Outline

- Course Basics
- Course Syllabus
- Guest Lectures
- Review of 102a
  - Signals and Systems in the Time Domain
    - Continuous and Discrete-Time Signals/Systems
    - Convolution
  - Continuous-Time Signals in the Frequency Domain
    - Fourier Series and Fourier Transforms
  - Discrete Time Signals in the Frequency Domain
    - Fourier Series and Fourier Transforms
Course Information*

People

• Instructor: Andrea Goldsmith, andrea@ee, Packard 371, OHs: MWF 11:30-12:30 and by appt.

• TAs: Mainak Chowdhury (mainakch@stanford.edu) Alon Kipnis (kipnisal@stanford.edu) Jeremy Kim (jpkim@stanford.edu)
  • Discussion: TBD (tentative Monday afternoon)
  • OHs and email OHs: TDB
  • Piazza Website: piazza.com/stanford/spring2016/ee102b

• Class Administrator: Julia Gillespie, jvgill@stanford, Packard 365, 3-2681. Homework dropoff: Wed by 5 pm.

*See web or handout for more details
Course Information
Nuts and Bolts

● Prerequisites: EE102a or equivalent knowledge

● Required Reading: EE102b course reader
  ● Available at bookstore or on the class website

● Recommended/Supplemental texts

● Class Homepage: www.stanford.edu/class/ee102b
  ● All announcements, handouts, homeworks, etc. posted to website

● Mailing List: ee102b-spr1516-students@lists: all students on list. Mailing ee102b-spr1516-staff@lists reaches me and TAs.
Course Information Policies

- **Grading:** HW – 35%, Midterm – 25%, Final - 40%

- **HWs:** assigned Wed, due following Wed 5pm (starts 3/30)
  - Homeworks lose 33% credit after 5pm Wed, lowest HW dropped
  - Can collaborate on HWs with others, but need an individual writeup
  - Collaboration means all collaborators work out all problems together
  - Unpermitted collaboration or aid (e.g. solns for the book or from prior years) is an honor code violation and will be dealt with strictly.
  - Extra credit: MATLAB HWs problem may have extra credit extensions

- **Exams:**
  - Midterm week of 5/2. (It will be scheduled outside class time; the duration is 2 hours.) Final on 6/7 from 8:30am-11:30am.
  - Exams **must** be taken at scheduled time (with very few exceptions)

- **TGIF:** Breakfast treats at Friday lectures
Course Syllabus

Corresponds to Reader Topics

1. Review of 102a, examples and applications
2. Sampling and reconstruction
3. Communication systems
4. Finite impulse response discrete-time filters
5. Discrete Fourier transforms and applications
6. Laplace transforms and applications
7. Feedback control systems
8. Z transforms and discrete-time control systems
9. IIR filter design

Disclaimer: Last taught this class 21 years ago at Caltech (first class ever taught)
### Very Tentative Syllabus (24 lectures + 4 guest lectures)

<table>
<thead>
<tr>
<th>Topics</th>
<th>Recommended/Supplementary Reading</th>
<th>Lecture Dates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Course Overview, Review of 102a, Examples and Applications</td>
<td>OWN chps: 3-5, esp. Tables 3.1, 3.2, 4.1, 4.2, 5.1, 5.2, 5.3 (in today’s lecture)</td>
<td>3/28-3/30</td>
</tr>
<tr>
<td>2. Sampling and reconstruction</td>
<td>OWN 7.0-7.6; OS 4.0-4.6.</td>
<td>4/1-4/6</td>
</tr>
<tr>
<td>3. Communication systems</td>
<td>OWN 8.0-8.4</td>
<td>4/8-4/13</td>
</tr>
<tr>
<td>Guest Lecture: Sachin Katti (Full duplex wireless radios)</td>
<td></td>
<td>4/15</td>
</tr>
<tr>
<td>5. Discrete Fourier transforms and applications</td>
<td>OS 8.1-8.7, 10.1-10.2, 9.2.</td>
<td>4/22-4/27</td>
</tr>
<tr>
<td>Guest Lecture: Gordon Wetzstein (Computational Imaging)</td>
<td></td>
<td>4/29</td>
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</tbody>
</table>

**Midterm (2 hours) to be scheduled the week of 5/2**

| 6. Laplace transforms and applications                                | OWN 9.0-9.10, A.1-A.2, 6.5.                                                                       | 5/2-5/6         |

**Guest Lecture: Paul Nuyujukian (Brain-Machine Interfaces and Neuro Signal Processing)**

| 9. IIR Filter design                                                  |                                                                                                  |                 |
| Tour: Professor Oussama Khatib’s Robotics Lab                        |                                                                                                  | 6/1             |
| Course Summary                                                       |                                                                                                  | 6/3             |

**Final Exam June 7th, 8:30-11:30am**
Guest Lectures/Lab Tour

- Gordon Wetzstein: Computational Imaging
- Sachin Katti: Future Wireless Networks
- Paul Nuyujukian: Brain-Machine Interfaces
  https://www.youtube.com/watch?v=gvR0kHm9fwo
- Oussama Khatib’s Robotics Lab
Review of 102a

Only what you need for 102b

Or other emergencies...

- **Signals and Systems in the Time Domain**
  - Continuous- and Discrete-Time Signals
  - Linear systems
  - Convolution

- **Continuous-Time Signals in the Frequency Domain**
  - Fourier Series
  - Fourier Transform

- **Discrete Time Signals in the Frequency Domain**
  - Discrete Time Fourier Series
  - Discrete Time Fourier Transform
A continuous-time signal has values for all points in time in some (possibly infinite) interval.

A discrete-time signal has values at discrete points in time.
- Can be (finite/infinite precision) samples of a cts-time signal:

Signals can also be a function of space or of space and time.
- For example, images and videos
Types of Systems

- **Linear, Time-Invariant (LTI) Systems:** $y(t) = F[x(t)]$
  - Linear: $F[ax(t) + bw(t)] = aF[x(t)] + bF[w(t)]$
  - Time-Invariant: $y(t) = F[x(t)] \rightarrow y(t - \tau) = F[x(t - \tau)]$

- **Continuous-time systems**
  - Continuous-time inputs and outputs
  - Examples: AM/FM radio, TV (not HD), circuits

- **Discrete-time systems**
  - Discrete-time inputs and outputs
  - Examples: computers, bank accounts, digital cameras

- **Hybrid systems**
  - Inputs and outputs are mixed discrete and continuous
  - Examples: Sampler, cell phone text
Why are Discrete-Time Signals and Systems Important?

Most signals in nature are continuous in time, space, or both.

Discretizes time (space) and values of continuous signals.

Building the right bridge avoids data overload.

Analog to Digital conversion (ADC) provides the bridge.

Data storage and processing (e.g., MATLAB) are digital.

010010011001
001000010000
1000100111...

### Important Continuous Signals

- **Sinuoidal signals**
  - \( \cos(\omega t) = \sin(\omega t + \pi/2) \)

- **Complex Exponentials:**
  - \( e^{j\omega t + \theta} = \cos(\omega t + \theta) + j\sin(\omega t + \theta) \)

- **Exponential signals:**
  - \( e^{-t}, e^t \)

- **Unit step and unit ramp**
  - \( u(t), r(t) \)

- **Impulse functions:** \( \delta(t) \)
Rectangles and Triangles

- **Rectangular signals**

  \[ \Pi(x) = \begin{cases} 1 & |x| \leq \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases} \]

- **Triangular Signals**

  \[ \Lambda(x) = \begin{cases} 1 - |x| & |x| \leq 1 \\ 0 & |x| > 1 \end{cases} \]

  \[ \Lambda(x) = \Pi(x) * \Pi(x) = \int_{-\infty}^{\infty} \Pi(x') \Pi(x - x') dx' \]
Fourier Series

- Exponentials/Sinusoids are basis functions for periodic signals
- Can represent periodic signal in terms of FS coefficients
- Complex coefficients are frequency components of signal

\[
x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}
\]

\[
a_k = \frac{1}{T_0} \int_{T_0}^{T_0} x(t) e^{-jk\omega_0 t} dt
\]

Infinite Frequency Content
<table>
<thead>
<tr>
<th>Property</th>
<th>Section</th>
<th>Periodic Signal</th>
<th>Fourier Series Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$x(t)$, $y(t)$ Periodic with period $T$ and fundamental frequency $\omega_0 = 2\pi/T$</td>
<td>$a_k$, $b_k$</td>
</tr>
<tr>
<td></td>
<td>3.5.1</td>
<td>$Ax(t) + By(t)$</td>
<td>$Aa_k + Bb_k$</td>
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<tr>
<td>Linearity</td>
<td></td>
<td></td>
<td>$a_k e^{-j\omega_0 t}$</td>
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<tr>
<td></td>
<td>3.5.2</td>
<td>$x(t - t_0)$</td>
<td>$a_{k-M}$</td>
</tr>
<tr>
<td>Time Shifting</td>
<td></td>
<td>$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$</td>
<td>$a_{-k}$</td>
</tr>
<tr>
<td>Frequency Shifting</td>
<td></td>
<td></td>
<td>$a_k$</td>
</tr>
<tr>
<td>Conjugation</td>
<td>3.5.6</td>
<td>$x^*(t)$</td>
<td>$A*_{-k}$</td>
</tr>
<tr>
<td>Time Reversal</td>
<td>3.5.3</td>
<td>$x(-t)$</td>
<td>$a_{-k}$</td>
</tr>
<tr>
<td>Time Scaling</td>
<td>3.5.4</td>
<td>$x(\alpha t), \alpha &gt; 0$ (periodic with period $T/\alpha$)</td>
<td>$T a_k b_k$</td>
</tr>
<tr>
<td>Periodic Convolution</td>
<td></td>
<td>$\int_T x(\tau)y(t - \tau) d\tau$</td>
<td>$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$</td>
</tr>
<tr>
<td>Multiplication</td>
<td>3.5.5</td>
<td>$x(t)y(t)$</td>
<td>$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$</td>
</tr>
<tr>
<td>Differentiation</td>
<td></td>
<td>$\frac{dx(t)}{dt}$</td>
<td>$a_k = a_{-k}$</td>
</tr>
<tr>
<td>Integration</td>
<td></td>
<td>$\int_{-\infty}^{\infty} x(t) dt$ (finite valued and periodic only if $a_0 = 0$)</td>
<td>$\Re{a_k} = \Re{a_{-k}}$</td>
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<td></td>
<td></td>
<td></td>
<td>$\Im{a_k} = -\Im{a_{-k}}$</td>
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<tr>
<td></td>
<td>3.5.6</td>
<td>$x(t)$ real</td>
<td>$a_k$ real and even</td>
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<tr>
<td>Conjugate Symmetry for</td>
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<td></td>
<td>$a_k$ purely imaginary and odd</td>
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<tr>
<td>Real Signals</td>
<td></td>
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<td>$\Re{a_k}$</td>
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<tr>
<td></td>
<td>3.5.6</td>
<td>$x(t)$ real and even</td>
<td>$\Im{a_k}$</td>
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<tr>
<td>Real and Even Signals</td>
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<td>$j\Im{a_k}$</td>
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<td></td>
<td>3.5.6</td>
<td>$x(t)$ real and odd</td>
<td>$\Re{a_k}$</td>
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<tr>
<td>Real and Odd Signals</td>
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<td>$\Im{a_k}$</td>
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<tr>
<td>Even-Odd Decomposition</td>
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<td></td>
<td>$\Re{a_k}$</td>
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<tr>
<td>of Real Signals</td>
<td></td>
<td></td>
<td>$j\Im{a_k}$</td>
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<tr>
<td>Parseval's Relation for</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Periodic Signals</td>
<td></td>
<td>$\frac{1}{T} \int_T</td>
<td>x(t)</td>
</tr>
</tbody>
</table>
Fourier Transform

- Represents spectral (frequency) components of a signal
- Signal uniquely represented in time or frequency domain
  - Fourier transform and inverse is a 1-1 mapping between domains
- Fourier transform generally complex
  - Characterized by amplitude and phase

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega \quad \leftrightarrow \quad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt
\]

\[
x(t) = A\Pi(t/T) \iff X(j\omega) = AT\text{sinc}(0.5\omega T / \pi) = AT \sin(0.5\omega T) / 0.5\omega T
\]
<table>
<thead>
<tr>
<th>Section</th>
<th>Property</th>
<th>Aperiodic Signal</th>
<th>Fourier Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.3.1</td>
<td>Linearity</td>
<td>$ax(t) + by(t)$</td>
<td>$aX(j\omega) + bY(j\omega)$</td>
</tr>
<tr>
<td>4.3.2</td>
<td>Time Shifting</td>
<td>$x(t - t_0)$</td>
<td>$e^{-j\omega t_0}X(j\omega)$</td>
</tr>
<tr>
<td>4.3.6</td>
<td>Frequency Shifting</td>
<td>$e^{j\omega_0 t}x(t)$</td>
<td>$X(j(\omega - \omega_0))$</td>
</tr>
<tr>
<td>4.3.3</td>
<td>Conjugation</td>
<td>$x^*(t)$</td>
<td>$X^*(-j\omega)$</td>
</tr>
<tr>
<td>4.3.5</td>
<td>Time Reversal</td>
<td>$x(-t)$</td>
<td>$X(-j\omega)$</td>
</tr>
<tr>
<td></td>
<td>Time and Frequency Scaling</td>
<td>$x(at)$</td>
<td>$\frac{1}{</td>
</tr>
<tr>
<td>4.4</td>
<td>Convolution</td>
<td>$x(t) \ast y(t)$</td>
<td>$X(j\omega)Y(j\omega)$</td>
</tr>
<tr>
<td>4.5</td>
<td>Multiplication</td>
<td>$x(t)y(t)$</td>
<td>$\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta)Y(j(\omega - \theta))d\theta$</td>
</tr>
<tr>
<td>4.3.4</td>
<td>Differentiation in Time</td>
<td>$\frac{d}{dt} x(t)$</td>
<td>$j\omega X(j\omega)$</td>
</tr>
<tr>
<td>4.3.6</td>
<td>Integration</td>
<td>$\int_{-\infty}^{t} x(t)dt$</td>
<td>$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$</td>
</tr>
<tr>
<td>4.3.6</td>
<td>Differentiation in Frequency</td>
<td>$tx(t)$</td>
<td>$\frac{j}{d\omega}X(j\omega)$</td>
</tr>
<tr>
<td>4.3.3</td>
<td>Conjugate Symmetry for Real Signals</td>
<td>$x(t)$ real</td>
<td>$X(j\omega) = X^*(-j\omega)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\Re{X(j\omega)} = \Re{X(-j\omega)}$</td>
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<td>$\Im{X(j\omega)} = -\Im{X(-j\omega)}$</td>
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<td></td>
<td>$\angle X(j\omega) = -\angle X(-j\omega)$</td>
</tr>
<tr>
<td>4.3.3</td>
<td>Symmetry for Real and Even Signals</td>
<td>$x(t)$ real and even</td>
<td>$X(j\omega)$ real and even</td>
</tr>
<tr>
<td>4.3.3</td>
<td>Symmetry for Real and Odd Signals</td>
<td>$x(t)$ real and odd</td>
<td>$X(j\omega)$ purely imaginary and odd</td>
</tr>
<tr>
<td>4.3.3</td>
<td>Even-Odd Decomposition for Real Signals</td>
<td>$x_e(t) = \delta v{x(t)}$</td>
<td>$[x(t)$ real]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x_o(t) = \delta d{x(t)}$</td>
<td>$[x(t)$ real]</td>
</tr>
<tr>
<td>4.3.7</td>
<td>Parseval’s Relation for Aperiodic Signals</td>
<td>$\int_{-\infty}^{\infty}</td>
<td>x(t)</td>
</tr>
<tr>
<td>Signal</td>
<td>Fourier transform</td>
<td>Fourier series coefficients (if periodic)</td>
<td></td>
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<tr>
<td>--------------------------------</td>
<td>--------------------------------------------</td>
<td>-----------------------------------------</td>
<td></td>
</tr>
<tr>
<td>$\sum_{k=-\infty}^{\infty} a_k e^{jkw_0}$</td>
<td>$2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - kw_0)$</td>
<td>$a_k$</td>
<td></td>
</tr>
<tr>
<td>$e^{jw_0t}$</td>
<td>$2\pi \delta(\omega - w_0)$</td>
<td>$a_1 = 1$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$a_k = 0$, otherwise</td>
<td></td>
</tr>
<tr>
<td>$\cos \omega_0 t$</td>
<td>$\pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$</td>
<td>$a_1 = a_{-1} = \frac{i}{2}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$a_k = 0$, otherwise</td>
<td></td>
</tr>
<tr>
<td>$\sin \omega_0 t$</td>
<td>$\frac{\pi}{j} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$</td>
<td>$a_1 = -a_{-1} = \frac{-i}{2j}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$a_k = 0$, otherwise</td>
<td></td>
</tr>
<tr>
<td>$x(t) = 1$</td>
<td>$2\pi \delta(\omega)$</td>
<td>$a_0 = 1$, $a_k = 0$, $k \neq 0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(this is the Fourier series representation for any choice of $T &gt; 0$)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Periodic square wave</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x(t) = \begin{cases} 1, &amp;</td>
<td>t</td>
<td>&lt; T_1 \ 0, &amp; T_1 &lt;</td>
<td>t</td>
</tr>
<tr>
<td>and</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x(t + T) = x(t)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sum_{n=-\infty}^{\infty} \delta(t - nT)$</td>
<td>$\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T})$</td>
<td>$a_k = \frac{1}{T}$ for all $k$</td>
<td></td>
</tr>
<tr>
<td>$x(t) = \begin{cases} 1, &amp;</td>
<td>t</td>
<td>&lt; T_1 \ 0, &amp;</td>
<td>t</td>
</tr>
<tr>
<td>$\sin \frac{Wh_t}{\pi t}$</td>
<td>$X(j\omega) = \begin{cases} 1, &amp;</td>
<td>\omega</td>
<td>&lt; W \ 0, &amp;</td>
</tr>
<tr>
<td>$\delta(t)$</td>
<td>$1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$u(t)$</td>
<td>$\frac{1}{j\omega} + \pi \delta(\omega)$</td>
<td></td>
<td></td>
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<tr>
<td>$\delta(t - t_0)$</td>
<td>$e^{-j\omega t_0}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e^{-\alpha t} u(t), \Re(\alpha) &gt; 0$</td>
<td>$\frac{1}{a + j\omega}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$te^{-\alpha t} u(t), \Re(\alpha) &gt; 0$</td>
<td>$\frac{1}{(a + j\omega)^2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{e^{-\alpha t}}{(t+1)} e^{-\alpha t} u(t), \Re(\alpha) &gt; 0$</td>
<td>$\frac{1}{(a + j\omega)^n}$</td>
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</tbody>
</table>
Convolution

\[ y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau \]

Flip \( x(t) \), “drag it” through \( h(t) \), compute area at each drag location
(can reverse roles of \( x(t) \) and \( h(t) \))

Easy to flip

\( x(t) \)

\( h(t) \)

https://www.youtube.com/watch?v=jwlfSIBNqP8

- Multiplication in time \( \Leftrightarrow \) Convolution in Frequency
- Convolution in time \( \Leftrightarrow \) Multiplication in Frequency
Key FT Properties: Filtering

- Filtering signals in time $\Leftrightarrow$ multiplication in frequency of signal FT and filter frequency response

\[
\sum_k a_k e^{jk\omega t} \xrightarrow{X(j\omega)} h(t) \xrightarrow{H(j\omega)} y(t) = h(t) * x(t) \quad \Rightarrow \quad \sum_k a_k H(jk\omega_0) e^{jk\omega_0 t} = Y(j\omega) = H(j\omega)X(j\omega)
\]

- Filtering in time is convolution:
  \[
x(t) * h(t) = \int_{-\infty}^{\infty} x(t) h(t - \tau) d\tau
\]
- $h(t)$ is “impulse response” of filter
  - Output for spike input: hard to measure in practice
- Often easier to analyze filtering in the frequency domain
- Exponentials are eigenfunctions of LTI filters
Key FT Properties: Modulation

- Frequency shifting
  - Modulation (multiplying a time signal by an exponential) leads to a frequency shift.

\[
x(t) e^{j\omega_c t} \quad \Rightarrow \quad y(t) = x(t) e^{-j\omega_c t}
\]

Hotel California

\[
x(t) \quad \Rightarrow \quad y(t) = x(t) e^{-j\omega_c t} \quad \Rightarrow \quad x(t)
\]

\[ f_c = \frac{\omega_c}{2\pi} = 106.7 \]

Will cover AM/FM/Digital modulation a bit later in the course
Sampling

- Sampling (Time):
  
  \[ x(t) = \sum_n \delta(t-nT_s) \]

  Creates a discrete signal
  \[ x[n]=x_s(nT_s) \]

- Sampling (Frequency)
  
  \[ X(j\omega) * \frac{2\pi}{T_s} \sum_n \delta(t-n/T_s) = X_s(j\omega) \]
Nyquist Sampling Theorem

- A bandlimited signal \([-2\pi B, 2\pi B]\) radians is completely described by samples every \(.5/B\) secs.
  - Nyquist rate is \(2B\) samples/sec

- Recreate signal from its samples by using a low pass filter in the frequency domain
  - Sinc interpolation in time domain
  - Undersampling creates aliasing

More sophisticated sampling/ADC ideas starting in Friday’s lecture
Common Discrete Time Signals

Sampled versions of common continuous time signals

- **Sinuoidal signals**
  \[ x[n] = x(t)|_{t=nT} = A\cos(\omega_0 n T + \theta) \]

- **Exponential signals:**

- **Unit step, ramp, and impulse**
  \[ u[n] = \begin{cases} 
  0 & n < 0 \\
  1 & n \geq 0 
\end{cases} \]
  \[ r[n] = n \cdot u[n] = \begin{cases} 
  0 & n < 0 \\
  n & n \geq 0 
\end{cases} \]
  \[ \delta[n] = \begin{cases} 
  0 & n \neq 0 \\
  1 & n = 0 
\end{cases} \]
Discrete Time Fourier Series

- Discrete-time exponentials are basis functions for discrete-time periodic signals
- Can represent DT periodic signal in terms of FS coefficients

\[
x[n] = \frac{1}{N} \sum_{k=-N_0}^{N_0} a_k e^{j k \omega_0 n} = \frac{1}{N} \sum_{k=-N_0}^{N_0} a_k e^{j (2\pi/N_0) n} \quad \Leftrightarrow \quad a_k = \frac{1}{N} \sum_{n=-N_0}^{N_0} x[n] e^{-j k \omega_0 n}
\]
TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

<table>
<thead>
<tr>
<th>Property</th>
<th>Periodic Signal</th>
<th>Fourier Series Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x[n]$ Periodic with period $N$ and $y[n]$ fundamental frequency $\omega_0 = 2\pi/N$</td>
<td>$a_k$ Periodic with $b_k$ period $N$</td>
</tr>
<tr>
<td>Linearity</td>
<td>$Ax[n] + By[n]$</td>
<td>$Aa_k + Bb_k$</td>
</tr>
<tr>
<td>Time Shifting</td>
<td>$x[n - n_0]$</td>
<td>$a_k e^{-j(2\pi/N)n_0}$</td>
</tr>
<tr>
<td>Frequency Shifting</td>
<td>$e^{jM(2\pi/N)n}x[n]$</td>
<td>$a_{k-M}$</td>
</tr>
<tr>
<td>Conjugation</td>
<td>$x^*[n]$</td>
<td>$a_{-k}^*$</td>
</tr>
<tr>
<td>Time Reversal</td>
<td>$x[-n]$</td>
<td>$a_{-k}$</td>
</tr>
<tr>
<td>Time Scaling</td>
<td>$x_{(m)}[n] = \begin{cases} x[n/m], &amp; \text{if } n \text{ is a multiple of } m \ 0, &amp; \text{if } n \text{ is not a multiple of } m \end{cases}$</td>
<td>$\frac{1}{m}a_k$ (viewed as periodic with period $mN$)</td>
</tr>
<tr>
<td>Periodic Convolution</td>
<td>$\sum_{r=-(N)}^{N} x[r]y[n - r]$</td>
<td>$Na_kb_k$</td>
</tr>
<tr>
<td>Multiplication</td>
<td>$x[n]y[n]$</td>
<td>$\sum_{l=-(N)}^{N} a_lb_{k-l}$</td>
</tr>
<tr>
<td>First Difference</td>
<td>$x[n] - x[n - 1]$</td>
<td>$(1 - e^{-j(2\pi/N)})a_k$</td>
</tr>
<tr>
<td>Running Sum</td>
<td>$\sum_{k=-\infty}^{\infty} x[k]$</td>
<td>$\left(\frac{1}{1 - e^{-j(2\pi/N)}}\right)a_k$</td>
</tr>
<tr>
<td>Conjugate Symmetry for</td>
<td>$x[n]$ real</td>
<td>$a_k = a_{-k}^*$</td>
</tr>
<tr>
<td>Real Signals</td>
<td></td>
<td>$\Re{a_k} = \Re{a_{-k}}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\Im{a_k} = -\Im{a_{-k}}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\angle a_k = -\angle a_{-k}$</td>
</tr>
<tr>
<td>Real and Even Signals</td>
<td>$x[n]$ real and even</td>
<td>$a_k$ real and even</td>
</tr>
<tr>
<td>Real and Odd Signals</td>
<td>$x[n]$ real and odd</td>
<td>$a_k$ purely imaginary and odd</td>
</tr>
<tr>
<td>Even-Odd Decomposition of</td>
<td>$\begin{cases} x_e[n] = \delta v(x[n]) &amp; [x[n]$ real] \ x_o[n] = \delta o(x[n]) &amp; [x[n]$ real] \end{cases}$</td>
<td>$\Re{a_k}$</td>
</tr>
<tr>
<td>Real Signals</td>
<td></td>
<td>$j\Im{a_k}$</td>
</tr>
<tr>
<td>Parseval’s Relation for</td>
<td>$\frac{1}{N} \sum_{n=-(N)}^{(N)}</td>
<td>x[n]</td>
</tr>
</tbody>
</table>
Discrete Time Fourier Transform

- Take a non-periodic discrete signal $x[n]$
  - Periodize (repeat) it every $N$, yields periodic signal $\tilde{x}(n)$
  - The periodic signal has a discrete-time Fourier series
- Letting $N \to \infty$ yields discrete-time Fourier Transform

\[
\tilde{x}[n] = \frac{1}{N} \sum_{k=\lfloor N \rfloor} a_k e^{j(2\pi/N)n}
\]

\[
a_k = \frac{1}{N} \sum_{n=\lfloor N \rfloor} \tilde{x}[n] e^{-jk(2\pi/N)n}
\]

\[
x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\Omega}) e^{j\Omega} d\omega
\]

\[
X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}
\]

Periodic in $2\pi$
Example: Rectangular Pulse

\[ x[n] = \begin{cases} 
1 & |n| \leq N_1 \\
0 & |n| > N_1 
\end{cases} \]

\[ X(e^{j\Omega}) = \frac{\sin \left( \frac{\Omega}{2}(2N_1 + 1) \right)}{\sin \left( \frac{\Omega}{2} \right)} \]

\[ N_1 = 2, \quad 2N_1 + 1 = 5 \]
### Table 5.1 Properties of the Discrete-Time Fourier Transform

<table>
<thead>
<tr>
<th>Section</th>
<th>Property</th>
<th>Aperiodic Signal</th>
<th>Fourier Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.3.2</td>
<td>Linearity</td>
<td>(x[n])</td>
<td>(X(e^{j\omega})) periodic with (Y(e^{j\omega})) period (2\pi)</td>
</tr>
<tr>
<td>5.3.3</td>
<td>Time Shifting</td>
<td>(ax[n] + by[n])</td>
<td>(aX(e^{j\omega}) + bY(e^{j\omega}))</td>
</tr>
<tr>
<td>5.3.3</td>
<td>Frequency Shifting</td>
<td>(x[n - n_0])</td>
<td>(e^{-j\omega n_0}X(e^{j\omega}))</td>
</tr>
<tr>
<td>5.3.4</td>
<td>Conjugation</td>
<td>(x^*[n])</td>
<td>(X^*(e^{-j\omega}))</td>
</tr>
<tr>
<td>5.3.5</td>
<td>Time Reversal</td>
<td>(x[-n])</td>
<td>(X(e^{-j\omega}))</td>
</tr>
<tr>
<td>5.3.7</td>
<td>Time Expansion</td>
<td>(x_{(\ell)}[n] = \begin{cases} x[n/k] &amp; \text{if } n = \text{multiple of } k \ 0 &amp; \text{if } n \neq \text{multiple of } k \end{cases})</td>
<td>(X(e^{jk\omega}))</td>
</tr>
<tr>
<td>5.4</td>
<td>Convolution</td>
<td>(x[n] * y[n])</td>
<td>(X(e^{j\omega})Y(e^{j\omega}))</td>
</tr>
<tr>
<td>5.5</td>
<td>Multiplication</td>
<td>(x[n]y[n])</td>
<td>(\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta)</td>
</tr>
<tr>
<td>5.3.5</td>
<td>Differencing in Time</td>
<td>(x[n] - x[n - 1])</td>
<td>((1 - e^{-j\omega})X(e^{j\omega}))</td>
</tr>
<tr>
<td>5.3.5</td>
<td>Accumulation</td>
<td>(\sum_{k=-\infty}^{n} x[k])</td>
<td>(\frac{1}{1 - e^{-j\omega}}X(e^{j\omega}))</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(+ \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k))</td>
</tr>
<tr>
<td>5.3.8</td>
<td>Differentiation in Frequency</td>
<td>(nx[n])</td>
<td>(j \frac{dX(e^{j\omega})}{d\omega})</td>
</tr>
<tr>
<td>5.3.4</td>
<td>Conjugate Symmetry for Real Signals</td>
<td>(x[n]) real</td>
<td>(X(e^{j\omega}) = X^*(e^{-j\omega}))</td>
</tr>
<tr>
<td>5.3.4</td>
<td>Symmetry for Real, Even Signals</td>
<td>(x[n]) real and even</td>
<td>(\Re{X(e^{j\omega})} = \Re{X(e^{-j\omega})})</td>
</tr>
<tr>
<td>5.3.4</td>
<td>Symmetry for Real, Odd Signals</td>
<td>(x[n]) real and odd</td>
<td>(\Im{X(e^{j\omega})} = -\Im{X(e^{-j\omega})})</td>
</tr>
<tr>
<td>5.3.4</td>
<td>Even-odd Decomposition of Real Signals</td>
<td>(x_e[n] = \Re{x[n]}) [(x[n]) real]</td>
<td>(</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(x_o[n] = \Im{x[n]}) [(x[n]) real]</td>
<td>(\Im{X(e^{j\omega})})</td>
</tr>
<tr>
<td>5.3.9</td>
<td>Parseval's Relation for Aperiodic Signals</td>
<td>(\sum_{n=-\infty}^{+\infty}</td>
<td>x[n]</td>
</tr>
</tbody>
</table>

*OWN pg. 391
Uses \(\omega\), not \(\Omega\)*
<table>
<thead>
<tr>
<th>Signal</th>
<th>Fourier Transform</th>
<th>Fourier Series Coefficients (if periodic)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sum_{k=N_1}^{\infty} a_k e^{j(kN_1/N \omega)}$</td>
<td>$2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$</td>
<td>$a_k$</td>
</tr>
</tbody>
</table>
| $e^{j\omega_0 n}$ | $2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l)$ | (a) $\omega_0 = \frac{2\pi m}{N}$  
$\quad a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$  
(b) $\omega_0$ irrational $\Rightarrow$ The signal is aperiodic |
| $\cos \omega_0 n$ | $\pi \sum_{l=-\infty}^{\infty} \left(\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\right)$ | (a) $\omega_0 = \frac{2\pi m}{N}$  
$\quad a_k = \begin{cases} 1, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$  
(b) $\omega_0$ irrational $\Rightarrow$ The signal is aperiodic |
| $\sin \omega_0 n$ | $\pi \sum_{l=-\infty}^{\infty} \left(\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\right)$ | (a) $\omega_0 = \frac{2\pi r}{N}$  
$\quad a_k = \begin{cases} 1, & k = r, r \pm N, r \pm 2N, \ldots \\ -1, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$  
(b) $\omega_0$ irrational $\Rightarrow$ The signal is aperiodic |
| $x[n] = 1$ | $2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l)$ | $a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ |
| Periodic square wave $x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & N_1 < |n| \leq N/2 \end{cases}$ and $x[n+N] = x[n]$ | $2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$ | $a_k = \begin{cases} \sin[(2\pi k/N)(N_1 + 1/2)]/N \sin(2\pi k/2N), & k \neq 0, \pm N, \pm 2N, \ldots \\ 2N_1 + 1, & k = 0, \pm N, \pm 2N, \ldots \end{cases}$ |
| $\sum_{k=-\infty}^{\infty} \delta[n-kN]$ | $2\pi \frac{\sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k/N)}{N}$ | $a_k = \frac{1}{N}$ for all $k$ |
| $a^n u[n], \quad |a| < 1$ | $\frac{1}{1 - ae^{-j\omega}}$ | — |
| $x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_1 \end{cases}$ | $\sin[\omega(N_1 + 1/2)]/\sin(\omega / 2)$ | — |
| $\frac{\sin \omega_0 n}{\omega_0} = \frac{\sin(\omega_0 n)}{\omega_0}$  
$0 < \omega < \pi$ | $X(\omega) = \begin{cases} 1, & 0 \leq |\omega| \leq W \\ 0, & W < |\omega| < \pi \end{cases}$  
$X(\omega)$ periodic with period $2\pi$ | — |
| $\delta[n]$ | $1$ | — |
| $u[n]$ | $\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi \delta(\omega - 2\pi k)$ | — |
| $\delta[n-n_0]$ | $e^{-j\omega n_0}$ | — |
| $(n+1)a^n u[n], \quad |a| < 1$ | $\frac{1}{(1 - ae^{-j\omega})^2}$ | — |
| $(n + r - 1)! \quad \frac{1}{n!(r-1)!} e^{j\omega n} \quad |a| < 1$ | $\ast$ | — |
### Summary of Transforms

*Duality will be discussed next lecture*

#### Fourier Series

<table>
<thead>
<tr>
<th>Continuous Time</th>
<th>Discrete Time</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Time Domain</strong></td>
<td><strong>Frequency Domain</strong></td>
</tr>
<tr>
<td>$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 t}$</td>
<td>$a_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t)e^{-j\omega_0 t} , dt$</td>
</tr>
<tr>
<td>Continuous</td>
<td>Discrete</td>
</tr>
<tr>
<td>Periodic</td>
<td>Aperiodic</td>
</tr>
</tbody>
</table>

**Duality**

#### Fourier Transform

<table>
<thead>
<tr>
<th>Continuous Time</th>
<th>Discrete Time</th>
</tr>
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<tbody>
<tr>
<td><strong>Time Domain</strong></td>
<td><strong>Frequency Domain</strong></td>
</tr>
<tr>
<td>$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} , d\omega$</td>
<td>$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} , dt$</td>
</tr>
<tr>
<td>Continuous</td>
<td>Discrete</td>
</tr>
<tr>
<td>Aperiodic</td>
<td>Aperiodic</td>
</tr>
</tbody>
</table>

**Duality**

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CLOSED CONNECTIONS BETWEEN THE CTFT, DTFT, CTFS, DTFS

- **x(t): Aperiodic Continuous**
  - Continuous-time signal

- **x[n]: Aperiodic Discrete**
  - Discrete-time signal

- **x(t): Periodic Continuous**
  - Periodic continuous signal

- **x[n]: Periodic Discrete**
  - Periodic discrete signal

**More details next lecture**

**CTFT:** $X(j\omega)$:
- Aperiodic Continuous

**Aliased version of $X(j\omega)$**
- From sampling $x(t)$ every $T_s = T/(2N+1)$, then scale $\omega$ axis by $2\pi/T_s$ to get $\Omega$ axis

**DTFT:** $X(e^{j\Omega})$:
- Periodic Continuous

**CTFS:** $a_k$
- Aperiodic Discrete

**DTFS:** $a_k$
- Periodic Discrete

**Samples of $X(j\omega)$**
- Every $2\pi/T_0$

**Samples of $X(e^{j\Omega})$**
- Every $2\pi/N_0$
Energy, Power, and Parseval’s Relation

Energy signals have zero power; Power signals have infinite energy

- **Continuous Time Aperiodic Signals**

\[
E = \int_{-\infty}^{\infty} |x(t)|^2 dt
\]

\[
P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt
\]

\[
\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega
\]

- **Discrete Time Aperiodic Signals**

\[
E = \lim_{N \to \infty} \sum_{n=-N}^{N} |x[n]|^2
\]

\[
P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2
\]

\[
\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi}^{2\pi} |X(e^{j\omega})|^2 d\omega
\]

Periodic: \[
\frac{1}{N} \sum_{n=\langle N \rangle}^{\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle}^{\langle N \rangle} |a_k|^2
\]
Applications

- Communications
  - Radio
  - TV/HDTV
  - Telephone
  - Cellular
  - WiFi
  - Ethernet
  - Bluetooth
  - DSL and Cable
  - GPS
  - Satellite Radio
  - Satellite TV
  - Deep Space Communication
  - Walkie Talkies
  - ...

- Signal Processing
  - Smart phones
  - Cameras
  - Videocams
  - Virtual Reality
  - Computer graphics
  - Gaming
  - Audio Enhancement
  - Speech recognition
  - Finance (market predictions and trading)
  - Noise reduction
  - ...

- Control
  - Biological Systems
  - Robots
  - Segway
  - Airplanes
  - (Self-driving) cars
  - Smart cities
  - Video games
  - Heating and Air Conditioning
  - Industrial Automation
  - Cruise control
  - Congestion control (networks/cars)
  - Amplifiers
  - ...

...
Main Points

- Continuous signals/systems appear in nature
- Data processing and storage typically done digitally, hence signal and system inputs/outputs often discrete
- Continuous-time signals represented in frequency via Fourier series (periodic signals) or Fourier transforms (aperiodic)
- Discrete-time signals represented in frequency via Fourier series (periodic signals) or Fourier transforms (aperiodic)
- Strong connections between continuous-time and Discrete-time Fourier Series and Transforms
- Energy and power can be computed in time or frequency domain
- Widespread applications for signals and systems analysis tools