Lecture 7
Circuit analysis via Laplace transform

• analysis of general LRC circuits
• impedance and admittance descriptions
• natural and forced response
• circuit analysis with impedances
• natural frequencies and stability
initial current: $i(0)$

KCL, KVL, and branch relations yield: $-u + Li' + y = 0, \ y = Ri$

take Laplace transforms to get

$$-U + L(sI - i(0)) + Y = 0, \ \ Y = RI$$

solve for $Y$ to get

$$Y = \frac{U + Li(0)}{1 + sL/R} = \frac{1}{1 + sL/R}U + \frac{L}{1 + sL/R}i(0)$$
in the time domain:

\[ y(t) = \frac{1}{T} \int_{0}^{t} e^{-\tau/T} u(t - \tau) \, d\tau + Ri(0)e^{-t/T} \]

where \( T = L/R \)

two terms in \( y \) (or \( Y \)):

- first term corresponds to solution with zero initial condition
- first term is convolution of source with a function
- second term corresponds to solution with zero source

we’ll see these are general properties . . .
Analysis of general LRC circuits

consider a circuit with \( n \) nodes and \( b \) branches, containing

- independent sources
- linear elements (resistors, op-amps, dep. sources, . . .)
- inductors & capacitors
such a circuit is described by three sets of equations:

- KCL: \[ Ai(t) = 0 \] (\( n - 1 \) equations)
- KVL: \[ v(t) = A^T e(t) \] (\( b \) equations)
- branch relations (\( b \) equations)

where

- \( A \in \mathbb{R}^{(n-1) \times b} \) is the reduced node incidence matrix
- \( i \in \mathbb{R}^b \) is the vector of branch currents
- \( v \in \mathbb{R}^b \) is the vector of branch voltages
- \( e \in \mathbb{R}^{n-1} \) is the vector of node potentials
Branch relations

- independent voltage source: $v_k(t) = u_k(t)$
- resistor: $v_k = R i_k$
- capacitor: $i_k = C v'_k$
- inductor: $v_k = L i'_k$
- VCVS: $v_k = a v_j$
- and so on (current source, VCCS, op-amp, ...)

thus:

circuit equations are a set of $2b + n - 1$ (linear) algebraic and/or differential equations in $2b + n - 1$ variables
Laplace transform of circuit equations

most of the equations are the same, e.g.,

- KCL, KVL become $AI = 0$, $V = A^T E$
- independent sources, e.g., $v_k = u_k$ becomes $V_k = U_k$
- linear static branch relations, e.g., $v_k = R_i_k$ becomes $V_k = RI_k$

the differential equations become algebraic equations:

- capacitor: $I_k = sCV_k - Cv_k(0)$
- inductor: $V_k = sLI_k - Li_k(0)$

thus, in frequency domain,

- circuit equations are a set of $2b + n - 1$ (linear) algebraic equations in $2b + n - 1$ variables
thus, LRC circuits can be solved **exactly like static circuits**, except

- all variables are Laplace transforms, not real numbers
- capacitors and inductors have branch relations $I_k = sCV_k - Cv_k(0)$,
  $V_k = sLI_k - Li_k(0)$

**interpretation:** an inductor is like a “resistance” $sL$, in series with an independent voltage source $-Li_k(0)$

a capacitor is like a “resistance” $1/(sC)$, in parallel with an independent current source $-Cv_k(0)$

- these “resistances” are called *impedances*
- these sources are impulses in the time domain which set up the initial conditions
Impedance and admittance

circuit element or device with voltage $v$, current $i$

\[
\begin{array}{c}
i \\
+ \\
\downarrow \\
v \\
- \\
i
\end{array}
\]

the relation $V(s) = Z(s)I(s)$ is called an \textbf{impedance description} of the device

- $Z$ is called the (s-domain) \textit{impedance} of the device
- in the time domain, $v$ and $i$ are related by convolution: $v = z * i$

similarly, $I(s) = Y(s)V(s)$ is called an \textbf{admittance description} ($Y = 1/Z$)
Examples

- a resistor has an impedance $R$
- an inductor with zero initial current has an impedance $Z(s) = sL$
  (admittance $1/(sL)$)
- a capacitor with zero initial voltage has an impedance $Z(s) = 1/(sC)$
  (admittance $sC$)

*cf.* impedance in SSS analysis with phasors:

- resistor: $V = RI$
- inductor: $V = (j\omega L)I$
- capacitor: $V = (1/j\omega C)I$

$s$-domain and phasor impedance agree for $s = j\omega$, but are not the same
we can express the branch relations as

\[ M(s)I(s) + N(s)V(s) = U(s) + W \]

where

- \( U \) is the independent sources
- \( W \) includes the terms associated with initial conditions
- \( M \) and \( N \) give the impedance or admittance of the branches

For example, if branch 13 is an inductor,

\[ (sL)I_{13}(s) + (-1)V_{13}(s) = Li_{13}(0) \]

(this gives the 13th row of \( M, N, U, \) and \( W \))
we can write circuit equations as one big matrix equation:

\[
\begin{bmatrix}
A & 0 & 0 \\
0 & I & -A^T \\
M(s) & N(s) & 0
\end{bmatrix}
\begin{bmatrix}
I(s) \\
V(s) \\
E(s)
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
U(s) + W
\end{bmatrix}
\]
hence,

\[
\begin{bmatrix}
I(s) \\
V(s) \\
E(s)
\end{bmatrix} = \begin{bmatrix}
A & 0 & 0 \\
0 & I & -A^T \\
M(s) & N(s) & 0
\end{bmatrix}^{-1} \begin{bmatrix}
0 \\
0 \\
U(s) + W
\end{bmatrix}
\]

in the time domain,

\[
\begin{bmatrix}
i(t) \\
v(t) \\
e(t)
\end{bmatrix} = \mathcal{L}^{-1} \left( \begin{bmatrix}
A & 0 & 0 \\
0 & I & -A^T \\
M(s) & N(s) & 0
\end{bmatrix}^{-1} \begin{bmatrix}
0 \\
0 \\
U(s) + W
\end{bmatrix} \right)
\]

- this gives a **explicit solution** of the circuit
- these equations are **identical** to those for a linear static circuit (except instead of real numbers we have Laplace transforms, \(i.e.,\) complex-valued functions of \(s\))
- hence, much of what you know extends to this case
Natural and forced response

let’s express solution as

\[
\begin{bmatrix}
i(t) \\
v(t) \\
e(t)
\end{bmatrix} = \mathcal{L}^{-1}\left(\begin{bmatrix}
A & 0 & 0 \\
0 & I & -A^T \\
M(s) & N(s) & 0
\end{bmatrix}^{-1} \begin{bmatrix}
0 \\
0 \\
U(s)
\end{bmatrix}\right)
\]

\[
+ \mathcal{L}^{-1}\left(\begin{bmatrix}
A & 0 & 0 \\
0 & I & -A^T \\
M(s) & N(s) & 0
\end{bmatrix}^{-1} \begin{bmatrix}
0 \\
0 \\
W
\end{bmatrix}\right)
\]

thus circuit response is equal to:

- the natural response, \textit{i.e.}, solution with independent sources off, plus
- the forced response, \textit{i.e.}, solution with zero initial conditions
• the forced response is linear in $U(s)$, \textit{i.e.}, the independent source signals
• the natural response is linear in $W$, \textit{i.e.}, the inductor & capacitor initial conditions
Back to the example

initial current: $i(0)$

natural response: set source to zero, get LR circuit with solution

$$y_{\text{nat}}(t) = Ri(0)e^{-t/T}, \quad T = L/R$$

forced response: assume zero initial current, replace inductor with impedance $Z = sL$: 

Circuit analysis via Laplace transform
by voltage divider rule (for impedances), \( Y_{\text{frc}} = U \frac{R}{R + sL} \) (as if they were simple resistors!)

so \( y_{\text{frc}} = \mathcal{L}^{-1}\left(\frac{R}{R + sL}\right) \ast u, \ i.e., \)

\[
y_{\text{frc}}(t) = \frac{1}{T} \int_{0}^{t} e^{-\tau/T} u(t - \tau) \, d\tau
\]

all together, the voltage is \( y(t) = y_{\text{nat}}(t) + y_{\text{frc}}(t) \) (same as before)
Circuit analysis with impedances

for a circuit with

- linear static elements (resistors, op-amps, dependent sources, \ldots )
- independent sources
- elements described by impedances (inductors & capacitors with zero initial conditions, \ldots )

we can manipulate

- Laplace transforms of voltages, currents
- impedances

as if they were (real, constant) voltages, currents, and resistances, respectively
reason: they both satisfy the same equations

examples:

- series, parallel combinations
- voltage & current divider rules
- Thevenin, Norton equivalents
- nodal analysis
example:

\[ I_{in} \quad 2F \]

\[ + \]

\[ V_{in} \quad 1\Omega \]

\[ 3\Omega \]

\[ 4H \]

let's find input impedance, \( i.e., Z_{in} = \frac{V_{in}}{I_{in}} \)

by series/parallel combination rules,

\[
Z_{in} = \frac{1}{2s} + (1\parallel 4s) + 3 = \frac{1}{2s} + \frac{4s}{1 + 4s} + 3
\]

we have

\[
V_{in}(s) = \left( \frac{1}{2s} + \frac{4s}{1 + 4s} + 3 \right) I_{in}(s)
\]

provided the capacitor & inductor have zero initial conditions
**example:** nodal analysis

\[ GE = I_{\text{src}} \]

- \( I_{\text{src}} \) is total of current sources flowing into nodes
- \( G_{ii} \) is sum of admittances tied to node \( i \)
- \( G_{ij} \) is minus the sum of all admittances between nodes \( i \) and \( j \)

Circuit analysis via Laplace transform
for this example we have:

\[
\begin{bmatrix}
1 + 2s + \frac{1}{3} & -(2s + \frac{1}{3}) \\
-(2s + \frac{1}{3}) & \frac{1}{3} + 2s + \frac{1}{4} + \frac{1}{5s}
\end{bmatrix}
\begin{bmatrix}
E_1(s) \\
E_2(s)
\end{bmatrix}
= 
\begin{bmatrix}
I_{in}(s) \\
0
\end{bmatrix}
\]

(which we could solve . . . )
example: Thevenin equivalent

\[ 2H \]

\[ 0 \quad 0 \quad 0 \]

A

\[ 1 - e^{-t} \]

\[ + \]

\[ - \]

\[ 1\Omega \]

B

voltage source is \( \frac{1}{s} - \frac{1}{s+1} = \frac{1}{s(s+1)} \) in \( s \)-domain

Thevenin voltage is open-circuit voltage, \( i.e., \)

\[ V_{th} = \frac{1}{s(s+1)} \frac{1}{1 + 2s} \]

Thevenin impedance is impedance looking into terminals with source off, \( i.e., \)

\[ Z_{th} = 1 || 2s = \frac{2s}{1 + 2s} \]
Thevenin equivalent circuit is:

\[ V_{th}(s) = \frac{1}{s(s + 1)(1 + 2s)} \]

\[ Z_{th}(s) = \frac{2s}{1 + 2s} \]
Natural frequencies and stability

we say a circuit is **stable** if its natural response decays (i.e., converges to zero as \( t \to \infty \)) for all initial conditions

in this case the circuit “forgets” its initial conditions as \( t \) increases; the natural response contributes less and less to the solution as \( t \) increases, i.e.,

\[
y(t) \to y_{fr}(t) \quad \text{as} \quad t \to \infty
\]

circuit is stable when poles of the natural response, called **natural frequencies**, have negative real part

these are given by the zeros of

\[
\text{det}
\begin{pmatrix}
A & 0 & 0 \\
0 & I & -A^T \\
M(s) & N(s) & 0
\end{pmatrix}
\]