Lecture 8
Transfer functions and convolution

- convolution & transfer functions
- properties
- examples
- interpretation of convolution
- representation of linear time-invariant systems

Convolution systems

convolution system with input u $\big(u(t) = 0,\ t < 0\big)$ and output y:

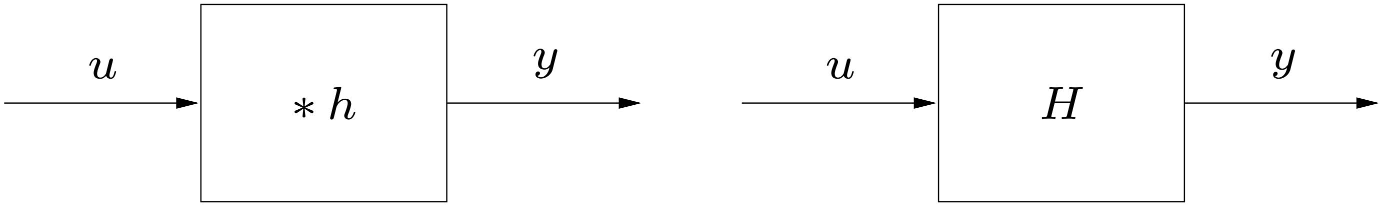
$$y(t) = \int_0^t h(\tau)u(t-\tau)\, d\tau = \int_0^t h(t-\tau)u(\tau)\, d\tau$$

abbreviated: $y = h * u$

in the frequency domain: $Y(s) = H(s)U(s)$

- H is called the *transfer function* (TF) of the system
- h is called the *impulse response* of the system

block diagram notation(s):



Properties

1. convolution systems are **linear**: for all signals u_1, u_2 and all α, $\beta \in \mathbf{R}$,

$$h * (\alpha u_1 + \beta u_2) = \alpha(h * u_1) + \beta(h * u_2)$$

2. convolution systems are **causal**: the output $y(t)$ at time t depends only on past inputs $u(\tau)$, $0 \le \tau \le t$

3. convolution systems are **time-invariant**: if we shift the input signal u over $T > 0$, *i.e.*, apply the input

$$\widetilde{u}(t) = \begin{cases} 0 & t < T \\ u(t-T) & t \ge 0 \end{cases}$$

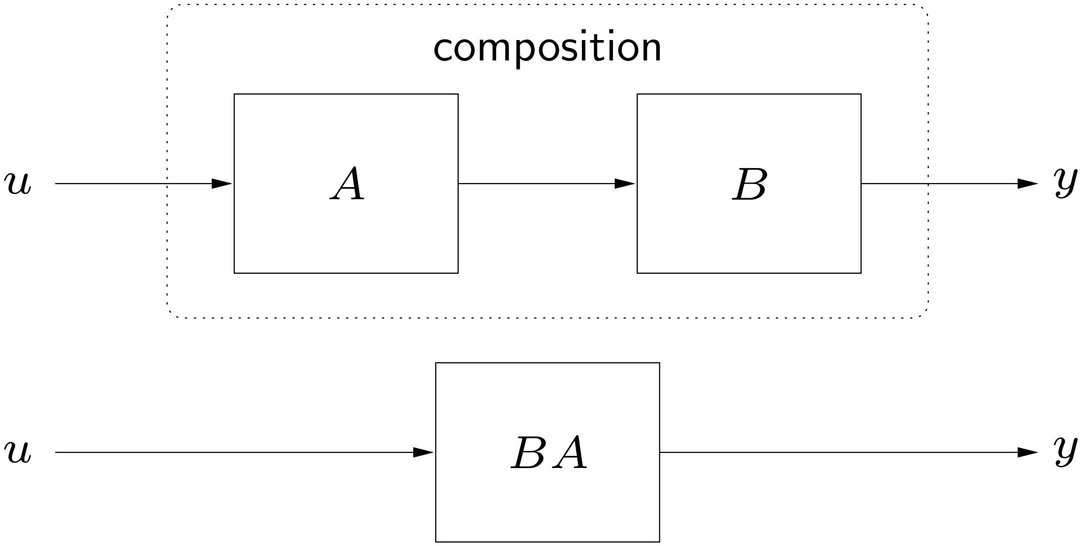
to the system, the output is

$$\widetilde{y}(t) = \begin{cases} 0 & t < T \\ y(t-T) & t \ge 0 \end{cases}$$

in other words: convolution systems commute with delay

4. **composition** of convolution systems corresponds to

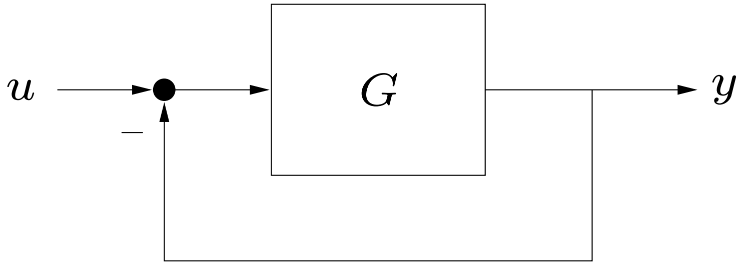
- multiplication of transfer functions
- convolution of impulse responses



ramifications:

- can manipulate block diagrams with transfer functions as if they were simple gains
- convolution systems commute with each other

Example: feedback connection



in **time domain**, we have complicated integral equation

$$y(t) = \int_0^t g(t-\tau)(u(\tau) - y(\tau))\, d\tau$$

which is not easy to understand or solve . . .
in **frequency domain**, we have $Y = G(U - Y)$; solve for Y to get

$$Y(s) = H(s)U(s), \quad H(s) = \frac{G(s)}{1 + G(s)}$$

(as if G were a simple scaling system!)

General examples

first order LCCODE: $y' + y = u$, $y(0) = 0$

take Laplace transform to get

$$Y(s) = \frac{1}{s+1} U(s)$$

transfer function is $1/(s+1)$; impulse response is e^{-t}

integrator: $y(t) = \int_0^t u(\tau)\, d\tau$

transfer function is $1/s$; impulse response is 1

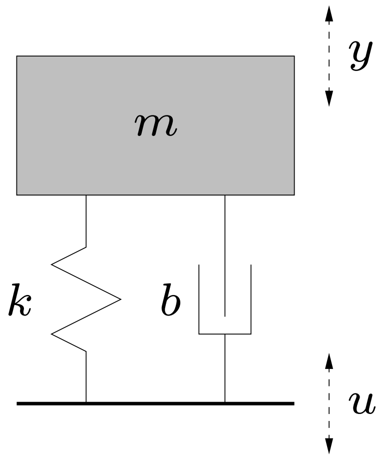
delay: with $T \geq 0$,

$$y(t) = \begin{cases} 0 & t < T \\ u(t-T) & t \geq T \end{cases}$$

impulse response is $\delta(t-T)$; transfer function is e^{-sT}

Vehicle suspension system

(simple model of) vehicle suspension system:



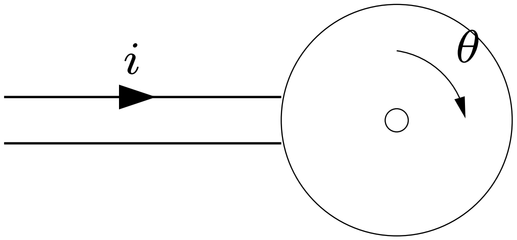
- input u is road height (along vehicle path); output y is vehicle height
- vehicle dynamics: $my'' + by' + ky = bu' + ku$

assuming $y(0) = 0$, $y'(0) = 0$, (and $u(0_-) = 0$),

$$(ms^2 + bs + k)Y = (bs + k)U$$

TF from road height to vehicle height is $H(s) = \dfrac{bs + k}{ms^2 + bs + k}$

DC motor



$$J\theta'' + b\theta' = ki$$

(J is rotational inertia of shaft & load; b is mechanical resistance of shaft & load; k is *motor constant*)

assuming $\theta(0) = \theta'(0) = 0$,

$$Js^2\Theta(s) + bs\Theta(s) = kI(s), \quad \Theta(s) = \frac{k}{Js^2 + bs}I(s)$$

$i.e.$, transfer function H from i to θ is

$$H(s) = \frac{k}{Js^2 + bs}$$

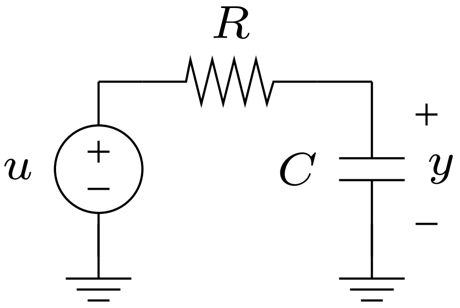
Circuit examples

consider a circuit with linear elements, zero initial conditions for inductors and capacitors,

- one independent source with value u
- y is a voltage or current somewhere in the circuit

then we have $Y(s) = H(s)U(s)$

example: RC circuit



$$RCy'(t) + y(t) = u(t), \quad Y(s) = \frac{1}{1+sRC}U(s)$$

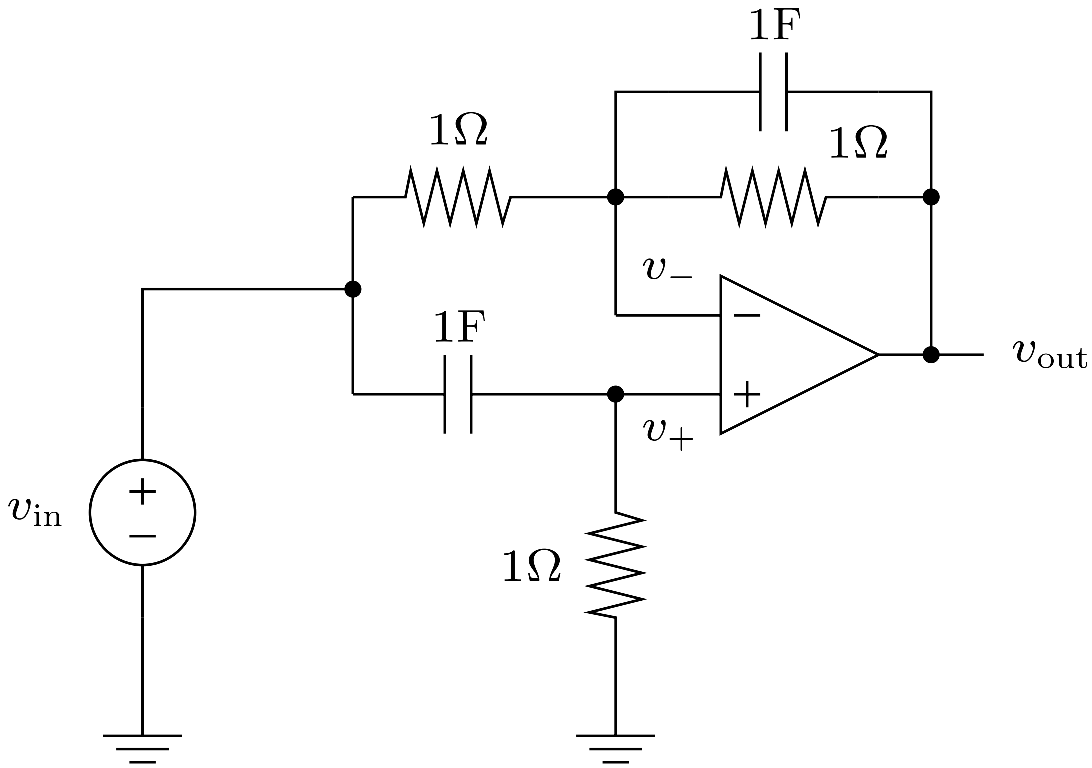
impulse response is $\mathcal{L}^{-1}\left(\frac{1}{1+sRC}\right) = \frac{1}{RC}e^{-t/RC}$

to find H: write circuit equations in frequency domain:

- resistor: $v(t) = Ri(t)$ becomes $V(s) = RI(s)$
- capacitor: $i(t) = Cv'(t)$ becomes $I(s) = sCV(s)$
- inductor: $v(t) = Li'(t)$ becomes $V(s) = sLI(s)$

in frequency domain, circuit equations become *algebraic* equations

example:



let's find TF from v_{in} to v_{out} (assuming zero initial voltages for capacitors)

- by voltage divider rule, $V_+ = V_{\mathrm{in}} \dfrac{1}{1+1/s} = V_{\mathrm{in}} \dfrac{s}{s+1}$
- current in lefthand resistor is (using $V_- = V_+$):

$$I = \frac{V_{\mathrm{in}} - V_-}{1\Omega} = \left(1 - \frac{s}{s+1}\right) V_{\mathrm{in}} = \frac{1}{s+1} V_{\mathrm{in}}$$

- I flows through $1\mathrm{F}\|1\Omega$, yielding voltage

$$V_{\mathrm{in}}\frac{1}{s+1}\,\frac{(1)(1/s)}{1+1/s}=V_{\mathrm{in}}\frac{1}{(s+1)^2}$$

- finally we have $V_{\mathrm{out}}=V_{-}-V_{\mathrm{in}}\dfrac{1}{(s+1)^2}=V_{\mathrm{in}}\dfrac{s^2+s-1}{(s+1)^2}$

so transfer function is

$$H(s)=\frac{s^2+s-1}{(s+1)^2}=1-\frac{1}{s+1}-\frac{1}{(s+1)^2}$$

impulse response is

$$h(t)=\mathcal{L}^{-1}(H)=\delta(t)-e^{-t}-te^{-t}$$

we have

$$v_{\mathrm{out}}(t)=v_{\mathrm{in}}(t)-\int_0^t(1+\tau)e^{-\tau}v_{\mathrm{in}}(t-\tau)\;d\tau$$

Interpretation of convolution

$$y(t) = \int_0^t h(\tau)u(t-\tau)\, d\tau$$

- $y(t)$ is current output; $u(t-\tau)$ is what the input was τ seconds ago
- $h(\tau)$ shows how much current output depends on what input was τ seconds ago

for example,

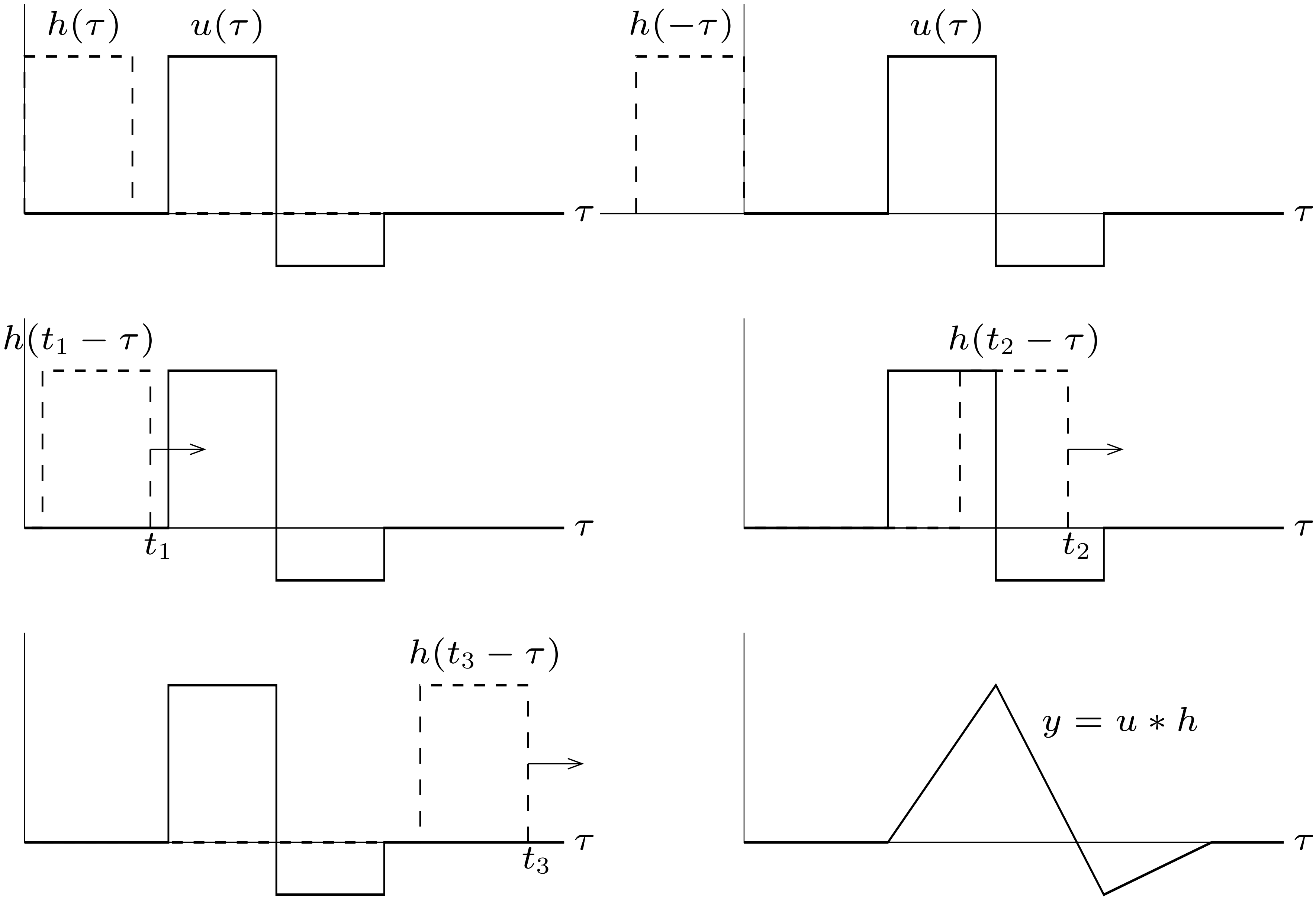
- $h(21)$ big means current output depends quite a bit on what input was, 21sec ago
- if $h(\tau)$ is small for $\tau > 3$, then $y(t)$ depends mostly on what the input has been over the last 3 seconds
- $h(\tau) \to 0$ as $\tau \to \infty$ means $y(t)$ depends less and less on remote past input

Graphical interpretation

$$y(t) = \int_0^t h(t-\tau)u(\tau)\, d\tau$$

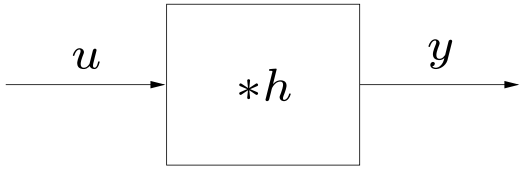
to find $y(t)$:

- flip impulse response $h(\tau)$ backwards in time (yields $h(-\tau)$)
- drag to the right over t (yields $h(t-\tau)$)
- multiply pointwise by u (yields $u(\tau)h(t-\tau)$)
- integrate over τ to get $y(t)$

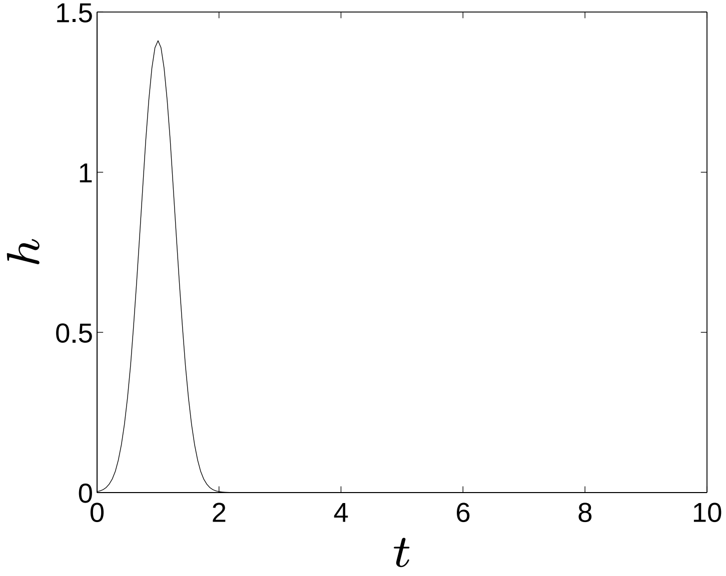

$h(\tau)$
$u(\tau)$
τ
$h(-\tau)$
$u(\tau)$
τ
$h(t_1-\tau)$
t_1
τ
$h(t_2-\tau)$
t_2
τ
$h(t_3-\tau)$
t_3
τ
$y=u*h$
τ

Example

communication channel, $e.g.$, twisted pair cable

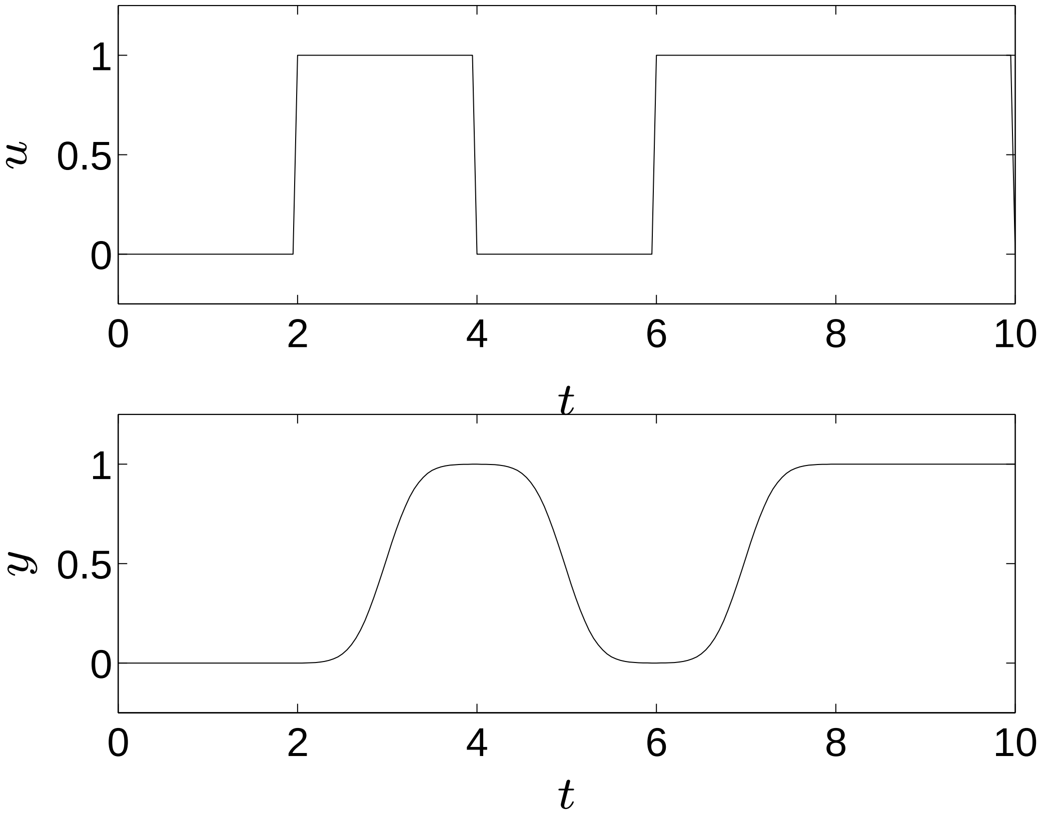


impulse response:



a delay ≈ 1, plus smoothing

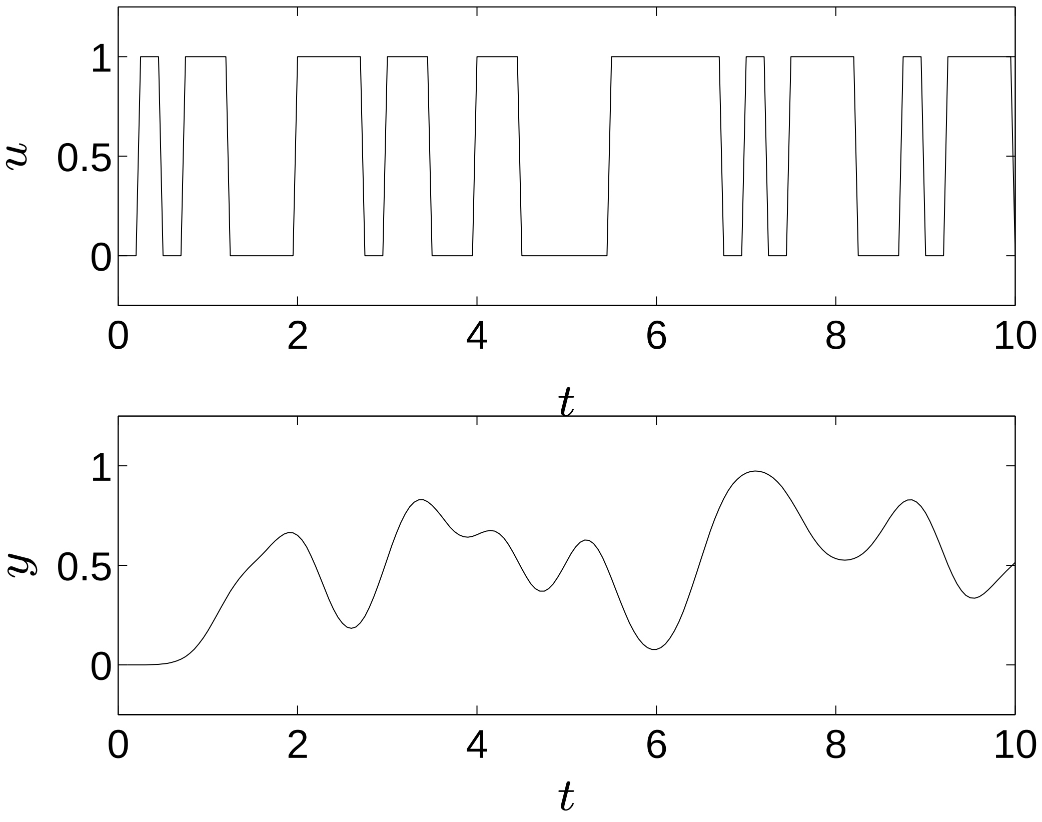
simple signalling at 0.5 bit/sec; Boolean signal $0, 1, 0, 1, 1, \ldots$



output is delayed, smoothed version of input

1's & 0's easily distinguished in y

simple signalling at 4 bit/sec; same Boolean signal



smoothing makes 1's & 0's very hard to distinguish in y

Linear time-invariant systems

consider a system $\mathcal{A}$ which is

- linear
- time-invariant (commutes with delays)
- causal ($y(t)$ depends only on $u(\tau)$ for $0 \leq \tau \leq t$)

called a *linear time-invariant* (LTI) causal system

we have seen that any convolution system is LTI and causal; the converse is also true: any LTI causal system can be represented by a convolution system

convolution/transfer function representation gives *universal description* for LTI causal systems

(precise statement & proof is not simple . . .)