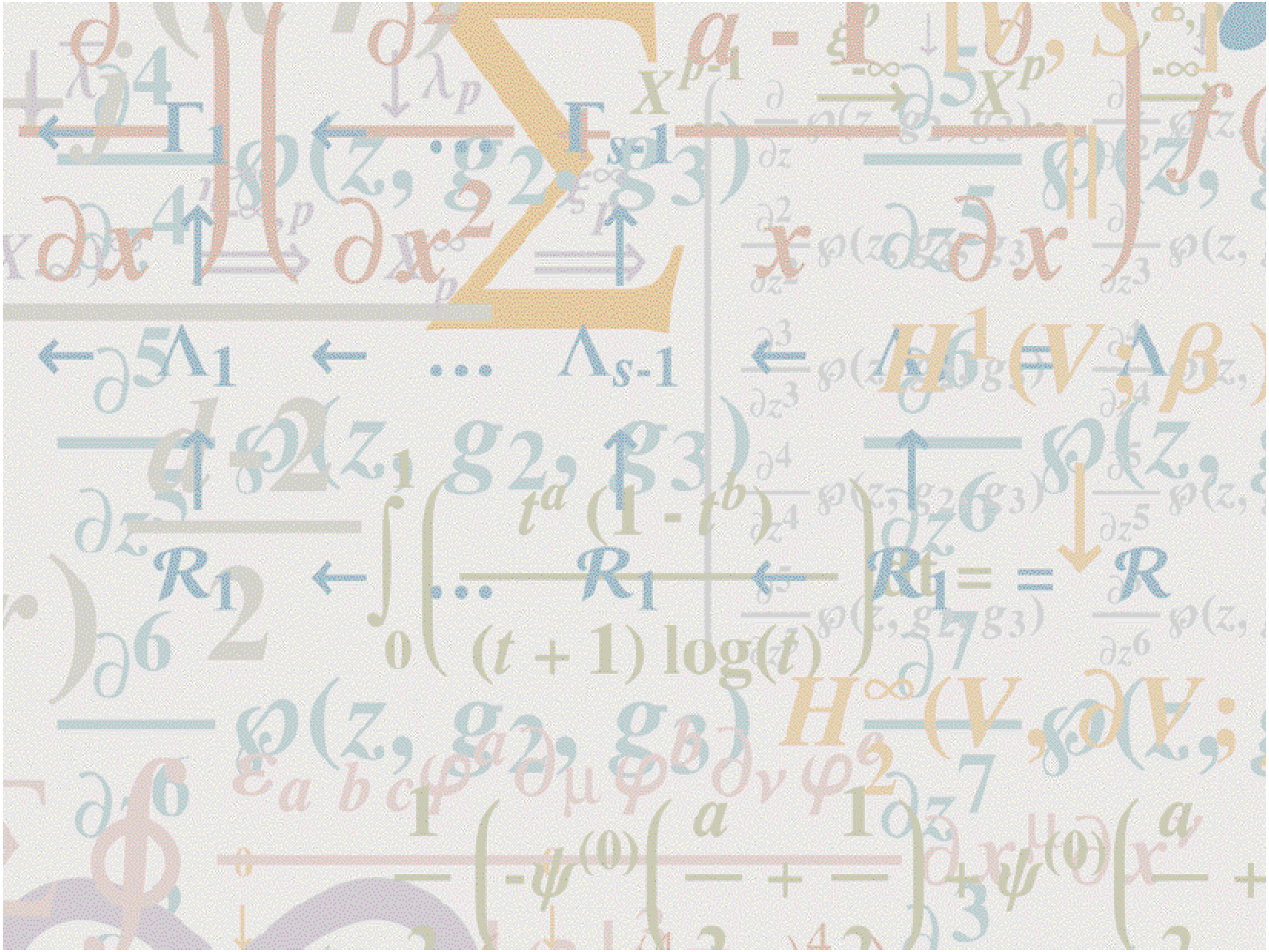
How did human beings acquire the ability to do mathematics?



"THE MATH GENE"

Meaning

An innate facility for mathematical thought, inherited at birth.

Mathematical ability:

My questions

Mathematical ability:
My questions

- **How did the human brain acquire this ability?**

Mathematical ability:
My questions

- **How did the human brain acquire this ability?**
- **When did it acquire the ability?**

Mathematical ability:
My questions

- **How did the human brain acquire this ability?**
- **When did it acquire the ability?**
- **What evolutionary advantage did the ability confer?**

My approach

My approach

- **Split mathematical ability into simpler mental capacities.**

My approach

- Split mathematical ability into simpler mental capacities.
- Ask what led to the human brain acquiring each capacity.

My approach

- **Split mathematical ability into simpler mental capacities.**
- **Ask what led to the human brain acquiring each capacity.**
- **Ask when each capacity was acquired.**

My approach

- **Split mathematical ability into simpler mental capacities.**
- **Ask what led to the human brain acquiring each capacity.**
- **Ask when each capacity was acquired.**
- **Ask how and when they came together to give mathematical thinking.**

What are the ingredients for a mathematical mind?

What are the ingredients for a mathematical mind?

1. Number sense

What are the ingredients for a mathematical mind?

1. Number sense
2. Numerical ability

What are the ingredients for a mathematical mind?

1. Number sense
2. Numerical ability
3. Spatial reasoning ability

What are the ingredients for a mathematical mind?

1. Number sense
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3. Spatial reasoning ability
4. A sense of cause and effect

What are the ingredients for a mathematical mind?

1. Number sense
2. Numerical ability
3. Spatial reasoning ability
4. A sense of cause and effect
5. The ability to construct and follow a causal chain of facts or events

What are the ingredients for a mathematical mind?

What are the ingredients for a mathematical mind?

6. Algorithmic ability

What are the ingredients for a mathematical mind?

6. Algorithmic ability
7. The ability to handle abstraction

What are the ingredients for a mathematical mind?

6. Algorithmic ability
7. The ability to handle abstraction
8. Logical reasoning ability

What are the ingredients for a mathematical mind?

6. Algorithmic ability
7. The ability to handle abstraction
8. Logical reasoning ability
9. Relational reasoning ability

The fundamental questions:

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- When did each of these nine mental capacities evolve?

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- When did each of these nine mental capacities evolve?
- What survival value did it offer?

The fundamental questions:

- When did each of these nine mental capacities evolve?
- What survival value did it offer?
- What brought them together to give the ability for mathematical thinking?

Number sense

Number sense

- *Sense of the size of a collection*

Number sense

- *Sense of the size of a collection*
- *Does not require numbers*

Number sense

- *Sense of the size of a collection*
- *Does not require numbers*
- *Possessed by many creatures*

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- *Possessed by many creatures*
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- *Karen Wynn (MIT, 1992)*

Numerical ability

Numerical ability

- *Requires numbers*

Numerical ability

- Requires numbers
- Only humans have it (apart from limited forms)

Numerical ability

- *Requires numbers*
- *Only humans have it (apart from limited forms)*
- *Depends on language*

Numerical ability

- *Requires numbers*
- *Only humans have it (apart from limited forms)*
- *Depends on language*
- *Stanislas Dehaene (MIT, 1997)*

Spatial reasoning ability

Spatial reasoning ability

- *Any creature that moves needs this ability*

Relational reasoning ability

Relational reasoning ability

- *We use this to understand human relationships*

Relational reasoning ability

- *We use this to understand human relationships*
- *It is the way evolution has equipped us to live by cooperation*

The ability to handle abstraction

The ability to handle abstraction

- *This is the key capacity to do mathematics*

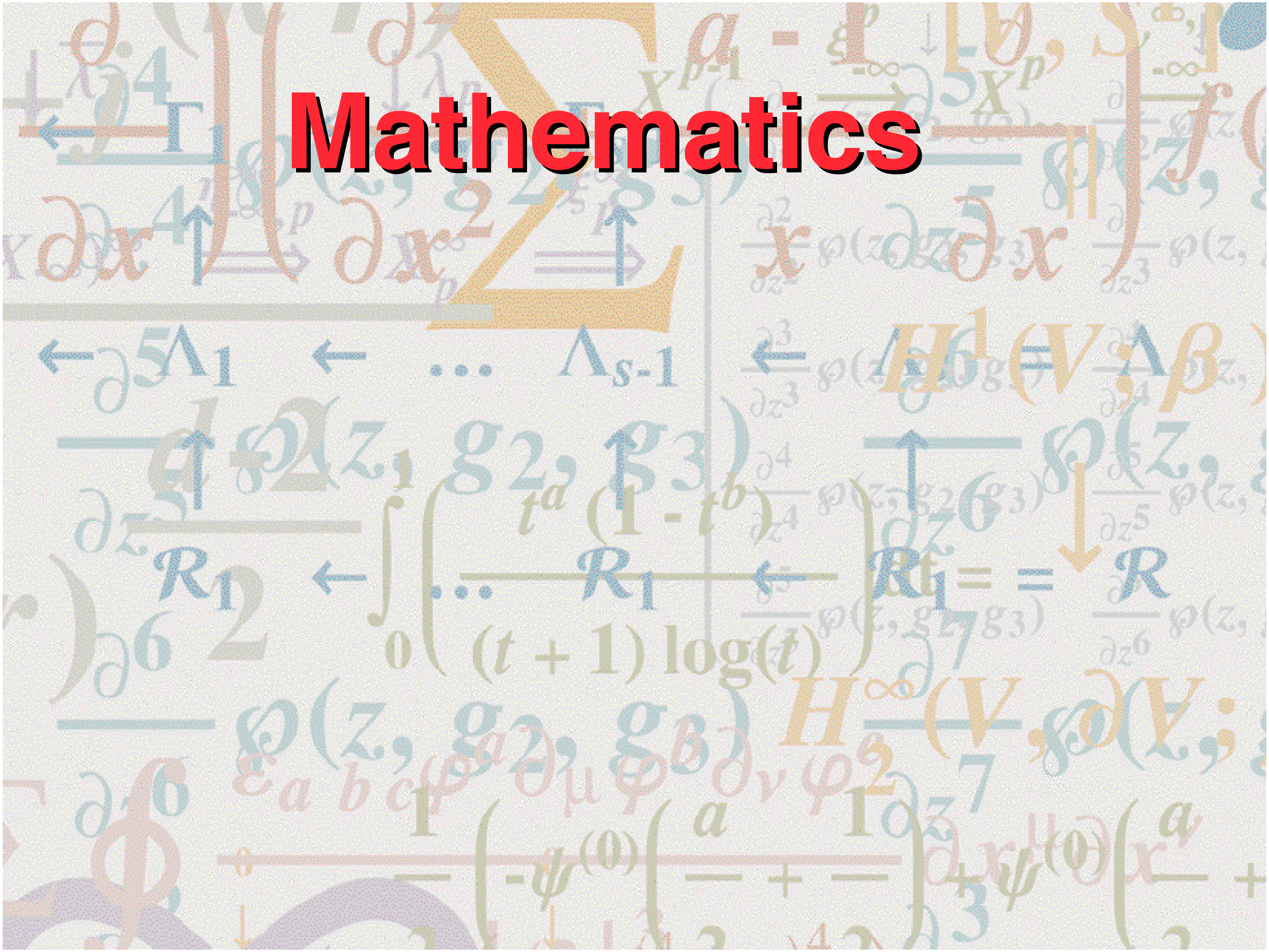
The ability to handle abstraction

- *This is the key capacity to do mathematics*
- *It is equivalent to having the capacity for language*

Devlin's hypothesis

The crucial step in the development of mathematical ability was handling increased abstraction — not a greater complexity of thought processes.

Mathematics



Mathematics

- To understand mathematics, you should view it as a fictional analogue of parts of the real world, both the physical world and the social world.

Mathematics

- To understand mathematics, you should view it as a fictional analogue of parts of the real world, both the physical world and the social world.
- We take mental capacities developed to negotiate the physical and the social world and apply them to reason about a fictional, abstract world our mind creates.

The secret to being able to do mathematics

The secret to being able to do mathematics

A mathematician is someone who views mathematics as a soap opera.

The mathematical soap opera

The mathematical soap opera

The “characters” in the mathematical soap opera are not people but mathematical objects — numbers, geometric figures, vectors, topological spaces, analytic functions, etc.

The mathematical soap opera

The facts and relationships of interest are not births, deaths, marriages, love affairs, and business relationships, but mathematical facts and relationships about mathematical objects.

Mathematical facts and relationships:

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- Are objects A and B equal?

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Mathematical facts and relationships:

- Are objects A and B equal?
- What is the relationship between objects X and Y?
- Find an object X having property P.
- Do all objects of type T have property P?
- **How many objects of type Z are there?**

The mathematical brain

The mathematical brain

- Mathematicians don’t have different brains.

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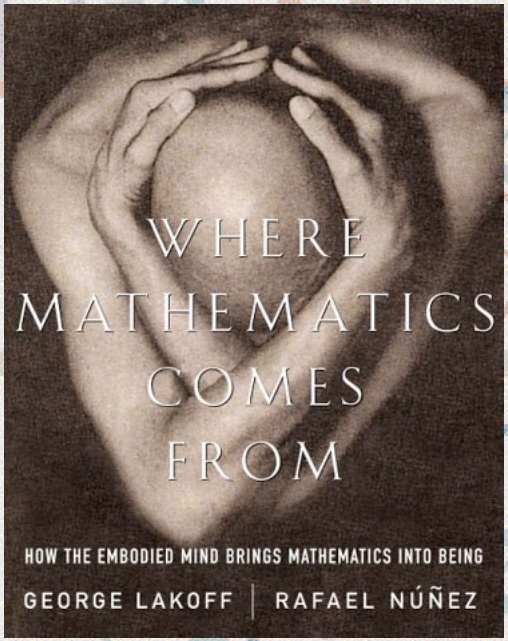
- They have found a way to use a standard-issue brain in a slightly different way.

The mathematical brain

- Mathematicians don't have different brains.
- They have found a way to use a standard-issue brain in a slightly different way.
- Mathematicians think about mathematical objects and the mathematical relationships between them using the same mental faculties that everyone uses to think about physical space and about other people.

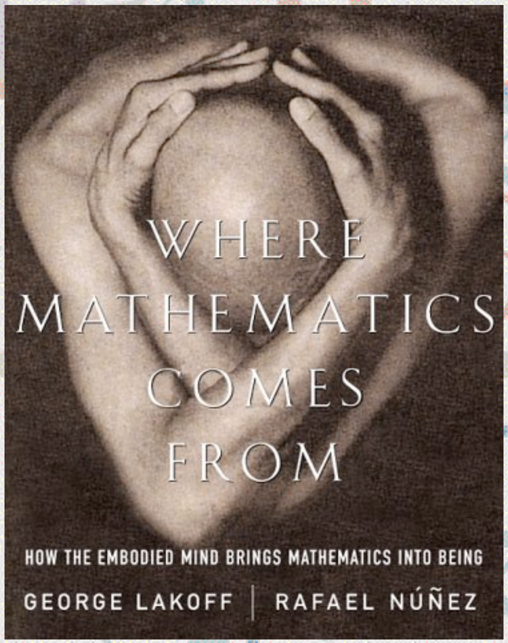
Exactly how does the everyday brain do mathematics?



Lakoff & Nuñez

- **Starting point:** the human brain evolved to help us negotiate the physical environment
- **Assumption:** mathematical thinking is derivative on everyday thinking

- **Method:** Show how each mathematical concept or process can be built up from everyday mental concepts or capacities by a chain of ever more abstract metaphors



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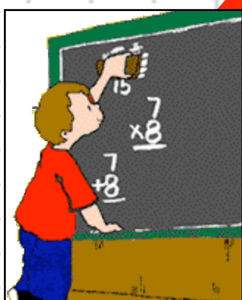
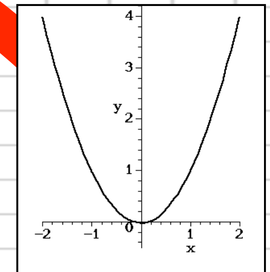
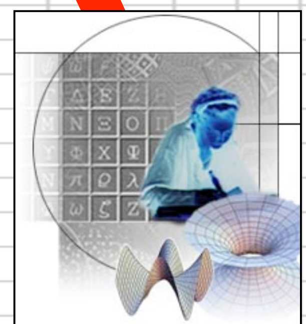
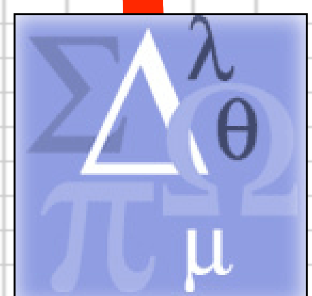
Lakoff & Nuñez



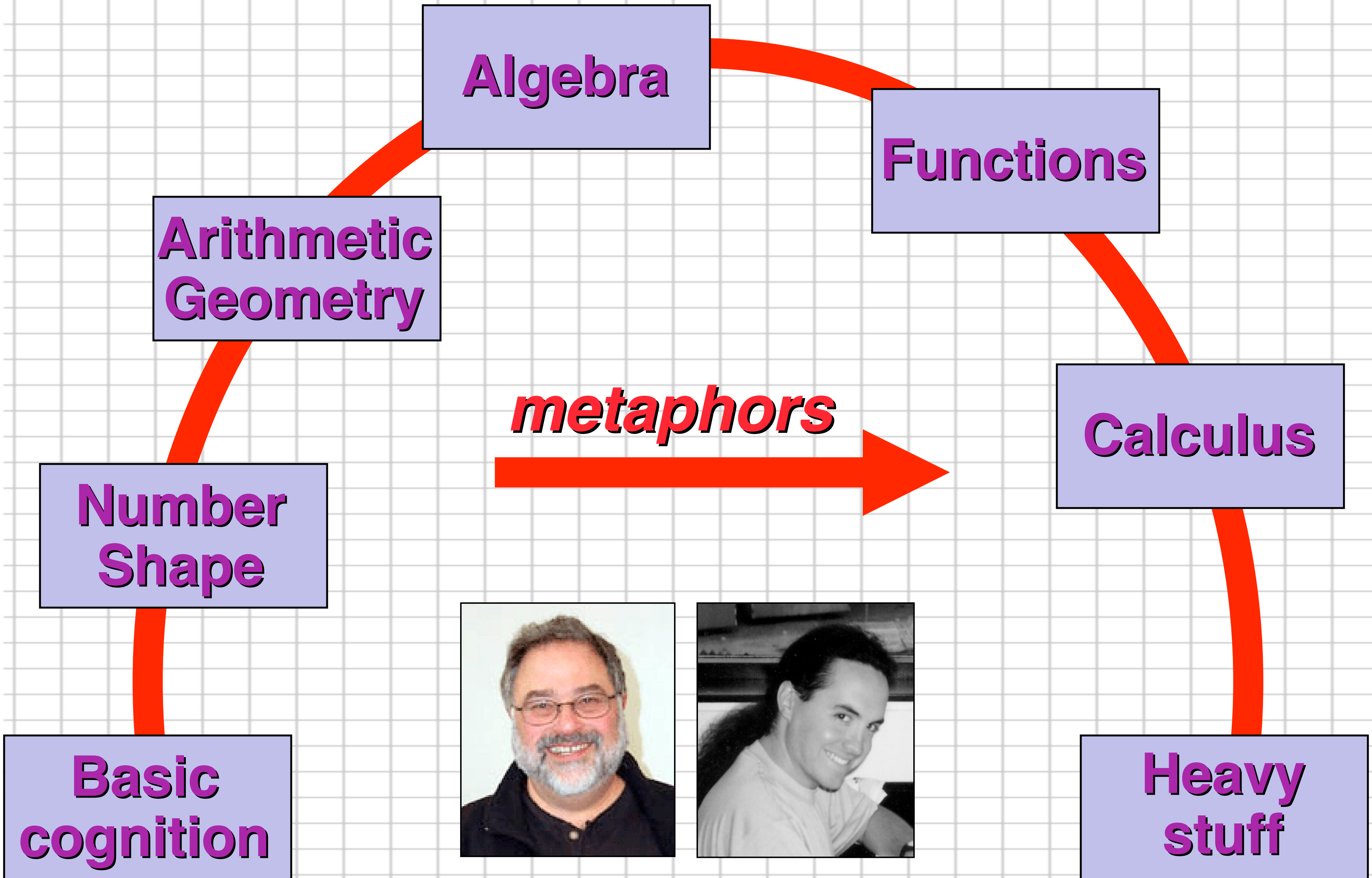




metaphors



Lakoff & Nuñez



Lakoff & Nuñez

Basic cognition

Number
Shape

Arithmetic
Geometry

Algebra

Functions

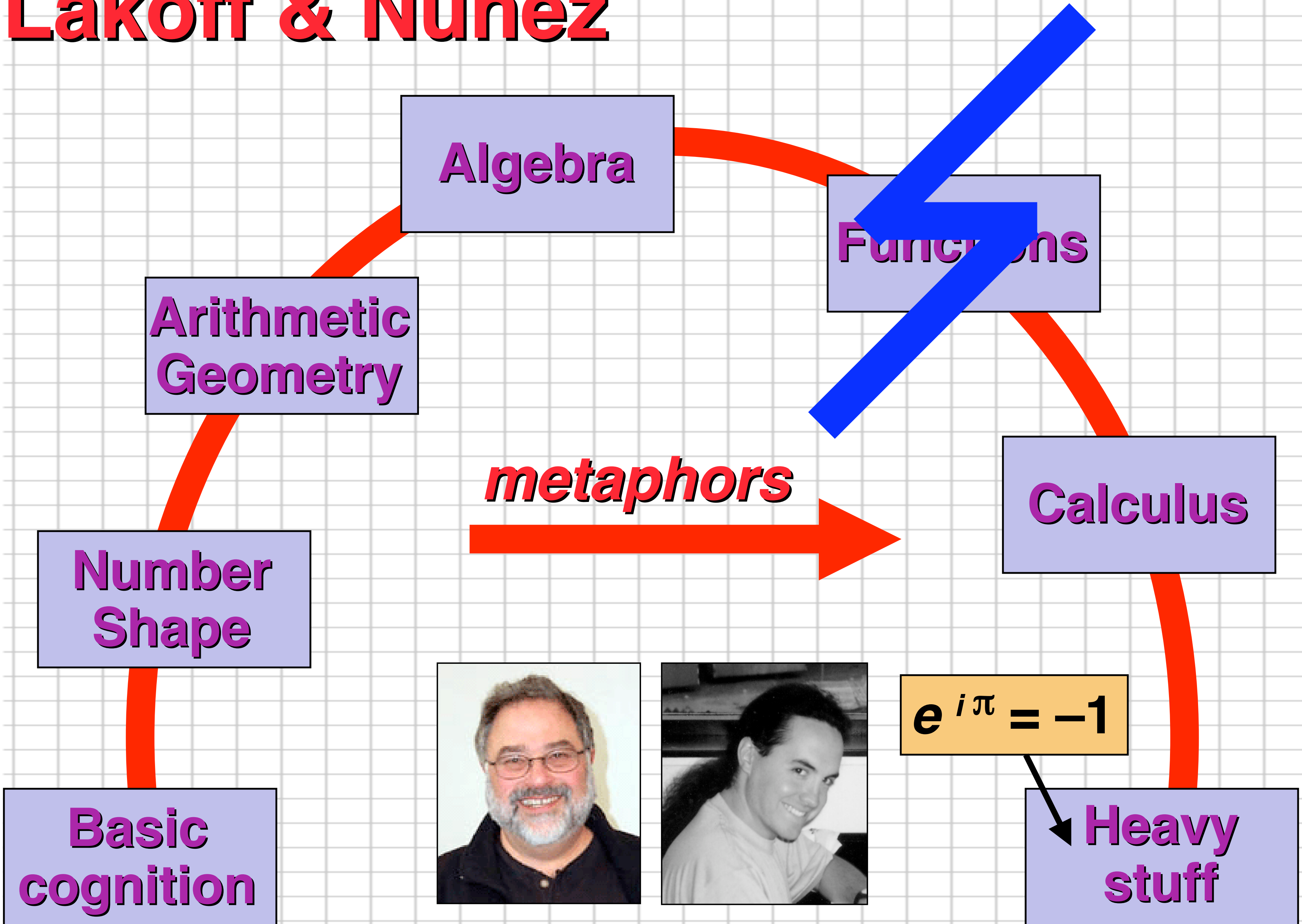
Calculus

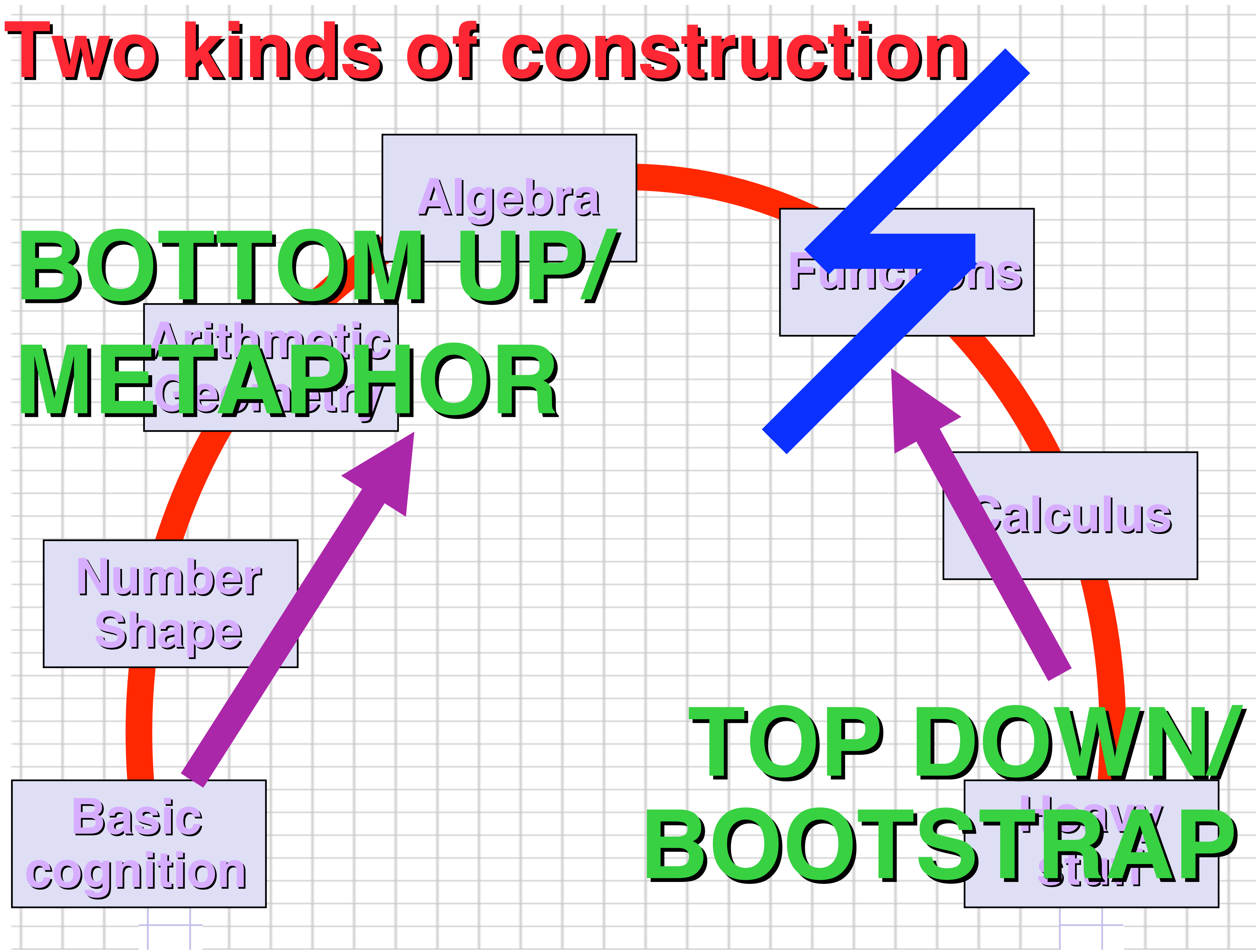
Heavy stuff

metaphors

$e^{i\pi} = -1$

Lakoff & Nuñez




Two kinds of construction
Algebra
BOTTOM UP/
METAPHOR
Arithmetic
Number
Shape
Basic
cognition
Calculus
TOP DOWN/
BOOTSTRAP

Uri Leron

NATURAL MATHEMATICS

Algebra

Functions

Arithmetic
Geometry

Number
Shape

Basic cognition

Calculus

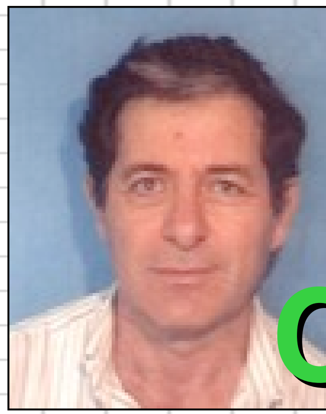
FORMAL MATHEMATICS

Heavy stuff

Uri Leron

FORMALIZED

COMMON SENSE



OFTEN

COUNTER TO

COMMON SENSE

Basic cognition

Heavy stuff

Natural mathematics

- Start with known concepts or processes
- Construct new ones understood in terms of the familiar ones
- With repeated practice, you eventually become familiar with the new ones, which become part of what you know
- Repeat the process

Natural mathematics	Formal mathematics
■ Start with known concepts or processes ■ Construct new ones understood in terms of the familiar ones ■ With repeated practice, you eventually become familiar with the new ones, which become part of what you know ■ Repeat the process	■ Learn the definition of a new concept or the rules of a new process in a purely formal way ■ "Play the game" until you become fluent in the formal manipulations ■ With repeated practice, the new concept or process will eventually acquire meaning

Natural mathematics	Formal mathematics
Meaning → Rules	Rules → Meaning

Natural mathematics

- Start with known concepts … processes …
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Meaning

Rules

Formal mathematics

- Learn the definition of a new … ew … purely formal way
- "Play the g… until you become flu… the formal manipulati…
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Rules

Meaning

Natural mathematics

- Example 1: Take a number and double it.
- Example 2: Take a square and rotate it through 45^o.
- New concept: A function is a process that takes a given object of a certain kind and transforms it.

Natural mathematics	Formal mathematics

Natural mathematics

- Example 1: Take a number and double it.
- Example 2: Take a square and rotate it through 45^o.
- New concept: A <u>function</u> is a process that takes a given object of a certain kind and transforms it.

Formal mathematics

- New concept: A <u>function</u> is a relation between sets *A* and *B* that to each object *a* in *A* assigns a unique object *b* in *B*.
- Example 1: Natural number doubling associates to each natural number *N* the natural number 2*N*.
- Example 2: 45^o rotation on all squares associates with any given square another square at 45^o to the first one.

Example of a function

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- The functions FIRST and REST act on sequences of numbers. FIRST strips off the first member of the sequence; REST gives the remainder of the sequence after the first number has been removed.

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- What is REST(S) ?

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- QUESTION: What is the value of S after the function FIRST has been applied to it?
- Most common answer: 13, 9, 121, 17, 16, 213
- This is the answer given by the natural math concept. It is "wrong".
- The correct answer is 5, 13, 9, 121, 17, 16, 213, i.e. S itself, since applying a function does not change the argument.

Natural mathematics	Formal mathematics
■ Example 1: Take a number and double it. ■ Example 2: Take a square and rotate it through 45°. ■ New concept: A function is a process that takes a given object of a certain kind and transforms it.	■ New concept: A function is a relation between sets *A* and *B* that to each object *a* in *A* assigns a unique object *b* in *B*. ■ Example 1: Natural number doubling associates to each natural number *N* the natural number 2*N*. ■ Example 2: 45° rotation on all squares associates with any given square another square at 45° to the first one.

Natural mathematics

- Example 1: Take a number and double it.
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Process

Formal mathematics

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Relation

Natural mathematics

- Example 1: Take a number and double it.
- Example 2: Take a square and rotate it through 45º.
- New concept: A function is a process that takes a given object of a certain kind and transforms it.

Process

The function acts on single objects

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Relation

The function "acts" on a set of objects

Example of a function

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References:

Uri Leron: *Mathematical Thinking and Human Nature*, in preparation.

http://www.icme-organisers.dk/tsg28/Leron Human Nature.doc

T Paz. *Natural Thinking vs. Formal Thinking: The Case of Functional Programming*, Ph.D. dissertation, Technion, Israel, 2003.

The notorious K–12 killer

Natural mathematics	Formal mathematics
■ Given B balls, of which R are red and the rest white, the proportion of red balls is R/B. ■ Hence: R/B + P/D = (R+P)/(B+D) ■ "Wrong."	■ Definition: R/B is that number which, when multiplied by B, gives R. ■ The "correct" rule: R/B + P/D = (RD+PB)/(BD)

The notorious university killer

Natural mathematics	Formal mathematics
■ Intuitive idea: A function $f : R \to R$ is called continuous if you can draw it without taking your pencil off the paper. ■ Definition: $f : R \to R$ is continuous if points close together are sent to points close together.	■ A function $f : R \to R$ is called continuous if for every real number a, for every positive real number ε, there is a positive real number δ such that $\lvert f(x) - f(a) \rvert < \varepsilon$ whenever $\lvert x - a \rvert < \delta$

CONCLUSION

Natural mathematics, including arithmetic (but not division by fractions), elementary geometry, basic algebra, and simple group theory, is within everyone's grasp and can be taught bottom up, starting with examples. Meaning comes first (and hence precedes mastery of technique).

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1. *Formalized common sense.*
2. *Meaning → rules.*
3. *Accessible to all.*

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1. *Can be counter to common sense.*
2. *Rules → meaning.*
3. *Perhaps cannot be fully mastered by all.*

THE END

The four levels of abstraction

Level 1. The objects thought about are real objects that are perceptually accessible.

Level 2. Involves real objects the thinker is familiar with but which are not perceptually accessible.

Level 3. The objects of thought may be real objects the individual has somehow learned of but never actually encountered, or imaginary versions of real objects, or imaginary variants of real objects, or imaginary combinations of real objects.

Level 4. The objects of thought are entirely abstract.

The four levels of abstraction
Some examples:

- Level 1. Physical activity.
- Level 2. Planning future action.
- Level 3. Reading a novel.
- Level 4. Mathematical thinking.

[redacted]