Problem Set #7
Due: Friday June 1st, 2018 at 5 PM.

1. Laplace Transform Convergence
   Determine whether each of the following statements is true or false. If a statement is true, construct a convincing argument for it. If it is false, give a counterexample.
   (a) The Laplace transform of $t^2u(t)$ does not converge anywhere on the $s$-plane.
   (b) The Laplace transform of $e^{t^2}u(t)$ does not converge anywhere on the $s$-plane.
   (c) The Laplace transform of $e^{j\omega_0 t}$ does not converge anywhere on the $s$-plane.
   (d) The Laplace transform of $e^{j\omega_0 t}u(t)$ does not converge anywhere on the $s$-plane.
   (e) The Laplace transform of $|t|$ does not converge anywhere on the $s$-plane.

2. Bode Plots
   Sketch the Bode plots for the following frequency responses:
   (a) $\frac{s-10}{10(s+1)}$
   (b) $\frac{100(s+10)(10s+1)}{(s+100)(s^2+s+1)}$

3. Cascade and parallel realizations
   This problem shows how a higher-order LTI system can be constructed from lower-order subsystems, using the equivalences of interconnected systems presented on page 183 of the Laplace Transform chapter in course reader. Consider an LTI system with transfer function of the form
   $H(s) = \frac{s-\beta_1}{(s-\gamma_1)(s-\gamma_2)}, \quad \Re(s) > \min\{\Re(\gamma_1), \Re(\gamma_2)\},$
   where $\gamma_1 \neq \gamma_2 \neq \beta_1$.
   (a) Find rational transfer functions $H_1(s)$ and $H_2(s)$ of order 1 (i.e., their numerator and denominator are polynomial in $s$ of degree up to 1), such that the following two diagrams are equivalent:

   \[ x(t) \xrightarrow{H_1(s)} x(t) \xrightarrow{H_2(s)} y(t) \]

   \[ x(t) \xrightarrow{H(s)} y(t) \]

   (b) Find rational transfer functions $H_1(s)$ and $H_2(s)$ of order 1 (i.e., their denominator is a degree 1 polynomial in $s$), such that the following two diagrams are equivalent:
4. **Stabilizing unstable system by feedback**

This problem makes reference to the Feedback Systems lecture notes, pages 197-198. \( H(s) \) consists of a proportional-plus-derivative transfer function.

\[
x(t) \rightarrow \begin{array}{c}
\text{y(t)}
\end{array}
\]

\[
\begin{array}{c}
\text{G(s) = 1/s}
\end{array}
\]

\[
\begin{array}{c}
\text{H(s) = K}
\end{array}
\]

(a) Assume \( K = 0 \). Find the response of the system to a step input \( x(t) = u(t) \). Is the system stable? Explain.

(b) Find a real \( K \) such that the system is stable.

(c) Find the response to a step input \( x(t) = u(t) \) with the \( K \) as in (b).

(d) **Extra credit:** using the final value theorem, find

\[
\lim_{t \to \infty} y(t)
\]

when \( x(t) = u(t) \) as in (c). Make sure your answer agrees with the limit evaluated directly from the step response in (c).

5. **Computing z-transforms**

For the following signals, determine bilateral z-transforms and regions of convergence. Note that if the region of convergence is null, the z-transform does not exist.

(a) \( x[n] = \delta[n] - 3u[n - 5] \).

(b) \( x[n] = \frac{\beta^n}{n!} u[n] \).

(c) \( \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{2}\right)^n u[-n] \).

(d) \( x[n] = a^n u[n - 1] \).

(e) \( x[n] = -a^n u[-n] \).
MATLAB Assignment

General Instructions

(30 pts)

Answer all questions asked. Your submission should include all m-file listings and plots requested. All plots should have a title and x- and y-axes properly labeled.

In the following problems, you will examine the pole locations for second-order systems of the form

\[ H(s) = \frac{\omega_n^2}{s^2 + 2\eta \omega_n s + \omega_n^2} \]

(1)

The values of the damping ratio \( \eta \) and undamped natural frequency \( \omega_n \) specify the locations of the poles, and consequently the behavior of this system. In this exercise, you will see how the locations of the poles affect the frequency response.

First, you will examine the pole locations and frequency responses for four different choices of \( \eta \) while \( \omega_n \) remains fixed at 1.

a) Define \( H_1(s) \) through \( H_4(s) \) to be the system functions that result from fixing \( \omega_n = 1 \) in Eq. (1) above while \( \eta \) is 0, \( \frac{1}{4} \), 1, and 2, respectively. Define \( a_1 \) through \( a_4 \) to be the coefficient vectors for the denominators of \( H_1(s) \) through \( H_4(s) \). Find and plot the locations of the poles for each of these systems. Note that the command \( \text{roots([a,b,c])} \) returns the values of the roots (in this case poles) of an equation of the form \( ax^2 + bx + c \). To plot the poles, display the imaginary part on the y-axis and the real part on the x-axis.

b) Define \( \omega = [\text{-}5:0.1:5] \) to be the frequencies at which you will compute the frequency responses of the four systems. Use the Matlab command \( \text{freqs} \) to compute and plot \( |H(j\omega)| \) for each of the four systems you defined in Part (a). How are the frequency responses for \( \eta < 1 \) qualitatively different from those for \( \eta \geq 1 \)? Can you explain how the pole locations for the systems cause this difference?

Next, you will trace the locations of the poles as you vary \( \eta \) and \( \omega_n \) and see how varying these parameters affects the frequency response of the system.

c) First, you will vary \( \eta \) in the range \( 0 \leq \eta \leq 10 \) while holding \( \omega_n = 1 \). Define \( \text{etarange}=\left[0 \ \text{logspace(-1,1,99)}\right] \) to get 100 logarithmically spaced points in \( 0 \leq \eta \leq 10 \). Define \( a_\text{eta} \) to be a \( 3 \times 100 \) matrix where each column is the denominator coefficients for \( H(s) \) when \( \eta \) has the value in the corresponding column of \( \text{etarange} \). Notice that only the middle coefficient is changing with each new value of \( \eta \). Define \( \text{etapoles} \) to be a \( 2 \times 100 \) matrix where each column is the roots of the corresponding column of \( a_\text{eta} \). On a single figure, plot the real versus imaginary parts for each row of \( \text{etapoles} \) and describe the loci they trace. On your plot, indicate the following points: \( \eta = 0, \frac{1}{4}, 1, \text{and 2} \) (you can use something like the \( \text{scatter} \) command to show these individual points). In order to get a square aspect ratio with equal length axes in your plot, you can type:
Describe qualitatively how you expect the frequency response to change as $\eta$ goes from 0 to 1 and then from 1 to 10.

d) In this problem, you will hold $\eta = \frac{1}{4}$ and examine the effect of increasing $\omega_n$ from 0 to 10. Define $\text{omegarange} = [0 \ \text{logspace}(-1, 1, 99)]$ to get 100 logarithmically spaced points in the region of interest. Define $a_\omega$ and $\omega_p$ analogously to the way you defined $a_\eta$ and $\eta_p$ in Part (c). On a single figure, plot the real versus imaginary parts for each column of $\omega_p$. How would you expect changing $\omega_n$ to change the frequency response $H(j\omega)$? Use $\text{freqs}$ to evaluate the frequency response when $\omega_n = 2$ and $\eta = \frac{1}{4}$ and plot the magnitude of this frequency response. Compare this with the plot you made in Part (b) for $\omega_n = 1$ and $\eta = \frac{1}{4}$. How are they different? Does this match what you expected from your plot of the loci traced by $\omega_p$?

e) Extra Credit: There is no reason that $\eta$ needs to be positive. Repeat Part (c) for $\eta$ between -10 and 0. When $\eta$ is negative, can the system described by $H(s)$ be both causal and stable? Also, plot the frequency response magnitude for the system when $\eta = -\frac{1}{4}$ and $\omega_n = 1$ using $\text{freqs}$. Is the system with the frequency response computed by $\text{freqs}$ causal? Also explain any similarities or differences between this plot and the frequency response magnitude plotted in Part (b) for $\eta = \frac{1}{4}$.