Lecture 2 Outline
“Fun” with Fourier

- Announcements:
  - Poll for discussion section and OHs: please respond
  - First HW posted 5pm tonight

- Duality Relationships
- Connections between Continuous/Discrete Time
- Filtering and Convolution
- Periodic Signals
- Energy, Power, and Parseval
# Duality in Fourier Transforms

<table>
<thead>
<tr>
<th>Continuous Time</th>
<th>Discrete Time</th>
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<tr>
<td><strong>Time Domain</strong></td>
<td><strong>Frequency Domain</strong></td>
</tr>
<tr>
<td><strong>Fourier Series</strong></td>
<td></td>
</tr>
<tr>
<td>[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 t} ]</td>
<td>[ a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-j\omega_0 t} dt ]</td>
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<tr>
<td>Continuous</td>
<td>Discrete</td>
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<tr>
<td>Periodic</td>
<td>duality</td>
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<tr>
<td><strong>Fourier Transform</strong></td>
<td></td>
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<tr>
<td>[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega ]</td>
<td>[ X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt ]</td>
</tr>
<tr>
<td>Continuous</td>
<td>Aperiodic</td>
</tr>
<tr>
<td>Aperiodic</td>
<td>duality</td>
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</table>

\[ x(t) \leftrightarrow X(j\omega) \quad \Rightarrow \quad X(t) \leftrightarrow 2\pi x(-j\omega) \]
CONNECTIONS BETWEEN THE CTFT, DTFT, CTFS, DTFS

Using rect↔sinc example

\( x(t) \): Aperiodic Continuous

\( x[n] \): Aperiodic Discrete

\( x(t) \): Periodic Continuous

\( x[n] \): Periodic Discrete

\( x(t) \): Aperiodic Continuous

\( X(j\omega) \): Aperiodic Continuous

\( X(e^{j\Omega}) \): Periodic Continuous

\( a_k \): Aperiodic Discrete

\( x[n] \): Periodic Discrete

\( X(e^{j\Omega}) \): Periodic Discrete

Using rect↔sinc example

Aliased version of \( X(j\omega) \) from sampling \( x(t) \) every \( T_s = T/(2N+1) \), then scale \( \omega \) axis by \( 2\pi/T_s \) to get \( \Omega \) axis

Samples of \( X(j\omega) \) every \( 2\pi/T_0 \)

Samples of \( X(e^{j\Omega}) \) every \( 2\pi/N_0 \)
Filtering and Convolution

- Filtering

\[ X(j\omega) \rightarrow H(j\omega) \rightarrow Y(j\omega) = H(j\omega)X(j\omega) \]

\[ X(e^{j\Omega}) \rightarrow H(e^{j\Omega}) \rightarrow Y(e^{j\Omega}) = H(e^{j\Omega})X(e^{j\Omega}) \]

- Example:

\[ x[n]: \text{Periodic Discrete} \]

- Important convolution examples
Periodic Signals

- **Continuous time:**
  - $x(t)$ periodic iff there exists a $T_0 > 0$ such that $x(t) = x(t + T_0)$ for all $t$
  - $T_0$ is called a period of $x(t)$; smallest such $T_0$ is fundamental period
  - If $x(t)$ periodic with period $T_1$, $y(t)$ periodic with period $T_2$, then $x(t) + y(t)$ is periodic with period $k_1 T_1 = k_2 T_2$ if such integers $k_i$ exist.

- **Discrete time:**
  - $x[n]$ periodic iff there exists a $N_0 > 0$ such that $x[n] = x[n + N_0]$ for all $n$
  - $N_0$ is called a period of $x(t)$; smallest such $N_0$ is fundamental period
  - If $x[n]$ periodic with period $N_1$, $y[n]$ periodic with period $N_2$, then $x[n] + y[n]$ is periodic with period $k_1 N_1 = k_2 N_2$ if such integers $k_i$ exist.

- **Filtering periodic signals**
  
  $\sum_k a_k e^{j k \omega t} \rightarrow H(j \omega) \rightarrow \sum_k a_k H(j k \omega_0) e^{j k \omega_0 t}$
  
  $\sum_k a_k e^{j k \omega n} \rightarrow H(e^{j \Omega}) \rightarrow \sum_k a_k H(j k \omega_0) e^{j k \omega_0 n}$
Energy, Power, and Parseval’s Relation

Energy signals have zero power; Power signals have infinite energy

- Continuous Time Aperiodic Signals

\[ E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \]

\[ P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \]

- Periodic: \[ \frac{1}{T} \int_{T} |x(t)|^2 \, dt = \sum_{k=-\infty}^{+\infty} |a_k|^2 \]

- Discrete Time Aperiodic Signals

\[ E = \lim_{N \to \infty} \sum_{n=-N}^{N} |x[n]|^2 \]

\[ P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \]

\[ \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\Omega})|^2 \, d\Omega \]

- Periodic: \[ \frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2 \]
Examples

- **Continuous-time**

\[
E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\frac{5T}{2}}^{\frac{5T}{2}} A^2 dt = A^2 T
\]

\[A \Pi(t / T) \Leftrightarrow A T \text{sinc}(0.5 \omega T / \pi)\]

Which would you rather integrate?

- **Discrete-time**

\[x[n]: \text{Periodic Discrete}\]

\[
E = \frac{1}{N} \sum_{n=-N}^{N} |x[n]|^2 = \frac{1}{N} \sum_{n=-N_1}^{N_1} 1 = (2N_1 + 1) / N
\]

Which would you rather sum?
Main Points

- Duality saves work! Only need to compute half of the Fourier Series and Transform pairs, use duality to get the other half.

- Fourier transforms of continuous aperiodic signals yield their discrete-time counterparts by sampling and appropriate scaling of $\Omega$ axis.

- Fourier transforms of continuous or discrete periodic signals obtained by sampling their aperiodic Fourier transforms

- Filtering convolves input signal with impulse response of filter. This becomes multiplication of corresponding Fourier Transforms
  - One domain typically easier to compute than the other

- Sum of 2 periodic signals has period equal to the LCM of the two periods.
  - Filtering a periodic signal preserves Fourier Series components in filter BW

- Energy and power can be computed in time or frequency domain