Lecture 17 Outline:
DFT Properties and Circular Convolution

● Announcements:
  ● Reading: “5: The Discrete Fourier Transform” pp. 7-11.
  ● HW 5 posted, short HW (2 analytical and 1 Matlab problem), due today 5pm. No late HWs as solutions will be available immediately.
  ● Midterm details on next page
  ● HW 6 will be posted Wed, due following Wed with free extension to Thurs.

● Properties of the DFT
  ● Circular time and frequency shift
  ● Circular Convolution
  ● Multiplication

● Circular Convolution Methods

● Linear vs. Circular Convolution
Midterm Details

- **Time/Location:** this Friday, May 6, 9:20am-11:20am in this room.
  - Students with class or exam conflicts have been contacted about logistics to start exam at 10:30 (go to room 200-203 - two floors up)

- Open book and notes – you can bring any written material you wish to the exam. Calculators and electronic devices not allowed.

- **Will cover all class material from Lectures 1-14.**
  - See lecture ppt slides for material in the reader that you are responsible for

- **Practice MT posted, worth 25 extra credit points for “taking” it**
  - Not graded. Can be turned in any time up until you take the exam
  - Solutions given when you “turn in” your answers (photo, email, hardcopy, etc.)
  - Have also posted additional practice problems and solutions

- **Instead of MT review, we will provide extra OHs (see calendar):**
  - My extra OHs: Wed 2-3:30pm, Thurs 3-4:30pm.
  - TA extra OHs: Tues 3-5pm, Wed 2-3pm, 5-6pm, Thur: 10:30-11:30am, 1:30-2:30pm, 5-6pm, 8-9pm. All in 3rd floor Packard kitchen area.
Review of Last Lecture

- **Discrete Fourier Series (DFS) Pair for Periodic Signals**
  \[
  \tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn} \quad \tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}
  \]

- **Discrete Fourier Transform (DFT) Pair**
  \[
  x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \quad X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}
  \]
  - \(x[n]\) and \(X[k]\) are one period of \(\tilde{x}[n]\) and \(\tilde{X}[k]\), respectively

- **DFT is DTFT sampled at \(N\) equally spaced frequencies between 0 and \(2\pi\):**
  \[
  X[k] = \mathcal{X}\left(e^{j\Omega}\right)_{\Omega=k\frac{2\pi}{N}} , \quad 0 \leq k \leq N - 1
  \]

- **The DFT and IDFT can be computed as matrix multiplications where matrix elements are different powers of \(W_N^{\pm kn}\)**
  \[
  W_N^{\pm kn} = e^{\pm j\frac{2\pi}{N}nk}
  \]
# Properties of the DFS/DFT

<table>
<thead>
<tr>
<th>Property</th>
<th>$N$-periodic sequence</th>
<th>$N$-periodic DFS</th>
<th>Property</th>
<th>$N$-point sequence</th>
<th>$N$-point DFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>$a \hat{x}_1[n] + b \hat{x}_2[n]$</td>
<td>$a\hat{X}_1[k] + b\hat{X}_2[k]$</td>
<td>Linearity</td>
<td>$ax_1[n] + bx_2[n]$</td>
<td>$aX_1[k] + bX_2[k]$</td>
</tr>
<tr>
<td>Duality</td>
<td>$\hat{X}[n]$</td>
<td>$N \hat{x}[-k]$</td>
<td>Duality</td>
<td>$X[n]$</td>
<td>$N x[(-k)_N]$</td>
</tr>
<tr>
<td>Time Shift</td>
<td>$\hat{x}[n-m]$</td>
<td>$W_N^{lm} \hat{X}[k]$</td>
<td>Circular Time Shift</td>
<td>$\hat{x}[((n-m)_N]$</td>
<td>$W_N^{lm} X[k]$</td>
</tr>
<tr>
<td>Frequency Shift</td>
<td>$W_N^{-lm} \hat{x}[n]$</td>
<td>$\hat{X}[k-l]$</td>
<td>Circular Frequency Shift</td>
<td>$W_N^{lm} x[n]$</td>
<td>$X[((k-l))_N]$</td>
</tr>
<tr>
<td>Periodic Convolution</td>
<td>$\sum_{m=0}^{N-1} \hat{x}_1[m] \hat{x}_2[n-m]$</td>
<td>$\hat{X}_1[k] \hat{X}_2[k]$</td>
<td>Circular Convolution</td>
<td>$\sum_{m=0}^{N-1} x_1[m] x_2[((n-m)_N]$</td>
<td>$X_1[k] X_2[k]$</td>
</tr>
<tr>
<td>Multiplication</td>
<td>$\hat{x}_1[n] \hat{x}_2[n]$</td>
<td>$\frac{1}{N} \sum_{l=0}^{N-1} \hat{x}_1[l] \hat{x}_2[k-l]$</td>
<td>Multiplication</td>
<td>$x_1[n] x_2[n]$</td>
<td>$\frac{1}{N} \sum_{l=0}^{N-1} x_1[l] x_2[((k-l))_N]$</td>
</tr>
<tr>
<td>Complex Conjugation</td>
<td>$\hat{x}^*[n]$</td>
<td>$\hat{X}^*[-k]$</td>
<td>Complex Conjugation</td>
<td>$x^*[n]$</td>
<td>$X^*[((-k))_N]$</td>
</tr>
</tbody>
</table>
## Properties (Continued)

<table>
<thead>
<tr>
<th>Time-Reversal and Complex Conjugation</th>
<th>( \bar{x}^*[n] )</th>
<th>( \bar{X}^*[k] )</th>
<th>Time-Reversal and Complex Conjugation</th>
<th>( x^*[((-n)_N)] )</th>
<th>( X^*[k] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real Part</td>
<td>Re{\bar{x}[n]}</td>
<td>( \bar{X}_{op}[k] = \frac{1}{2}(\bar{X}[k] + \bar{X}^*[−k]) )</td>
<td>Real Part</td>
<td>Re{x[n]}</td>
<td>( X_{op}[k] = \frac{1}{2} (X[k] + X^*[((-k)_N)]) )</td>
</tr>
<tr>
<td>Imaginary Part</td>
<td>j Im{\bar{x}[n]}</td>
<td>( \bar{X}_{op}[k] = \frac{1}{2}(\bar{X}[k] − \bar{X}^*[−k]) )</td>
<td>Imaginary Part</td>
<td>j Im{x[n]}</td>
<td>( X_{op}[k] = \frac{1}{2} (X[k] − X^*[((-k)_N)]) )</td>
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<tr>
<td>Even Part</td>
<td>( \bar{x}_{op}[n] = \frac{1}{2}(\bar{x}[n] + \bar{x}^*[−n]) )</td>
<td>Re{\bar{x}[k]}</td>
<td>Even Part</td>
<td>( x_{op}[n] = \frac{1}{2}(x[n] + x^*[((-n)_N)]) )</td>
<td>Re{X[k]}</td>
</tr>
<tr>
<td>Odd Part</td>
<td>( \bar{x}_{op}[n] = \frac{1}{2}(\bar{x}[n] − \bar{x}^*[−n]) )</td>
<td>j Im{\bar{x}[k]}</td>
<td>Odd Part</td>
<td>( x_{op}[n] = \frac{1}{2}(x[n] − x^*[((-n)_N]]) )</td>
<td>j Im{X[k]}</td>
</tr>
<tr>
<td>Symmetry for Real Sequence</td>
<td>( \bar{x}[n] = \bar{x}^*[n] )</td>
<td>( \bar{X}[k] = \bar{X}^*[−k] )</td>
<td>Symmetry for Real Sequence</td>
<td>( x[n] = x^*[n] )</td>
<td>( X[k] = X^*[((-k)_N)] )</td>
</tr>
<tr>
<td></td>
<td>( \left{ \begin{align*} \text{Re}{\bar{X}[k]} &amp;= \text{Re}{\bar{X}^<em>[−k]} \ \text{Im}{\bar{X}[k]} &amp;= -\text{Im}{\bar{X}^</em>[−k]} \end{align*} \right} )</td>
<td>( \left{ \begin{align*}</td>
<td>\bar{X}[k]</td>
<td>&amp;=</td>
<td>\bar{X}^*[−k]</td>
</tr>
<tr>
<td>Parseval’s Identity</td>
<td>( \sum_{n=0}^{N-1} \bar{x}_1[n] \bar{x}<em>2^*[n] = \frac{1}{N} \sum</em>{k=0}^{N-1} \bar{X}_1[k] \bar{X}^*_2[k] )</td>
<td>Parseval’s Identity</td>
<td>( \sum_{n=0}^{N-1} x_1[n] x_2^<em>[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_1[k] X^</em>_2[k] )</td>
<td>( \sum_{n=0}^{N-1}</td>
<td>x[n]</td>
</tr>
</tbody>
</table>
Circular Time/Frequency Shift

- Circular Time Shift (proved by DFS property of $\widetilde{x}[n]$)

\[
x[((n-m))_N] \leftrightarrow W_N^{-km} X[k] = e^{-j \frac{2\pi}{N} km} X[k]
\]

- Circular Frequency Shift (IDFS property of $\widetilde{X}[k]$)

\[
W_N^{ln} x[n] = e^{j \frac{2\pi}{N} ln} x[n] \leftrightarrow X[((k-l))_N]
\]
Circular Convolution

- Defined for two N-length sequences as

\[
x_1[n] \odot x_2[n] = \sum_{m=0}^{N-1} x_1[m] x_2[(n-m) \mod N] = \begin{cases} 
\sum_{m=0}^{N-1} \tilde{x}_1[m] \tilde{x}_2[n-m] & 0 \leq n \leq N - 1 \\
0 & \text{otherwise}
\end{cases}
\]

- Circular convolution in time is multiplication in frequency

\[
\sum_{m=0}^{N-1} x_1[m] x_2[(n-m) \mod N] \leftrightarrow X_1[k] X_2[k] \quad \text{Duality} \quad x_1[n] \cdot x_2[n] \leftrightarrow X_1[k] \odot X_2[k]
\]
Computing Circular Convolution;
Circular vs. Linear Convolution

- **Computing circular convolution:**
  - Linearly convolve $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$
  - Place sequences on circle in opposite directions, sum up all pairs, rotate outer sequence clockwise each time increment

$$ x_i[n] \oslash x_2[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m] \tilde{x}_2[n-m] \quad 0 \leq n \leq N-1 $$

- Otherwise: $0$

- Circular versus Linear Convolution

- Can obtain a linear convolution from a circular one by zero padding both sequences
Main Points

- DFS/DFT have similar properties as DTFS/DTFT but with modifications due to periodic/circular characteristics.
- A circular time shift leads to multiplication in frequency by a complex phase term.
- A circular frequency shift leads to a complex phase term multiplication with the original sequence (modulation).
- Circular convolution in time leads to multiplication of DFTs.
- Circular convolution can be computed based on linear convolution of periodized sequences or circle method.