Basic Graph Algorithms

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Outline

Graphs

- An abstract way of representing connectivity using nodes (also called vertices) and edges
- We will label the nodes from 1 to n
- m edges connect some pairs of nodes
  - Edges can be either one-directional (directed) or bidirectional
- Nodes and edges can have some auxiliary information

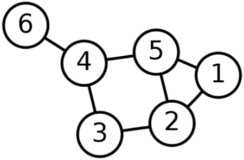


Figure from Wikipedia

Why Study Graphs?

- Lots of problems formulated and solved in terms of graphs
  - Shortest path problems
  - Network flow problems
  - Matching problems
  - 2-SAT problem
  - Graph coloring problem
  - Traveling Salesman Problem (TSP): *still unsolved!*
  - and many more...

Outline

Storing Graphs

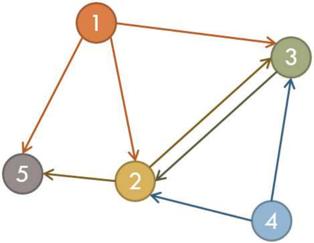
- Need to store both the set of nodes V and the set of edges E
  - Nodes can be stored in an array
  - Edges must be stored in some other way
- Want to support operations such as:
  - Retrieving all edges incident to a particular node
  - Testing if given two nodes are directly connected
- Use either adjacency matrix or adjacency list to store the edges

Adjacency Matrix

- An easy way to store connectivity information
  - Checking if two nodes are directly connected: $O(1)$ time
- Make an $n \times n$ matrix A
  - $a_{ij} = 1$ if there is an edge from i to j
  - $a_{ij} = 0$ otherwise
- Uses $\Theta(n^2)$ memory
  - Only use when n is less than a few thousands,
  - *and* when the graph is dense

Adjacency List

- Each node has a list of outgoing edges from it
  - Easy to iterate over edges incident to a certain node
  - The lists have variable lengths
  - Space usage: $\Theta(n + m)$

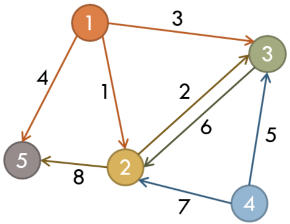


From	To		
1	2	3	5
2	3	5	
3	2		
4	2	5	
5			

Implementing Adjacency List

- Solution 1. Using linked lists
  - Too much memory/time overhead
  - Using dynamic allocated memory or pointers is bad
- Solution 2. Using an array of `vector`s
  - Easier to code, no bad memory issues
  - But very slow
- Solution 3. Using arrays (!)
  - Assuming the total number of edges is known
  - Very fast and memory-efficient

Implementation Using Arrays



ID	To	Next Edge ID
1	2	-
2	3	-
3	3	1
4	5	3
5	3	-
6	2	-
7	2	5
8	5	2

From	1	2	3	4	5
Last Edge ID	4	8	6	7	-

Implementation Using Arrays

- Have two arrays `E` of size m and `LE` of size n
  - `E` contains the edges
  - `LE` contains the starting pointers of the edge lists
- Initialize `LE[i] = -1` for all `i`
  - `LE[i] = 0` is also fine if the arrays are 1-indexed
- Inserting a new edge from `u` to `v` with ID `k`

```
E[k].to = v
E[k].nextID = LE[u]
LE[u] = k
```

Implementation Using Arrays

- Iterating over all edges starting at u:

```
for(ID = LE[u]; ID != -1; ID = E[ID].nextID)
  // E[ID] is an edge starting from u
```

- Once built, it's hard to modify the edges
  - The graph better be static!
  - But adding more edges is easy

Outline

Tree

- A connected acyclic graph
- Most important type of special graphs
  - Many problems are easier to solve on trees
- Alternate equivalent definitions:
  - A connected graph with $n - 1$ edges
  - An acyclic graph with $n - 1$ edges
  - There is exactly one path between every pair of nodes
  - An acyclic graph but adding any edge results in a cycle
  - A connected graph but removing any edge disconnects it

Other Special Graphs

- Directed Acyclic Graph (DAG): the name says what it is
  - Equivalent to a partial ordering of nodes
- Bipartite Graph: Nodes can be separated into two groups S and T such that edges exist between S and T only (no edges within S or within T)

Outline

Graph Traversal

- The most basic graph algorithm that visits nodes of a graph in certain order
- Used as a subroutine in many other algorithms

- We will cover two algorithms
  - Depth-First Search (DFS): uses recursion (stack)
  - Breadth-First Search (BFS): uses queue

Depth-First Search

DFS(v): visits all the nodes reachable from v in depth-first order

- Mark v as visited
- For each edge $v \to u$:
  - If u is not visited, call DFS(u)

- Use non-recursive version if recursion depth is too big (over a few thousands)
  - Replace recursive calls with a stack

Breadth-First Search

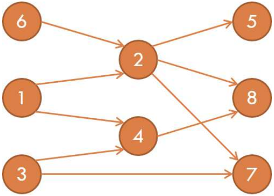
BFS(v): visits all the nodes reachable from v in breadth-first order

- Initialize a queue Q
- Mark v as visited and push it to Q
- While Q is not empty:
  - Take the front element of Q and call it w
  - For each edge $w \to u$:
    - If u is not visited, mark it as visited and push it to Q

Outline

Topological Sort

- Input: a DAG $G = (V, E)$
- Output: an ordering of nodes such that for each edge $u \to v$, u comes before v
- There can be many answers
  - *e.g.*, both $\{6, 1, 3, 2, 7, 4, 5, 8\}$ and $\{1, 6, 2, 3, 4, 5, 7, 8\}$ are valid orderings for the graph below



Topological Sort

- Any node without an incoming edge can be the first element
- After deciding the first node, remove outgoing edges from it
- Repeat!

- Time complexity: $O(n^2 + m)$
  - Too slow...

Topological Sort (faster version)

- Precompute the number of incoming edges $\deg(v)$ for each node v
- Put all nodes v with $\deg(v) = 0$ into a queue Q
- Repeat until Q becomes empty:
  - Take v from Q
  - For each edge $v \to u$:
    - Decrement $\deg(u)$ (essentially removing the edge $v \to u$)
    - If $\deg(u) = 0$, push u to Q
- Time complexity: $\Theta(n + m)$

Outline

Eulerian Circuit

- Given an undirected graph G
- Want to find a sequence of nodes that visits every edge exactly once and comes back to the starting point

- Eulerian circuits exist if and only if
  - G is connected
  - *and* each node has an even degree

Constructive Proof of Existence

- Pick any node in G and walk randomly without using the same edge more than once
- Each node is of even degree, so when you enter a node, there will be an unused edge you exit through
  - Except at the starting point, at which you can get stuck
- When you get stuck, what you have is a cycle
  - Remove the cycle and repeat the process in each connected component
  - Glue the cycles together to finish!

Related Problems

- Eulerian path: exists if and only if the graph is connected and the number of nodes with odd degree is 0 or 2.

- Hamiltonian path/cycle: a path/cycle that visits every *node* in the graph exactly once. Looks similar but very hard (still unsolved)!

Outline

Minimum Spanning Tree (MST)

- Given an undirected weighted graph $G = (V, E)$
- Want to find a subset of E with the minimum total weight that connects all the nodes into a tree

- We will cover two algorithms:
  - Kruskal's algorithm
  - Prim's algorithm

Kruskal's Algorithm

- Main idea: the edge $e^\star$ with the smallest weight has to be in the MST
- Simple proof:
  - Assume not. Take the MST T that doesn't contain $e^\star$.
  - Add $e^\star$ to T, which results in a cycle.
  - Remove the edge with the highest weight from the cycle.
    - The removed edge cannot be $e^\star$ since it has the smallest weight.
  - Now we have a better spanning tree than T
  - Contradiction!

Kruskal's Algorithm

- Another main idea: after an edge is chosen, the two nodes at the ends can be merged and considered as a single node (supernode)
- Pseudocode:
  - Sort the edges in increasing order of weight
  - Repeat until there is one supernode left:
    - Take the minimum weight edge $e^\star$
    - If $e^\star$ connects two different supernodes, then connect them and merge the supernodes (use union-find)
  - Otherwise, ignore $e^\star$ and try the next edge

Prim's Algorithm

- Main idea:
  - Maintain a set S that starts out with a single node s
  - Find the smallest weighted edge $e^\star = (u, v)$ that connects $u \in S$ and $v \notin S$
  - Add $e^\star$ to the MST, add v to S
  - Repeat until $S = V$

- Differs from Kruskal's in that we grow a single supernode S instead of growing multiple ones at the same time

Prim's Algorithm Pseudocode

- Initialize $S := \{s\}$, $D_v := \mathrm{cost}(s, v)$ for every v
  - If there is no edge between s and v, $\mathrm{cost}(s, v) = \infty$
- Repeat until $S = V$:
  - Find $v \notin S$ with smallest D_v
    - Use a priority queue or a simple linear search
  - Add v to S, add D_v to the total weight of the MST
  - For each edge (v, w):
    - Update $D_w := \min(D_w, \mathrm{cost}(v, w))$
- Can be modified to compute the actual MST along with the total weight

Kruskal's vs Prim's

- Kruskal's Algorithm
  - Takes $O(m \log m)$ time
  - Pretty easy to code
  - Generally slower than Prim's
- Prim's Algorithm
  - Time complexity depends on the implementation:
    - Can be $O(n^2 + m)$, $O(m \log n)$, or $O(m + n \log n)$
  - A bit trickier to code
  - Generally faster than Kruskal's

Outline

Strongly Connected Components (SCC)

- Given a *directed* graph $G = (V, E)$
- A graph is *strongly connected* if all nodes are reachable from every single node in V
- Strongly connected components of G are maximal strongly connected subgraphs of G
- The graph below has 3 SCCs: $\{a, b, e\}$, $\{c, d, h\}$, $\{f, g\}$

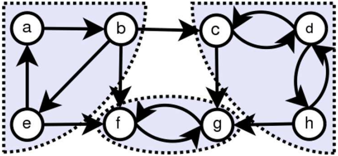


Figure from Wikipedia

Kosaraju's Algorithm

- Initialize counter $c := 0$
- While not all nodes are labeled:
  - Choose an arbitrary unlabeled node v
  - Start DFS from v
    - Check the current node x as visited
    - Recurse on all unvisited neighbors
    - After the DFS calls are finished, increment c and set the label of x as c
- Reverse the direction of all the edges
- For node v with label $n, n-1, \ldots, 1$:
  - Find all reachable nodes from v and group them as an SCC

Kosaraju's Algorithm

- We won't prove why this works
- Two graph traversals are performed
  - Running time: $\Theta(n + m)$

- Other SCC algorithms exist but this one is particularly easy to code
  - and asymptotically optimal