py)$. It is true that $\vdash \neg\forall x \, \neg(Px \to \forall y \, Py)$, but for every term t, $\nvdash (Pt \to \forall y \, Py)$.

Soundness and Completeness

An important question for any calculus is its relationship to the semantic notion of validity.

If only valid formulas are deducible, then the calculus is said to be *sound*.

If all valid formulas are deducible, then the calculus is said to be *complete*.

The existence of a sound and complete calculus for first-order logic is an important result which demonstrates that it is a reasonable model of mathematical thinking.

Soundness

Soundness Theorem

If $\Gamma \vdash \phi$, then $\Gamma \models \phi$.

Proof

The idea of the proof is that the logical axioms are logically valid, and that modus ponens preserves logical implications.

We first assume the axioms are valid and prove by induction that any formula ϕ deducible from Γ is logically implied by Γ.

- Case 1: if ϕ is a logical axiom, then by our assumption, $\models \phi$, and thus $\Gamma \models \phi$.
- Case 2: If $\phi \in \Gamma$, then clearly $\Gamma \models \phi$.
- Case 3: If ϕ is obtained by modus ponens from ψ and $\psi \to \phi$, then by the inductive hypothesis, $\Gamma \models \psi$ and $\Gamma \models (\psi \to \phi)$. It follows by the definition of $\models$ for $\to$ that $\Gamma \models \phi$.

Soundness

It remains to show that the axioms are valid. We will consider only Axiom Group 2 (the others are straightforward). First a lemma.

Substitution Lemma

If the term t is substitutable for the variable x in the *wff* ϕ, then for any model M and variable assignment s, $\models_M \phi_t^x[s]$ iff $\models_M \phi[s(x|\overline{s}(t))]$.

This lemma states that if we replace a variable x with a term t, the semantics are the same as if the variable assignment is modified so that x takes on the same value as the term t.

The proof is by induction on ϕ.

Now, consider Axiom Group 2: $\forall x\, \alpha \to \alpha_t^x$, where t is substitutable for x in α.

Assume $\models_M \forall x\, \alpha[s]$. We must show that $\models_M \alpha_t^x[s]$. We know from $\models_M \forall x\, \alpha[s]$ that for any $d \in dom(M)$, $\models_M \alpha[s(x|d)]$. In particular, if we let $d = \overline{s}(t)$, then we have $\models_M \alpha[s(x|\overline{s}(t))]$. But by the substitution lemma, this implies that $\models_M \alpha_t^x[s]$.

$\square$

Soundness Corollaries

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If $\vdash (\phi \leftrightarrow \psi)$, then ϕ and ψ are logically equivalent.

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Recall that a set Γ is *consistent* iff there is no formula ϕ such that both $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$.

Define Γ to be *satisfiable* iff there is some model M and variable assignment s such that $\models_M \Gamma[s]$.

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Corollary

If Γ is satisfiable, then Γ is consistent.

Completeness

Completeness Theorem (Gödel, 1930)

If $\Gamma \models \phi$, then $\Gamma \vdash \phi$.

This is equivalent to the following statement: any consistent set of formulas is satisfiable.