Lecture 22 Outline:
Laplace Examples, Inverse, Rational Form

- **Announcements:**
  - **Reading:** “6: The Laplace Transform” pp. 9-18.
  - HW 7 posted, due today

- More Laplace Transform Examples

- Inverse Laplace Transforms

- Rational Laplace Transforms

- ROCs for Right/Left/Two-sided Signals

- Magnitude/Phase of Fourier Transforms from Laplace
Review of Last Lecture

- Laplace transform generalizes Fourier Transform
  - Always exists within a Region of Convergence
  - Used to study systems/signals w/out Fourier Transforms

Defn: \( L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} \, dt \), \( s = \sigma + j\omega \); exists if \( \int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| \, dt < \infty \)

- Relation with Fourier Transform:
  \[
  L[x(t)] = F\left[ x(t)e^{-\alpha t} \right] \quad X(s)\big|_{s=j\omega} = X(j\omega) = F[x(t)]
  \]

- Region of Convergence (ROC):
  - All values of \( s=\sigma+j\omega \) such that \( L[x(t)] \) exists
  - Depends only on \( \sigma \)

- Example
  \[
  e^{-at}u(t) \xrightarrow{L} X(s) = \frac{1}{s + a}, \quad \text{Re}(s) > -a
  \]
Laplace Transform Examples

- **Rect function:** \( x_2(t) = \Pi\left(\frac{t}{2\tau}\right) = \begin{cases} 1 & |t| \leq \tau \\ 0 & |t| > \tau \end{cases} \)
  - Familiar friend; has a Fourier transform
  - Laplace Transform: \( X_2(s) = \int_{-\tau}^{\tau} e^{-st} dt = \frac{e^{s\tau} - e^{-s\tau}}{s} \)
  - ROC: Finite everywhere except possibly \( s=0 \): \( \lim_{s \to 0} X_2(s) = \lim_{s \to 0} \frac{d}{ds} \left( e^{s\tau} - e^{-s\tau} \right) = 2\tau \)
    - So ROC is entire \( s \)-plane: \( ROC=\{\text{all } s\} \): true for any finite duration absolutely integrable function
  - Fourier transform \( (j\omega \in ROC) \): \( X_2(j\omega) = \frac{e^{j\omega\tau} - e^{-j\omega\tau}}{j\omega} = 2\tau \frac{\sin \omega\tau}{\omega\tau} = 2\tau \text{sinc}\left(\frac{\omega\tau}{\pi}\right) \)

- **Left-Sided Real Exponential:** \( x_4(t) = -e^{-at}u(-t) \)
  - Laplace: \( X_4(s) = -\int_{-\infty}^{0} e^{-(s+a)t} dt \), converges if \( \text{Re}(s)<-a \)
    - \( X_4(s) = \frac{1}{s + a} \), \( \text{Re}(s) < -a \)
    - Same as for right-sided case but with a different ROC \( (\text{Re}(s)>-a) \)
  - Does not have a Fourier transform if \(-a<0\), else \( X_4(j\omega) = \frac{1}{j\omega + a} \)
Another Example: Two-Sided Real Exponential

\[ x_5(t) = e^{b|t|}, \quad b \text{ real} \]

- Can write as sum of two terms: \[ x_5(t) = e^{bt}u(t) + e^{-bt}u(-t) \]

\[ e^{bt}u(t) \leftrightarrow \frac{1}{s-b}, \quad \text{Re}(s) > b \]
\[ e^{-bt}u(-t) \leftrightarrow -\frac{1}{s+b}, \quad \text{Re}(s) < -b \]

\[
X_5(s) = \frac{1}{s-b} - \frac{1}{s+b} = \frac{2b}{(s-b)(s+b)} = \frac{2b}{s^2 - b^2}
\]

\[ \text{ROC} = \{\text{Re}(s) > b\} \cap \{\text{Re}(s) < -b\} \]
Inverse Laplace Transform

- **Use Fourier Relation:** \( X(s) = X(\sigma + j\omega) = F[x(t)e^{-\sigma t}] = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t} dt \)

- **Can apply inverse Fourier:**
  
  \[
  x(t)e^{-\sigma t} = F^{-1}[X(\sigma + j\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega \tau} d\omega
  \]

  - Multiply both sides by \( e^{-\sigma t} \):
    
    \[
    x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)\tau} d\omega
    \]

- **How to integrate over s-plane: change of variables**
  
  - \( s = \sigma + j\omega; \sigma \text{ fixed}, \ ds = dj\omega \):
    
    \[
    x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st} ds
    \]

  - Complex integration part of complex analysis (Math 116)
  - We need a different approach

- **Rational Laplace Transforms:**
  
  - Inverse obtained through partial fraction expansion
  - Allows analysis through pole-zero plots
Rational Laplace Transforms

- Numerator and Denominator are polynomials
  \[ X(s) = \frac{B(s)}{A(s)} = \frac{b_0 + b_1 s + \cdots + b_{M-1} s^{M-1} + b_M s^M}{a_0 + a_1 s + \cdots + a_{N-1} s^{N-1} + a_N s^N} \]

- Can factor as product of monomials
  \[ X(s) = \frac{b_M}{a_N} \frac{(s - \beta_1)(s - \beta_2) \cdots (s - \beta_{M-1})(s - \beta_M)}{(s - \gamma_1)(s - \gamma_2) \cdots (s - \gamma_{N-1})(s - \gamma_N)} \]

  - \( \beta \)'s are zeros (where \( X(s) = 0 \)), \( \gamma \)'s are poles (where \( X(s) = \infty \))
  - ROC cannot include any poles
  - If \( X(s) \) real, \( a \)'s and \( b \)'s are real \( \rightarrow \) all zeros of \( X(s) \) are real or occur in complex-conjugate pairs. Same for the poles.
  - \( b_m/a_n \), zeros, poles, and ROC fully specify \( X(s) \)

- Example: Two-sided Exponential
  \[ X_5(s) = \frac{1}{s-b} - \frac{1}{s+b} = \frac{2b}{(s-b)(s+b)} = \frac{2b}{s^2 - b^2} \]
More on Laplace

- ROC for Right, Left, and Two-Sided Signals
  - Right-sided: \(x(t) = 0\) for \(t < a\) for some \(a\)
    - ROC is to the right of the rightmost pole, e.g. RH exponential
  - Left-sided: \(x(t) = 0\) for \(t > a\) for some \(a\)
    - ROC is to the left of the leftmost pole, e.g. LH exponential
  - Two-sided: neither right or left sided
    - ROC is a vertical strip between two poles, e.g. 2-sided exponential

- Magnitude/Phase of Fourier from Laplace
  - Given Laplace in rational form with \(j\omega\) in ROC
    \[
    H(s) = \frac{b_M}{a_N} \prod_{k=1}^{N} (s - \gamma_k) \quad \implies \quad H(j\omega) = \frac{b_M}{a_N} \prod_{k=1}^{M} (j\omega - \beta_k) \prod_{k=1}^{N} (j\omega - \gamma_k) \\
    |H(j\omega)| = \frac{b_M}{a_N} \left| \prod_{k=1}^{M} (j\omega - \beta_k) \prod_{k=1}^{N} (j\omega - \gamma_k) \right| \\
    \angle H(j\omega) = \angle \left( \frac{b_M}{a_N} \right) + \sum_{k=1}^{M} \angle (j\omega - \beta_k) - \sum_{k=1}^{N} \angle (j\omega - \gamma_k)
    \]
  - Can obtain magnitude and phase from individual components using geometry, formulas, or Matlab
Main Points

- Laplace transform includes the ROC; different functions can have same Laplace transform with different ROCs.

- Inverse Laplace transform requires complex analysis to compute: need a different approach.

- Expressing the Laplace transform in rational form allows inverse via partial fraction expansion.
  - Also allows Laplace characterization via pole-zero plot.

- One-Sided signals have sided ROCs. Two sided signals have strips as ROCs.

- Can determine magnitude and phase of Fourier transform easily from rational form of Laplace.