Problem Set #8
Due: Thursday June 7, 2018 at 5 PM. This is a hard deadline and no late submissions will be accepted.

1. Infer the Z Transform of a signal
We are given the following five facts about a discrete-time signal $x[n]$ with transform $X(z)$

- $x[n]$ is real and right-sided.
- $X(z)$ has exactly two poles.
- $X(z)$ has two zeros at the origin.
- $X(z)$ has a pole at $z = \frac{1}{2} e^{j\pi/3}$
- $X(1) = \frac{8}{3}$

Determine $X(z)$ and specify its region of convergence.

2. Equations in z-transforms
Let the $z$-transform of $x[n]$ be given by

$$X(z) = \frac{1 + z^{-1}}{1 - \frac{1}{3} z^{-1}}.$$ 

Without computing the time-domain signal $x[n]$, use the $z$-transform tables of pairs and properties to find the $z$-transform of the following signals and the respective regions of convergence. Note that $x[n]$ may be right- or left-sided. Your answer should contain a total of 6 cases: for each of the three cases below, find $Y(z)$ for $x[n]$ being a right- and left-sided signal, respectively.

(a) $y[n] = x[n] - x[n - 2]$.
(b) $y[n] = (2 - n)x[n]$.
(c) $y[n] = \left(\frac{1}{3}\right)^n x[n]$. 

3. Stabilizing unstable system with feedback
Consider a digital filter structure as shown below:

(a) Find $H(z)$ for this causal filter. Plot the pole-zero pattern and indicate the ROC.
(b) For what values of $k$ is the system stable?
(c) Assume $k = 5$, which causes the system to be unstable. Now to make the system stable, we add a feedback as shown below:

Find a real value of $K_1$ such that the system is stable.

4. Continuous Time Filter approximation with the Z transform
The bilinear transformation may be used to obtain an Infinite Impulse Response discrete-time filter, the magnitude of whose frequency response is similar to the magnitude of the frequency response of a given continuous-time lowpass filter. In this problem, we illustrate the similarity through the example of a continuous-time second-order Butterworth filter with system function $H_c(s)$.

(a) Let

$$H_d(z) = H_c(s) \bigg|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$$

Show that

$$H_d(e^{j\Omega}) = H_c(j\omega) \bigg|_{\omega=\tan(\Omega/2)}$$

(b) Given that

$$H_c(s) = \frac{1}{(s+e^{j\pi/4})(s+e^{-j\pi/4})}$$

and that the corresponding filter is causal, verify that $H_c(0) = 1$, that $|H_c(j\omega)|$ decreases monotonically with increasing positive values of $\omega$, that $|H_c(j)^2| = 1/2$ (i.e., that $\omega_c = 1$ is the half-power frequency), and that $H_c(\infty) = 0$. 2
(c) Give an expression for $H_d(z)$, the Bilinear Z transform applied to $H_c(s)$ in part (b).

(d) Show that the following may be asserted about $H_d(z)$ and $H_d(e^{j\Omega})$:
   
   i. $H_d(z)$ has only two poles, both of which are inside the unit circle.
   
   ii. $H_d(e^{j0}) = 1$.
   
   iii. $|H_d(e^{j\Omega})|$ decreases monotonically as $\Omega$ goes from 0 to $\pi$.
   
   iv. The half-power frequency of $H_d(e^{j\Omega})$ is $\frac{\pi}{2}$.

(e) Find a difference equation relating $x[n]$ and $y[n]$, the input and output of the discrete IIR filter respectively, using your expression for $H_d(z)$.

(f) Write a Direct Form I realization of the filter.