Lecture 11
The Fourier transform

- definition
- examples
- the Fourier transform of a unit step
- the Fourier transform of a periodic signal
- properties
- the inverse Fourier transform

The Fourier transform

we'll be interested in signals defined for all t

the **Fourier transform** of a signal f is the function

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

- F is a function of a *real* variable ω; the function value $F(\omega)$ is (in general) a complex number

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt - j \int_{-\infty}^{\infty} f(t) \sin \omega t \, dt$$

- $|F(\omega)|$ is called the *amplitude spectrum* of f; $\angle F(\omega)$ is the *phase spectrum* of f
- notation: $F = \mathcal{F}(f)$ means F is the Fourier transform of f; as for Laplace transforms we usually use uppercase letters for the transforms ($e.g.$, $x(t)$ and $X(\omega)$, $h(t)$ and $H(\omega)$, etc.)

Fourier transform and Laplace transform

Laplace transform of f

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Fourier transform of f

$$G(\omega) = \int_{-\infty}^\infty f(t)e^{-j\omega t}\,dt$$

very similar definitions, with two differences:

- Laplace transform integral is over $0 \leq t < \infty$; Fourier transform integral is over $-\infty < t < \infty$
- Laplace transform: s can be any complex number in the region of convergence (ROC); Fourier transform: $j\omega$ lies on the imaginary axis

therefore,

- if $f(t) = 0$ for $t < 0$,
  - if the imaginary axis lies in the ROC of $\mathcal{L}(f)$, then

$$G(\omega) = F(j\omega),$$

    $i.e.$, the Fourier transform is the Laplace transform evaluated on the imaginary axis
  - if the imaginary axis is not in the ROC of $\mathcal{L}(f)$, then the Fourier transform doesn't exist, but the Laplace transform does (at least, for all s in the ROC)

- if $f(t) \neq 0$ for $t < 0$, then the Fourier and Laplace transforms can be very different

examples

- one-sided decaying exponential

$$f(t) = \begin{cases} 0 & t < 0 \\ e^{-t} & t \geq 0 \end{cases}$$

Laplace transform: $F(s) = 1/(s+1)$ with ROC $\{s \mid \Re s > -1\}$

Fourier transform is

$$\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt = \frac{1}{j\omega + 1} = F(j\omega)$$

- one-sided growing exponential

$$f(t) = \begin{cases} 0 & t < 0 \\ e^{t} & t \geq 0 \end{cases}$$

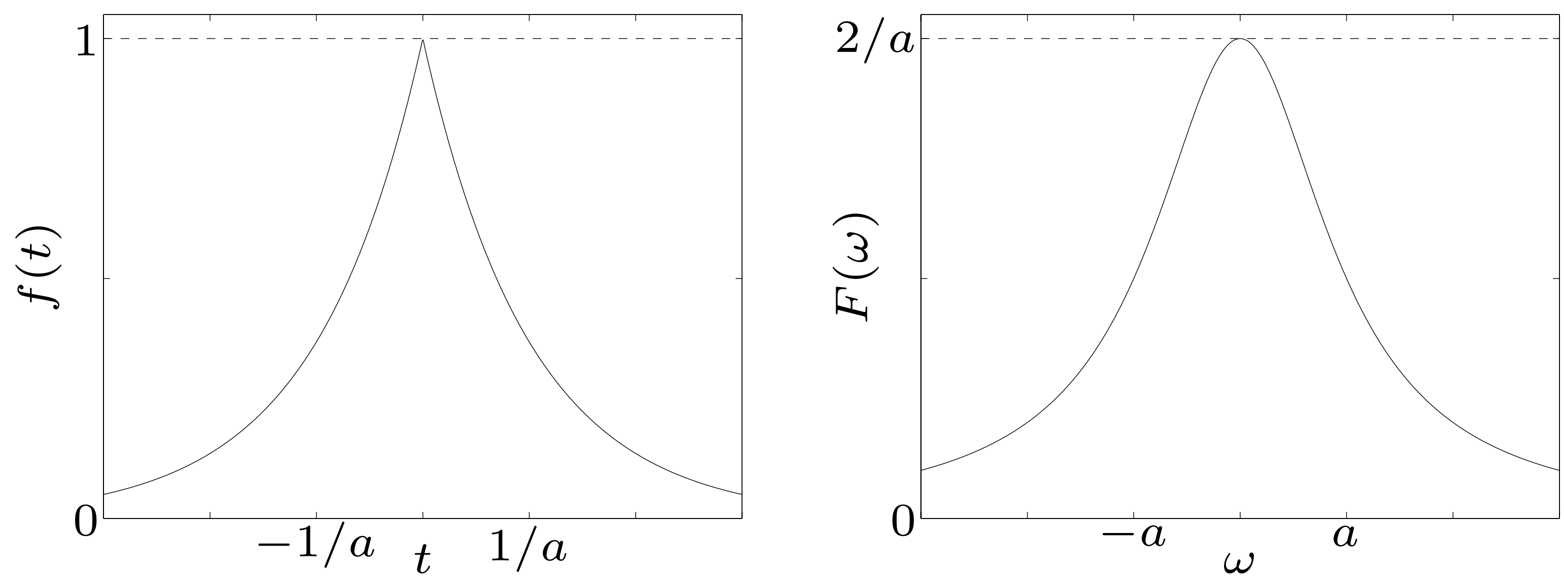
Laplace transform: $F(s) = 1/(s-1)$ with ROC $\{s \mid \Re s > 1\}$

f doesn't have a Fourier transform

Examples

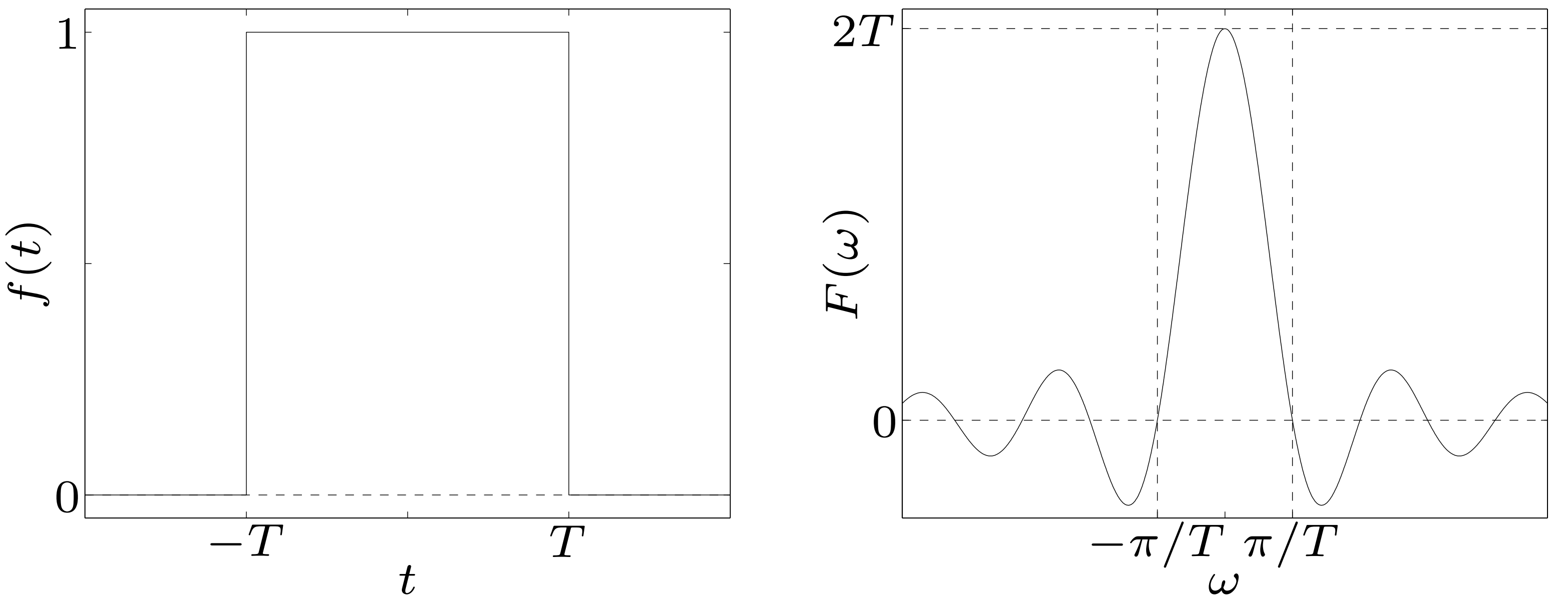
double-sided exponential: $f(t) = e^{-a|t|}$ (with $a > 0$)

$$\begin{aligned} F(\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\, dt &= \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt \\ &= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} \\ &= \frac{2a}{a^2 + \omega^2} \end{aligned}$$



rectangular pulse: $f(t) = \begin{cases} 1 & -T \leq t \leq T \\ 0 & |t| > T \end{cases}$

$$F(\omega) = \int_{-T}^{T} e^{-j\omega t}\, dt = \frac{-1}{j\omega}\left(e^{-j\omega T} - e^{j\omega T}\right) = \frac{2\sin\omega T}{\omega}$$

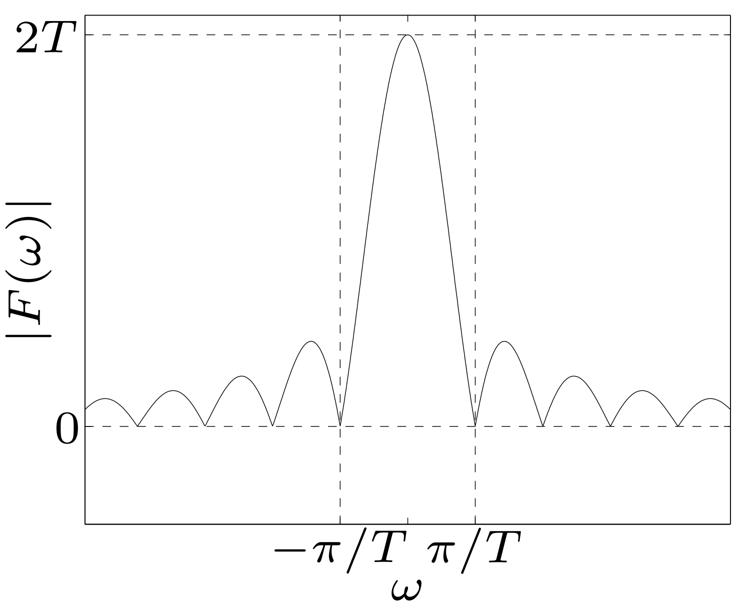


unit impulse: $f(t) = \delta(t)$

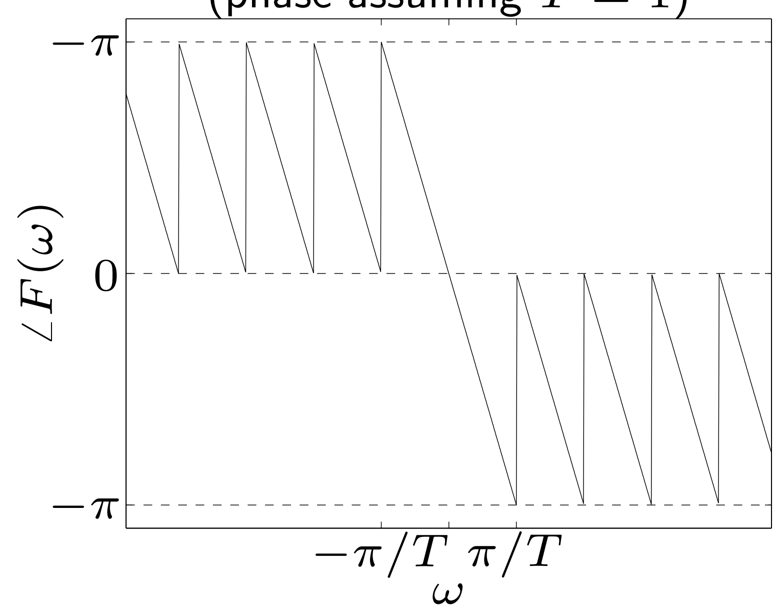
$$F(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt = 1$$

shifted rectangular pulse: $f(t) = \begin{cases} 1 & 1-T \leq t \leq 1+T \\ 0 & t < 1-T \text{ or } t > 1+T \end{cases}$

$$
\begin{aligned}
F(\omega) = \int_{1-T}^{1+T} e^{-j\omega t}\, dt &= \frac{-1}{j\omega}\left(e^{-j\omega(1+T)} - e^{-j\omega(1-T)}\right) \\
&= \frac{-e^{-j\omega}}{j\omega}\left(e^{-j\omega T} - e^{j\omega T}\right) \\
&= \frac{2\sin\omega T}{\omega} e^{-j\omega}
\end{aligned}
$$



(phase assuming $T = 1$)



Step functions and constant signals

by allowing impulses in $\mathcal{F}(f)$ we can define the Fourier transform of a step function or a constant signal

unit step

what is the Fourier transform of

$$f(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \; ?$$

the Laplace transform is $1/s$, but the imaginary axis is not in the ROC, and therefore the Fourier transform is *not* $1/j\omega$

in fact, the integral

$$\int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt = \int_0^{\infty} e^{-j\omega t}\, dt = \int_0^{\infty} \cos \omega t\, dt - j \int_0^{\infty} \sin \omega t\, dt$$

is not defined

however, we can interpret f as the limit for $\alpha \to 0$ of a one-sided decaying exponential

$$g_\alpha(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ 0 & t < 0, \end{cases}$$

($\alpha > 0$), which has Fourier transform

$$G_\alpha(\omega) = \frac{1}{a + j\omega} = \frac{a - j\omega}{a^2 + \omega^2} = \frac{a}{a^2 + \omega^2} - \frac{j\omega}{a^2 + \omega^2}$$

as $\alpha \to 0$,

$$\frac{a}{a^2 + \omega^2} \to \pi\delta(\omega), \qquad -\frac{j\omega}{a^2 + \omega^2} \to \frac{1}{j\omega}$$

let's therefore *define* the Fourier transform of the unit step as

$$F(\omega) = \int_0^\infty e^{-j\omega t}\, dt = \pi\delta(\omega) + \frac{1}{j\omega}$$

negative time unit step

$$f(t) = \begin{cases} 1 & t \leq 0 \\ 0 & t > 0 \end{cases}$$

$$F(\omega) = \int_{-\infty}^{0} e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{j\omega t}\, dt = \pi\delta(\omega) - \frac{1}{j\omega}$$

constant signals: $f(t) = 1$

f is the sum of a unit step and a negative time unit step:

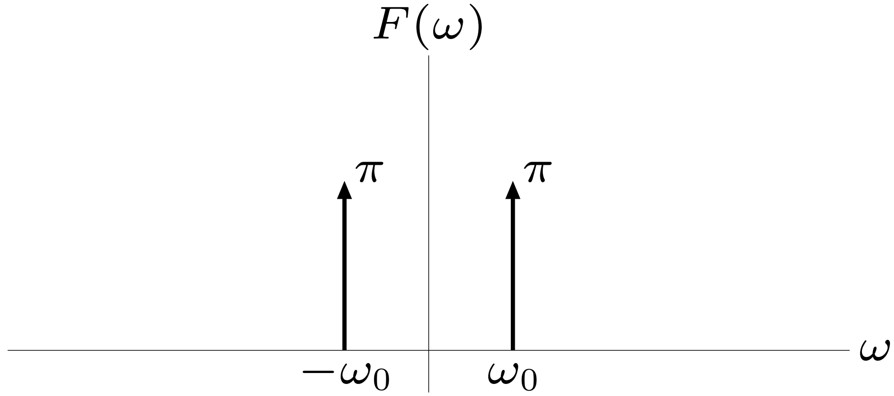
$$F(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}\, dt = \int_{-\infty}^{0} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-j\omega t}\, dt = 2\pi\delta(\omega)$$

Fourier transform of periodic signals

similarly, by allowing impulses in $\mathcal{F}(f)$, we can define the Fourier transform of a periodic signal

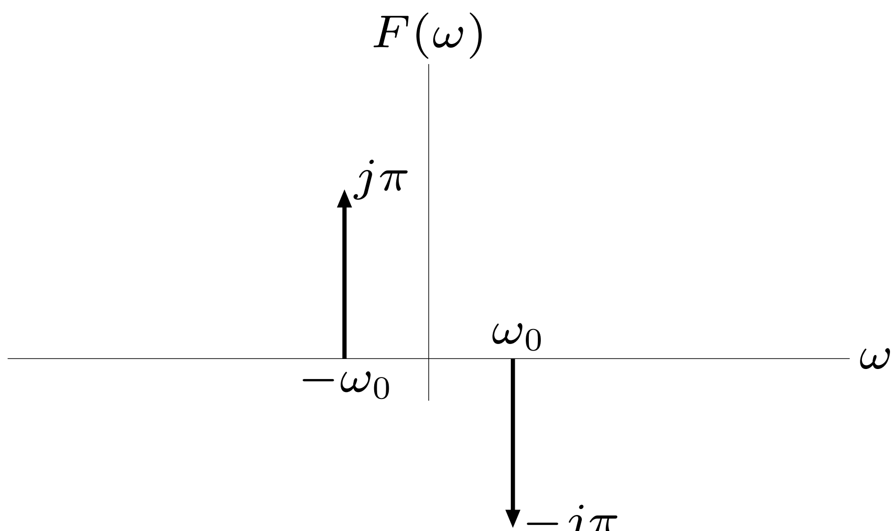
sinusoidal signals: Fourier transform of $f(t) = \cos \omega_0 t$

$$
\begin{aligned}
F(\omega) &= \frac{1}{2}\int_{-\infty}^{\infty} \left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) e^{-j\omega t}\, dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t} dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j(\omega+\omega_0)t} dt \\
&= \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)
\end{aligned}
$$



Fourier transform of $f(t) = \sin\omega_0 t$

$$
\begin{aligned}
F(\omega) &= \frac{1}{2j}\int_{-\infty}^{\infty}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)e^{-j\omega t}\,dt \\
&= \frac{1}{2j}\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t}dt + -\frac{1}{2j}\int_{-\infty}^{\infty} e^{-j(\omega_0+\omega)t}dt \\
&= -j\pi\delta(\omega-\omega_0) + j\pi\delta(\omega+\omega_0)
\end{aligned}
$$

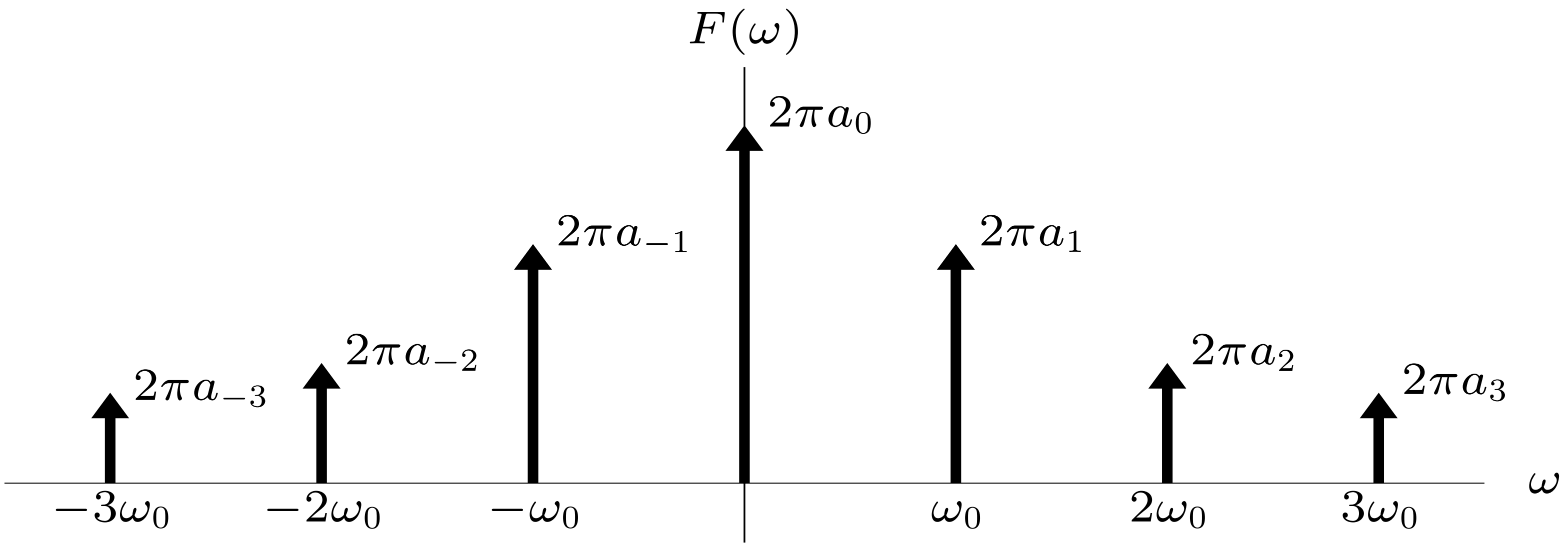


periodic signal $f(t)$ with fundamental frequency ω_0

express f as Fourier series

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$F(\omega) = \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} e^{j(k\omega_0 - \omega)t}\, dt = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$



Properties of the Fourier transform

linearity	$af(t) + bg(t)$	$aF(\omega) + bG(\omega)$
time scaling	$f(at)$	$\frac{1}{\lvert a\rvert}F(\frac{\omega}{a})$
time shift	$f(t-T)$	$e^{-j\omega T}F(\omega)$
differentiation	$\frac{df(t)}{dt}$	$j\omega F(\omega)$
	$\frac{d^k f(t)}{dt^k}$	$(j\omega)^k F(\omega)$
integration	$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$
multiplication with t	$t^k f(t)$	$j^k \frac{d^k F(\omega)}{d\omega^k}$
convolution	$\int_{-\infty}^{\infty} f(\tau)g(t-\tau)\, d\tau$	$F(\omega)G(\omega)$
multiplication	$f(t)g(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\widetilde{\omega})G(\omega-\widetilde{\omega})\, d\widetilde{\omega}$

Examples

sign function: $f(t) = \begin{cases} 1 & t \geq 0 \\ -1 & t < 0 \end{cases}$

write f as $f(t) = -1 + 2g(t)$, where g is a unit step at $t = 0$, and apply linearity

$$F(\omega) = -2\pi\delta(\omega) + 2\pi\delta(\omega) + \frac{2}{j\omega} = \frac{2}{j\omega}$$

sinusoidal signal: $f(t) = \cos(\omega_0 t + \phi)$

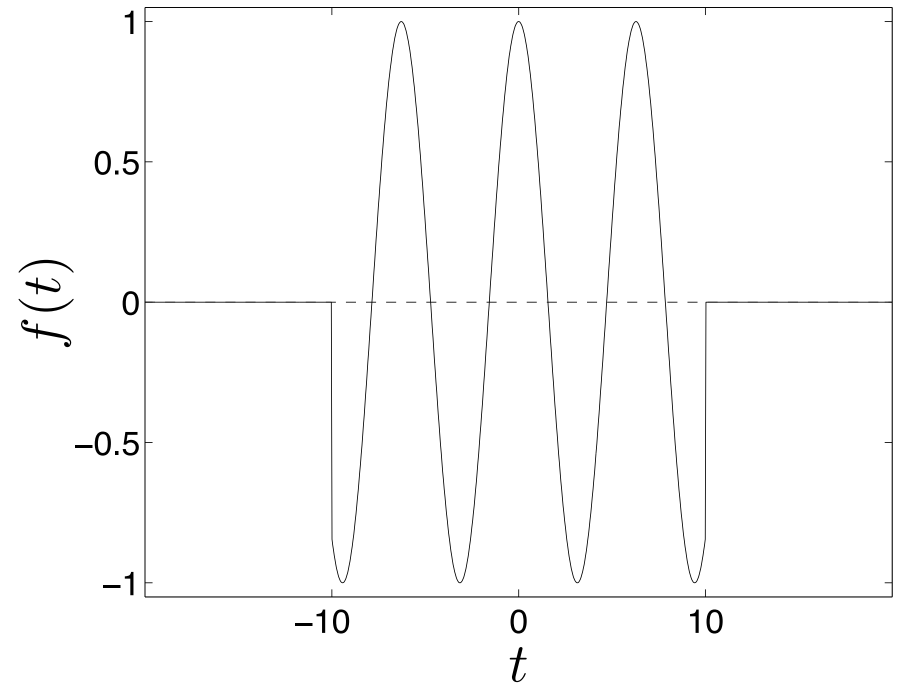
write f as

$$f(t) = \cos(\omega_0(t + \phi/\omega_0))$$

and apply time shift property:

$$F(\omega) = \pi e^{j\omega\phi/\omega_0}\left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)$$

pulsed cosine: $f(t) = \begin{cases} 0 & |t| > 10 \\ \cos t & -10 \le t \le 10 \end{cases}$



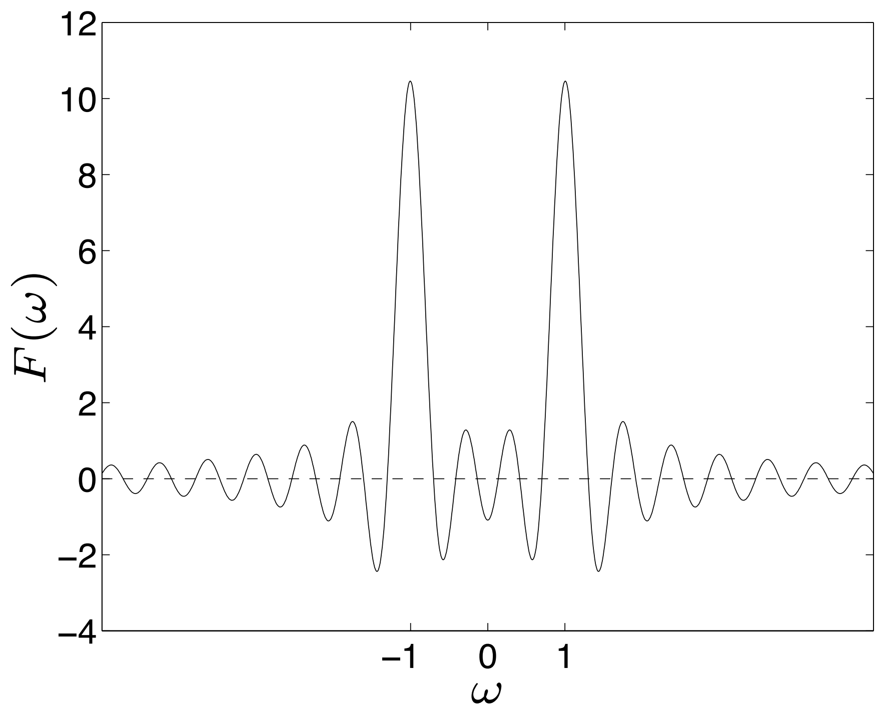
write f as a product $f(t) = g(t)\cos t$ where g is a rectangular pulse of width 20 (see page 12-7)

$$\mathcal{F}(\cos t) = \pi\delta(\omega - 1) + \pi\delta(\omega + 1), \quad \mathcal{F}(g(t)) = \frac{2\sin 10\omega}{\omega}$$

now apply multiplication property

$$F(j\omega) = \int_{-\infty}^{\infty} \frac{\sin 10\widetilde{\omega}}{\widetilde{\omega}} \left(\delta(\omega - \widetilde{\omega} - 1) + \delta(\omega - \widetilde{\omega} + 1)\right) d\widetilde{\omega}$$

$$= \frac{\sin(10(\omega - 1))}{\omega - 1} + \frac{\sin(10(\omega + 1))}{\omega + 1}$$



The inverse Fourier transform

if $F(\omega)$ is the Fourier transform of $f(t)$, *i.e.*,

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt$$

then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\, d\omega$$

let's check

$$\begin{aligned}
\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} F(\omega)e^{j\omega t}\, d\omega &= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \left(\int_{\tau=-\infty}^{\infty} f(\tau)e^{-j\omega\tau}\right) e^{j\omega t} d\omega \\
&= \frac{1}{2\pi}\int_{\tau=-\infty}^{\infty} f(\tau) \left(\int_{\omega=-\infty}^{\infty} e^{-j\omega(\tau-t)} d\omega\right) d\tau \\
&= \int_{-\infty}^{\infty} f(\tau)\delta(\tau-t)d\tau \\
&= f(t)
\end{aligned}$$